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The existence of periodic solutions of nonlinear differential equations of second order

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1 Introduction

It is well-known from qualitative theory of ordinary differential equations that the problem of existence of periodic solutions for various nonlinear differential equations of higher order continues to attract the attentions of many specialists despite its long history. In many works, several mathematics dealt with the problems by using Lyapunovs functions and Greens functions, obtained criteria for the existence of periodic solutions. In particular, for the use of Greens functions, ones can refer to Cronin [1] and Shadman [3].

In [2], the author used Leray-Schauder principle to show the existence of periodic solutions to second order nonlinear differential equations of the form

$$x'' + c(t)x' + f(t,x) = e(t).$$

Now, consider the real second order nonlinear differential equations of the type:

$$x'' + c(t)x' + f(t,x) = p(t,x,x'). \tag{1.1}$$

in which, $p(t,x,x', f(t,x)$ and $c(t)$ are continuous functions in their respective domains $[0,L) \times \mathbb{R}^2, [0,L) \times \mathbb{R}$ e $[0,L)$, respectively. Further, it is assumed that all initial value problems corresponding to equation (1.1) can be extended to $[0,L)$.

2 Mathematical Results

The following result is established.

Theorem 2.1. *We assume that the following conditions hold:*

1. $|f(t,x)| \leq \gamma|x| + \beta$ *for all* $t \in [0,L]$ *and* $x| < \infty$, *where* γ *and* β *are some non-negative constants.*

2. $|p(t,x,x'| \leq |e(t)|$ *for all* t, x *and* x' , *and* $e(t)$ *is a continuous function for all* $t \in [0,L]$.

3. $\left(\frac{L}{\pi}\right)^2 + \gamma_1 \left(\frac{L}{\pi}\right) < 1, \gamma_1 = \max|c(t)|$.

Then equation (1.1) possesses a solution satisfying

$$x^i(0) + x^i(L) = 0, \ (i = 0,1). \tag{2.2}$$

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Proof First, we show an estimate on the magnitude of the solutions of problem:

$$
\begin{gathered}
x'' = \mu \left[p(t, x, x') - f(t, x) \right], \ \mu \in [0, 1], \\
x^i(0) + x^i(L) = 0, \ (i = 0, 1).
\end{gathered} \tag{2.3}
$$

We assume that $x(t)$ is a function of class $C^{n-1}[0, \omega]$, such that $x(t+L) + x(t) = 0$ for all t, and we use Wirtingers inequalities in the following from:

$$
||x^{(i-1)}(t)||_2 \leq \left(\frac{L}{\pi} \right)^{n-i+1} ||x^{(n)}(t)||_2, \quad (i = 1, 2, \ldots, n), \tag{2.4}
$$

$$
||.||_2 = \left[\int_0^L |.|^2 dt \right]^2 .
$$

Now, we suppose that is a solution of the problem given by (2.3). In view of the assumptions of the theorem, it is easily followed from (2.3) that

$$
|x''(t)| \leq \gamma_1 |x'| + \mu \left[|e(t)| + \gamma |x(t)| + \beta \right].
$$

Hence, by using the Minkowskis inequality

$$
||x''(t)||_2 \leq \gamma_1 ||x'||_2 + \mu \left\{ ||e(t)||_2 + \gamma ||x(t)||_2 + \beta \sqrt{L} \right\},
$$

it can be seen from Wirtingers inequality that

$$
||x''(t)||_2 \leq \gamma_1 \left(\frac{L}{\pi} \right) ||x'||_2 + \mu \left\{ ||e(t)||_2 + \gamma \left(\frac{L}{\pi} \right)^2 ||x''(t)||_2 + \beta \sqrt{L} \right\},
$$

where

$$
[1 - \gamma_1 \left(\frac{L}{\pi} \right) - \mu \gamma \left(\frac{\omega}{\pi} \right)^2]||x''(t)||_2 \leq \mu \left\{ ||e(t)||_2 + \beta \sqrt{L} \right\}.
$$

Making use of assumption 3 of the theorem and in view of the fact $0 \leq \ \leq 1$, we obtain

$$
||x''(t)||_2 \leq \frac{||e(t)||_2 + \beta \sqrt{L}}{1 - \gamma_1 \left(\frac{L}{\pi} \right) - \mu \gamma \left(\frac{L}{\pi} \right)^2}. \tag{2.5}
$$

Now, we write

$$
x^{(i-1)}(t) = x^{(i-1)}(0) + \int_0^t x^{(i)}(\tau) d\tau, \ (i = 1, 2).
$$

For $t = \omega$, we have

$$
x^{(i-1)}(L) = x^{(i-1)}(0) + \int_0^L x^{(i)}(\tau) d\tau, \ (i = 1, 2). \tag{2.6}
$$

By noting the equality $x^{(i)}(0) + x^{(i)}(L) = 0$, we get

$$
x^{(i-1)}(0) = -x^{(i-1)}(L) = -\frac{1}{2} \int_0^L x^{(i)}(\tau) d\tau, \ (i = 1, 2).
$$

Referências

[1] Cronin, J.*Fixed points and topological degree in nonlinear analysis. Mathematical Surveys*, No. 11, American Mathematical Society, Providence, R.I. 1964.

[2] Villanueva Z. David. -*Periodic solutions and fixed points methods. Thesis Dourorado-USP*, 2006.

[3] Mehri, B. and Shadman, D.- *On the existence of periodic solutions of a certain class of second order nonlinear diferential equation*, Math. Inequal. Appl. 1 (1998), no. 3, 431-436.