Representations of compact Lie groups and their orbit spaces

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Encounters in Geometry
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- **Main definition:** $\rho_i : G_i \to O(V_i)$ for $i = 1, 2$, are called quotient-equivalent if V_1/G_1, V_2/G_2 are isometric.

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- **Question.** Does a minimal reduction always come from a minimal generalized section?

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- Not true in general; $x \in X$ is an orbifold point iff the slice repr at $p \in \pi^{-1}(x)$ is polar [LT]: $X_{orb} \supset X_{reg}$
- We have a classification of when $S(V)/G$ is a good Riemannian orbifold.

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or a pseudo-Hopf action which is a doubling representation

U_2	$\mathbf{C}^2 \oplus \mathbf{C}^2$
Sp_2	$\mathbf{H}^2 \oplus \mathbf{H}^2$
$Sp_2 U_1$	$\mathbf{H}^2 \oplus \mathbf{H}^2$

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  - If ρ is irreducible then its cohomogeneity is 3, its connected minimal reduction is $(U_1, \mathbf{C} \oplus \mathbf{C})$, and it is one of

$SO_2 \times Spin_9$	$\mathbf{R}^2 \otimes_{\mathbf{R}} \mathbf{R}^{16}$	$-$
$U_2 \times Sp_n$	$\mathbf{C}^2 \otimes_{\mathbf{C}} \mathbf{C}^{2n}$	$n \geq 2$
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    - Representations of cohomogeneity 3

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$U_1 \times SU_n \times U_1$	$\mathbf{C}^n \oplus \mathbf{C}^n$	$n \geq 2$
$Sp_1 \times Sp_n \times Sp_1$	$\mathbf{H}^n \oplus \mathbf{H}^n$	$n \geq 2$

    or an orbit-equivalent subgroup action, with conn min reduction $(U_1, \mathbf{C} \oplus \mathbf{C})$.

Non-reduced case

- **Theorem.** Let $\rho : G \to O(V)$ be non-reduced and non-polar, where G is connected. Assume that $X = S(V)/G$ is a good Riemannian orbifold.
  - If ρ is irreducible then its cohomogeneity is 3, its connected minimal reduction is $(U_1, \mathbf{C} \oplus \mathbf{C})$, and it is one of

$SO_2 \times Spin_9$	$\mathbf{R}^2 \otimes_{\mathbf{R}} \mathbf{R}^{16}$	–
$U_2 \times Sp_n$	$\mathbf{C}^2 \otimes_{\mathbf{C}} \mathbf{C}^{2n}$	$n \geq 2$
$Sp_1 \times Sp_n$	$S^3(\mathbf{C}^2) \otimes_{\mathbf{H}} \mathbf{C}^{2n}$	$n \geq 2$

  - If ρ is not irreducible then it has exacly two irreducible summands, and we have the following possibilities:
    - Representations of cohomogeneity 3

SO_n	$\mathbf{R}^n \oplus \mathbf{R}^n$	$n \geq 3$
$U_1 \times SU_n \times U_1$	$\mathbf{C}^n \oplus \mathbf{C}^n$	$n \geq 2$
$Sp_1 \times Sp_n \times Sp_1$	$\mathbf{H}^n \oplus \mathbf{H}^n$	$n \geq 2$

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    - Representations of cohomogeneity 4

SU_n	$\mathbf{C}^n \oplus \mathbf{C}^n$	$n \geq 3$
U_n	$\mathbf{C}^n \oplus \mathbf{C}^n$	$n \geq 3$
$Sp_n Sp_1$	$\mathbf{H}^n \oplus \mathbf{H}^n$	$n \geq 3$
$U_1 \times Sp_n \times U_1$	$\mathbf{H}^n \oplus \mathbf{H}^n$	$n \geq 2$
$Spin_9$	$\mathbf{R}^{16} \oplus \mathbf{R}^{16}$	–

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      with connected minimal reduction $(U_2, \mathbf{C}^2 \oplus \mathbf{C}^2)$.
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$U_1 \times Sp_n \times U_1$	$\mathbf{H}^n \oplus \mathbf{H}^n$	$n \geq 2$
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    with connected minimal reduction $(U_2, \mathbf{C}^2 \oplus \mathbf{C}^2)$.
    - $(Sp_n, \mathbf{H}^n \oplus \mathbf{H}^n)$ $(n \geq 3)$ of cohom 6 with conn min reduction $(Sp_2, \mathbf{H}^2 \oplus \mathbf{H}^2)$.
    - $(Sp_nU_1, \mathbf{H}^n \oplus \mathbf{H}^n)$ $(n \geq 3)$ of cohom 5 with conn min reduct $(Sp_2U_1, \mathbf{H}^2 \oplus \mathbf{H}^2)$.

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  - The group G is the semisimple factor of an irreducible polar representation of Hermitian type such that action of G is not orbit-equivalent to the polar representation:

SU_n	$S^2\mathbf{C}^n$	$n \geq 3$
SU_n	$\Lambda^2\mathbf{C}^n$	$n = 2p \geq 6$
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- Discuss case $k = 1$.

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