Edges, Orbifolds,

and

Seiberg-Witten Theory

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Stony Brook University

Encounters in Geometry
Cabo do Frio, Brazil
June 3, 2013

Main reference:

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arXiv:1305.1960 [math.DG].

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cf. joint paper with Atiyah:

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cf. joint paper with Atiyah:

arXiv:1203.6389 [math.DG],
to appear in Math. Proc. Cambr. Phil. Soc.

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Proof. Bianchi identity $\Longrightarrow \nabla \cdot \mathring{r} = (\frac{1}{2} - \frac{1}{n}) ds.$

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Anderson, Bando-Kasue-Nakajima, ...

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But today we will focus on codimension 2 case ...

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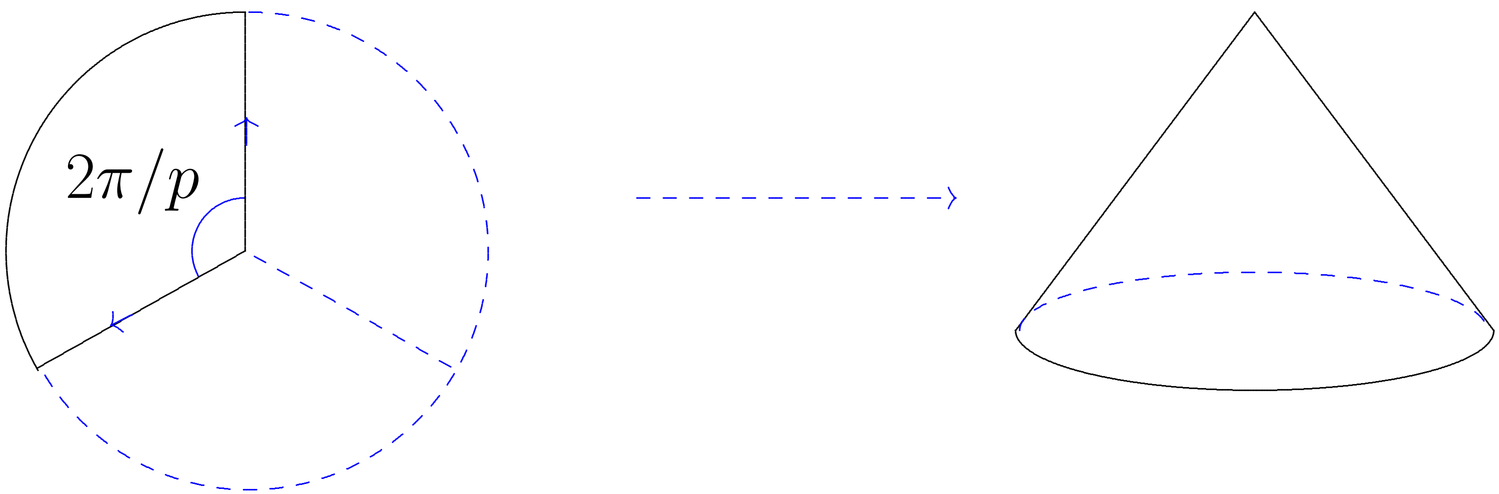

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Strategy: Deform cone angle $2\pi\beta$ with $\beta \in (0, 1/p]$.

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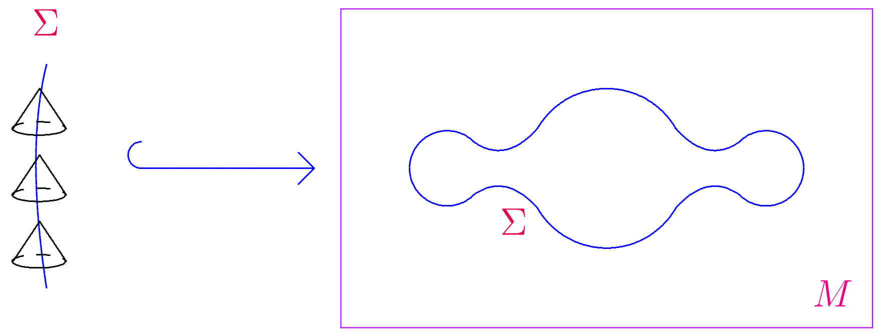

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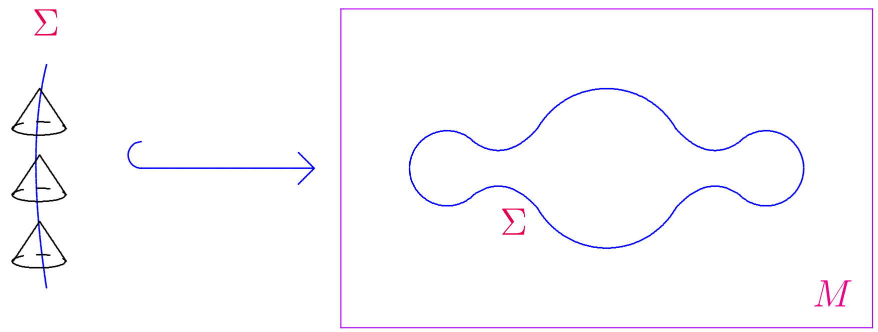

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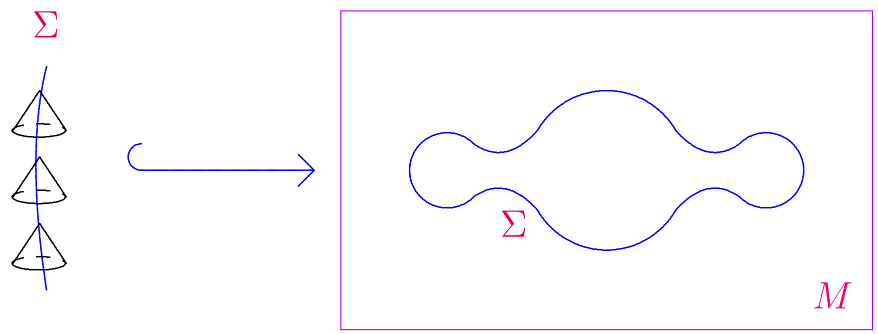

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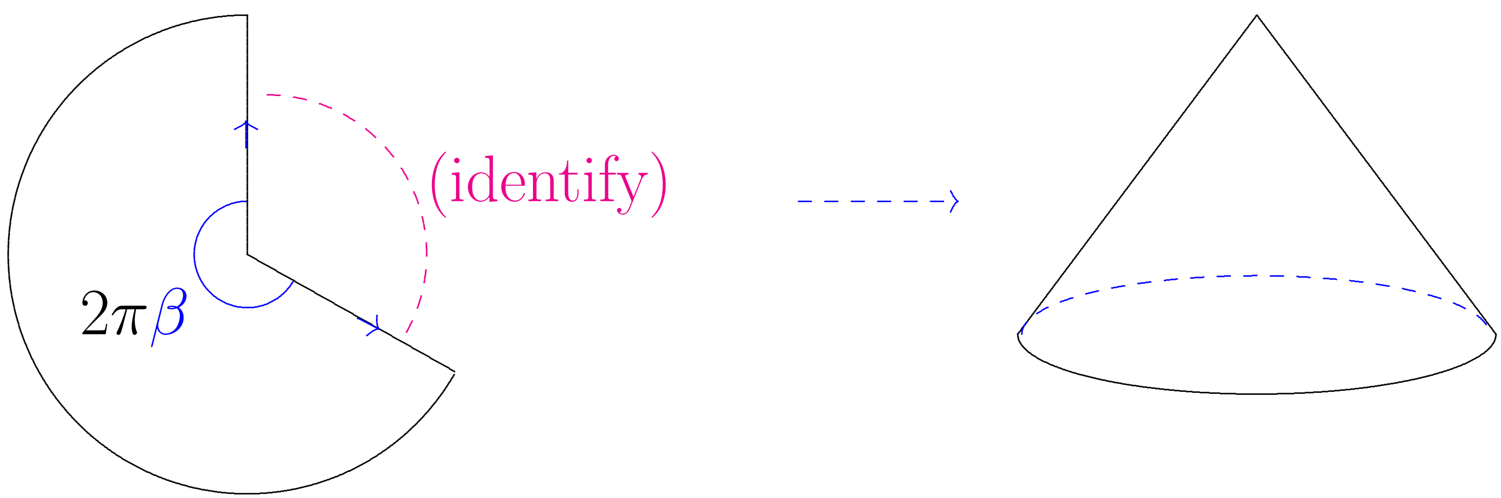

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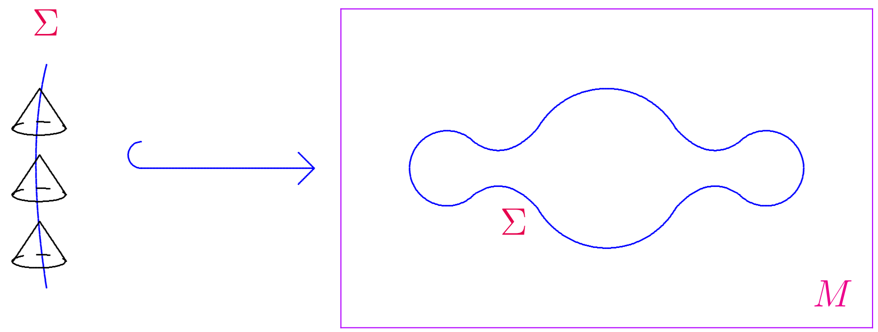

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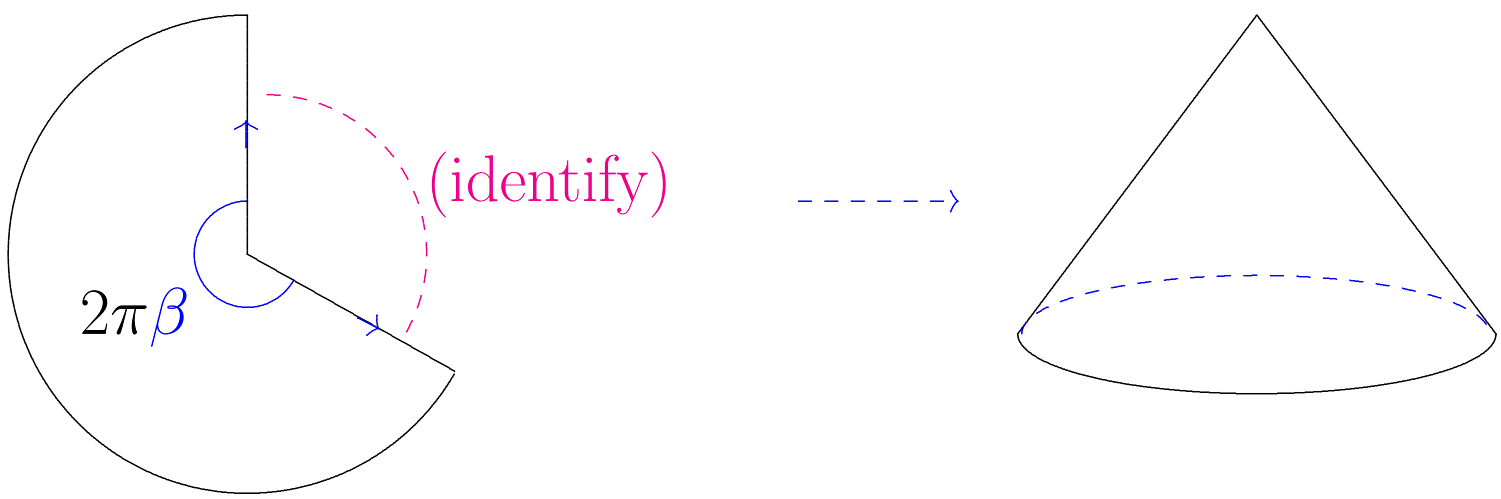

Program for $c_1 > 0$ assumes $[\Sigma] \propto c_1$.

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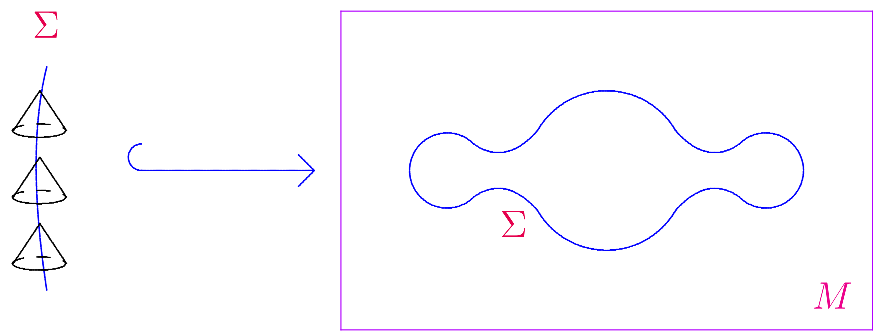
K-stability is a criterion which is formulated purely in terms of algebraic geometry. It concerns singular limits of embeddings $(M, J) \hookrightarrow \mathbb{CP}_N$.

Edge-Cone Metrics:



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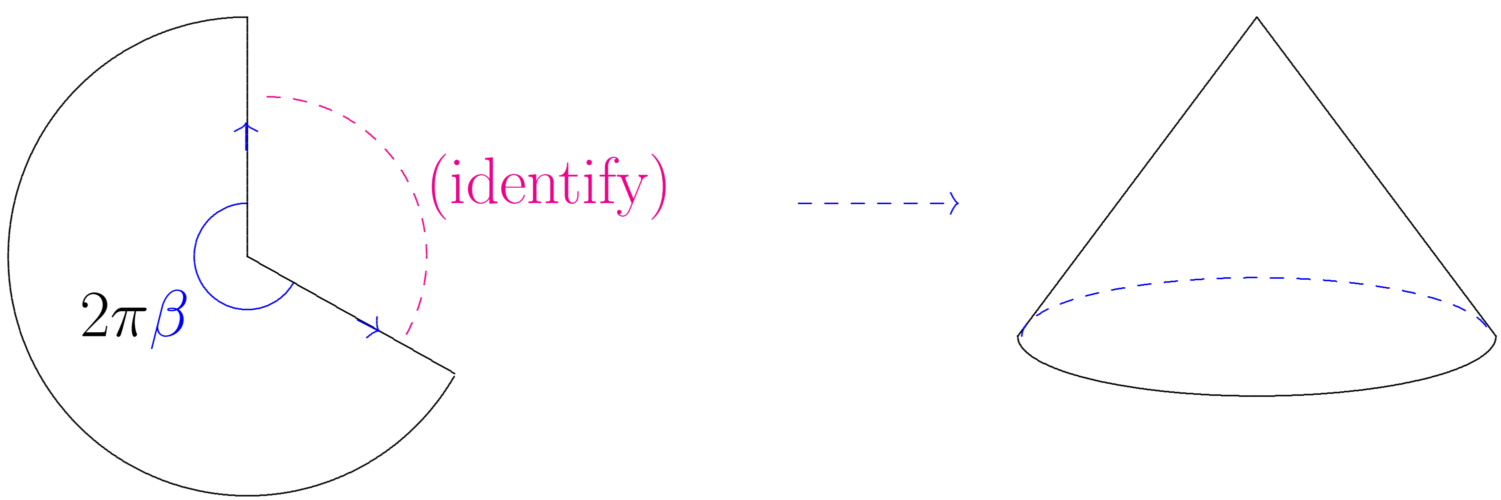

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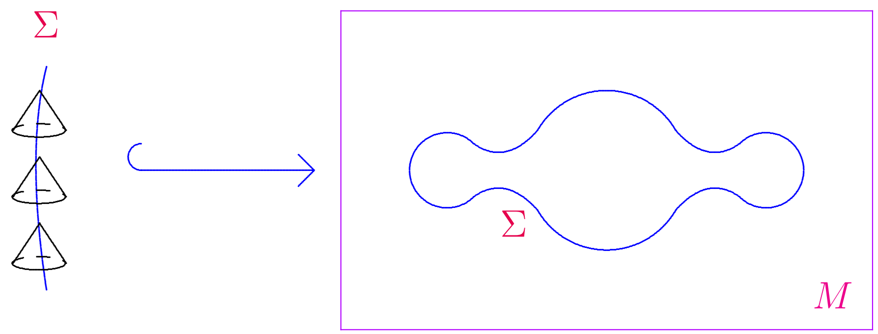

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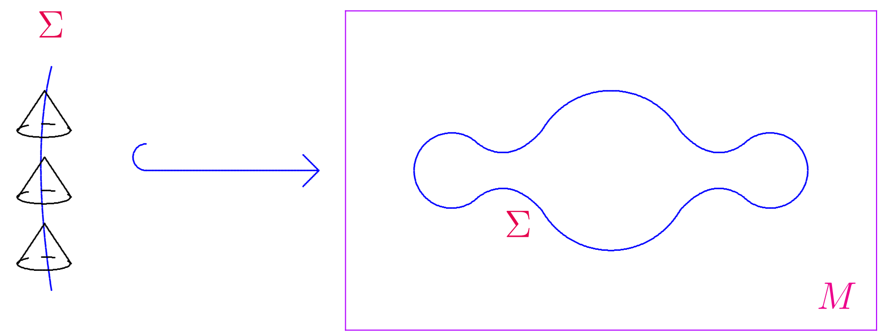

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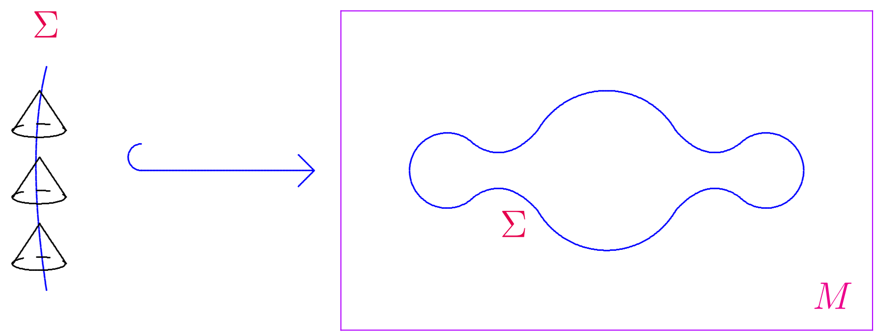

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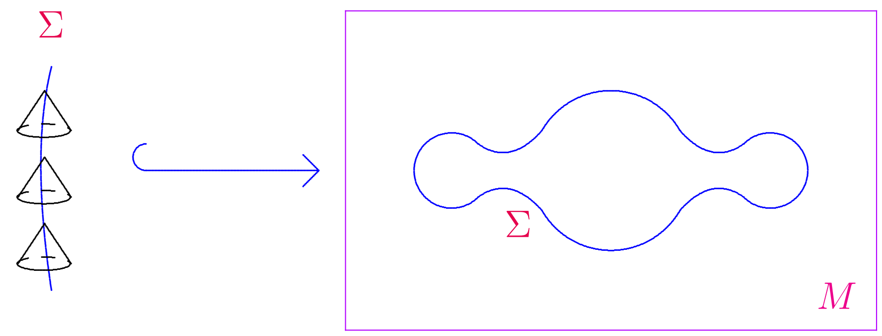

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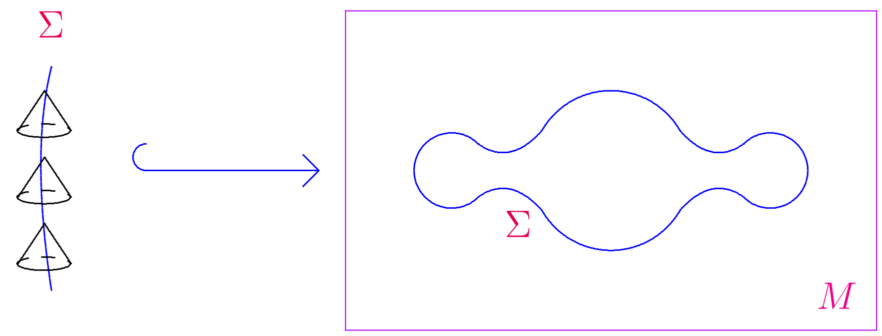

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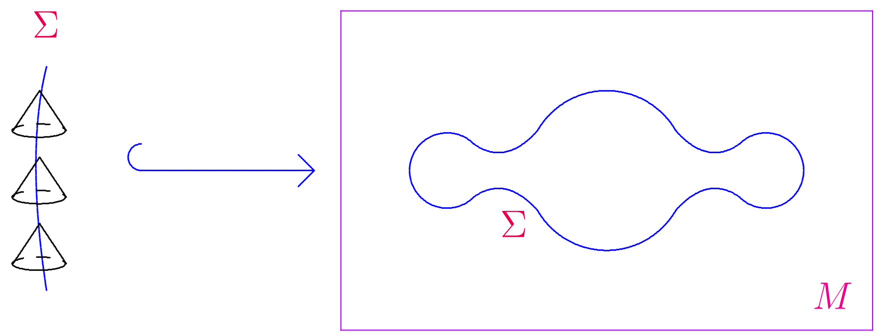

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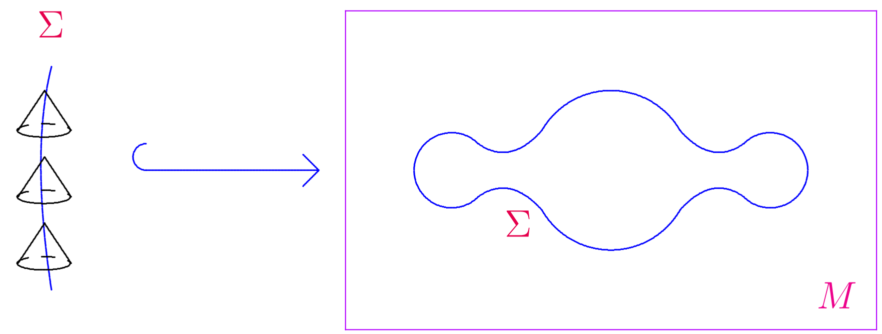

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But converse true in Einstein case. . .

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Begin with review of smooth case. . .

Two homotopy invariants

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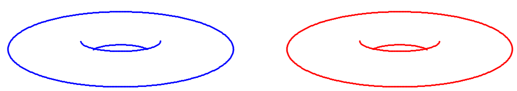

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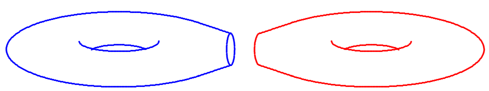

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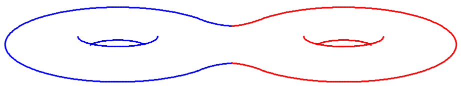

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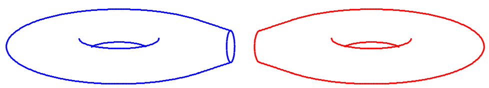

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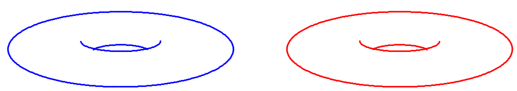

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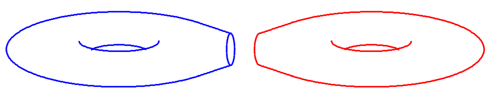

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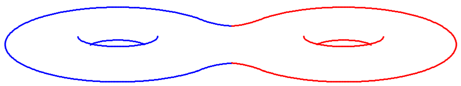

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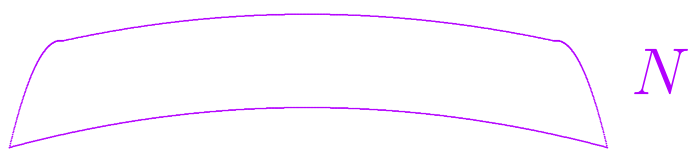

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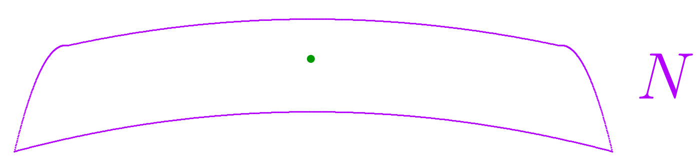

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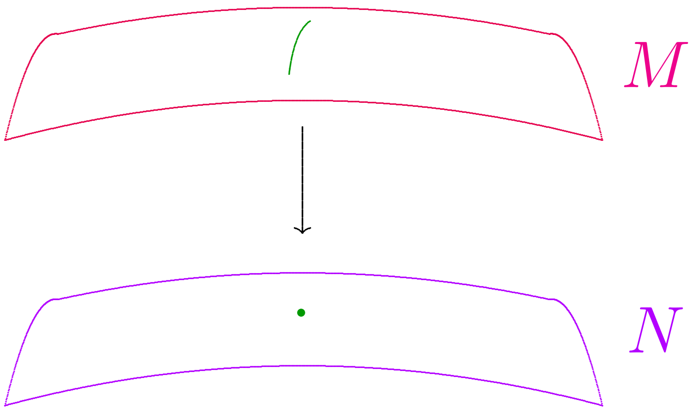

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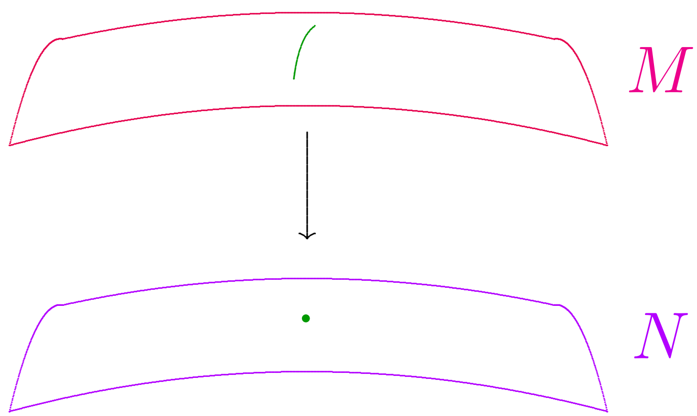

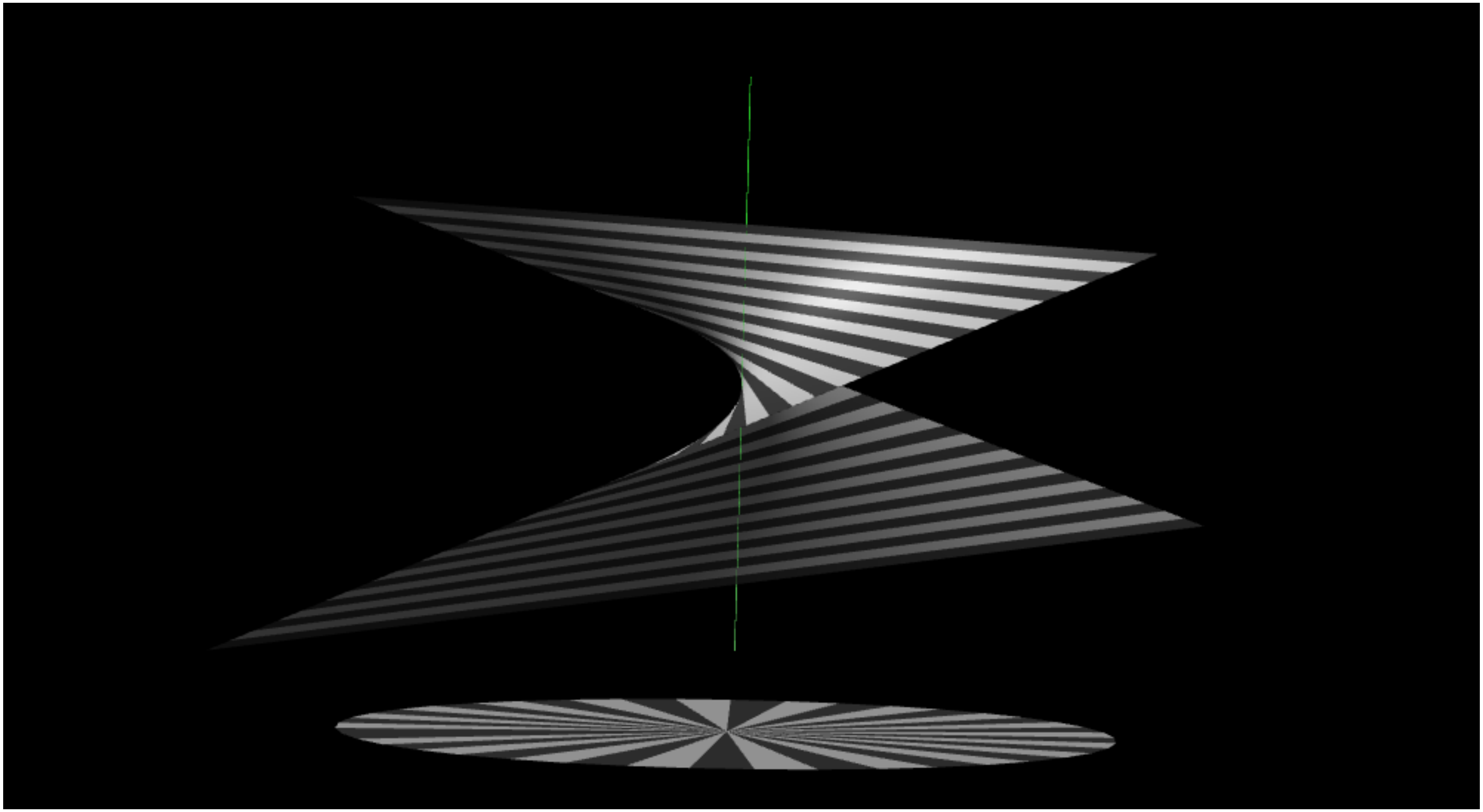

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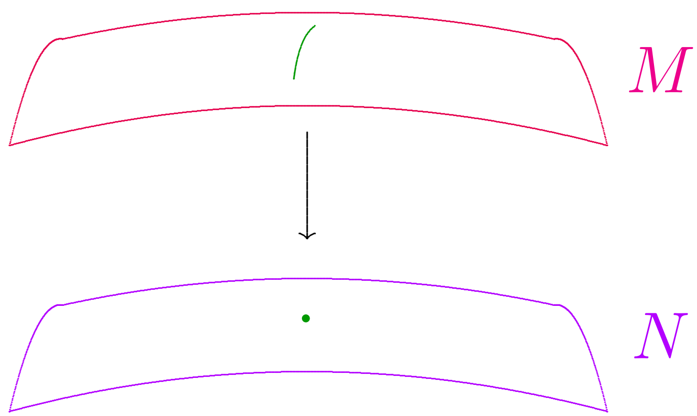

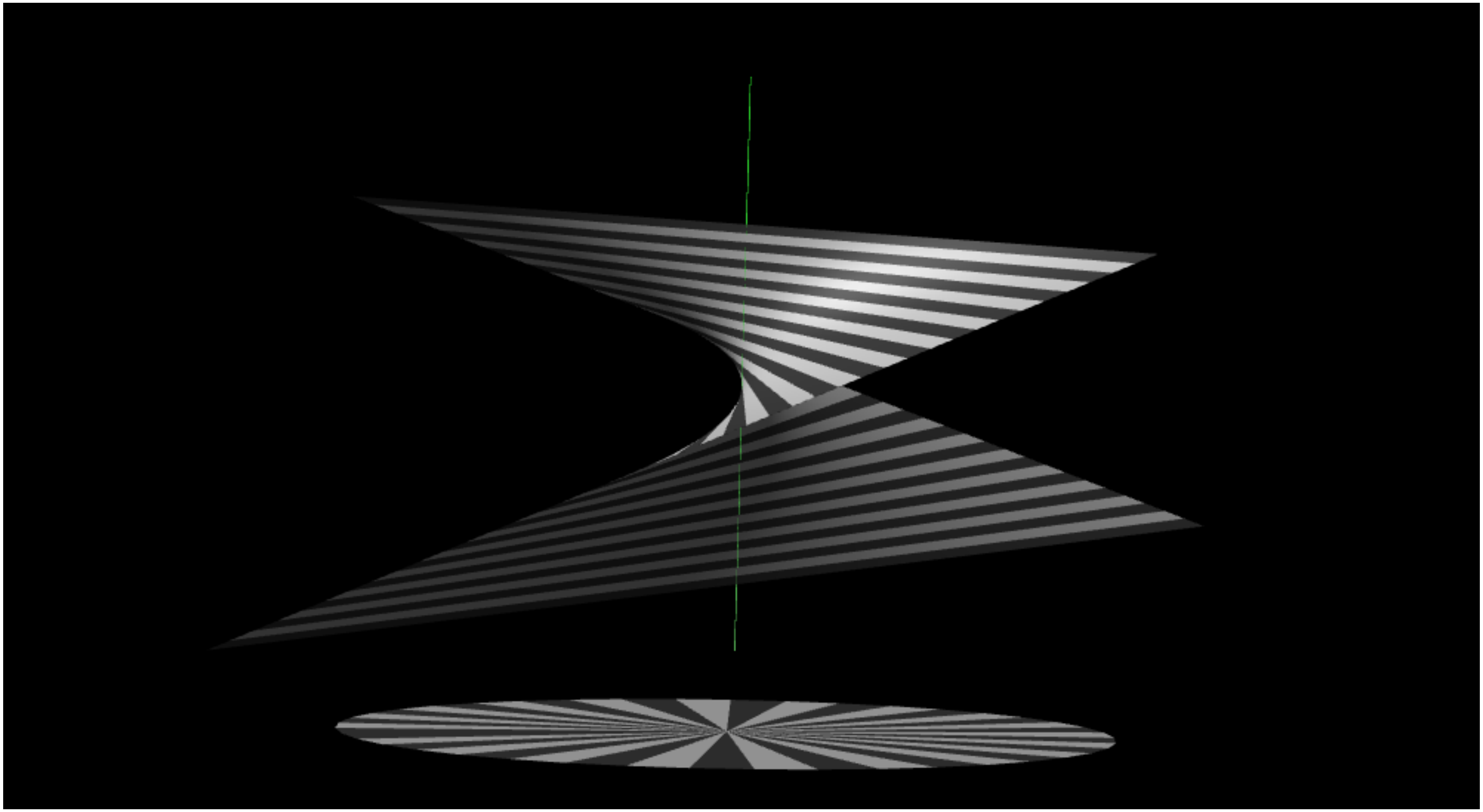

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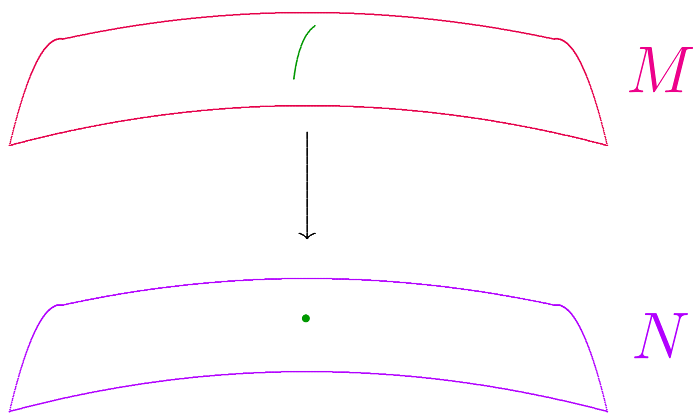

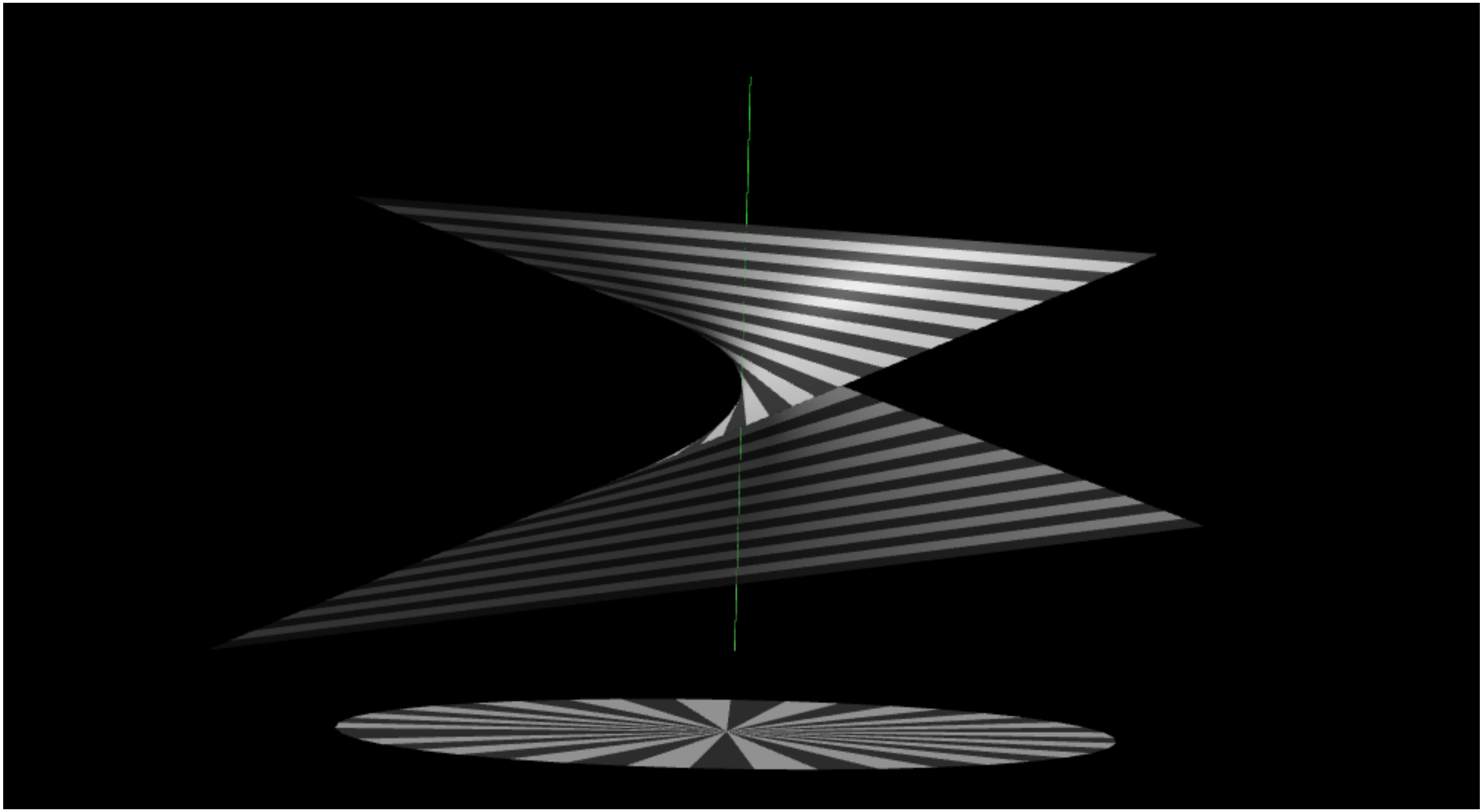

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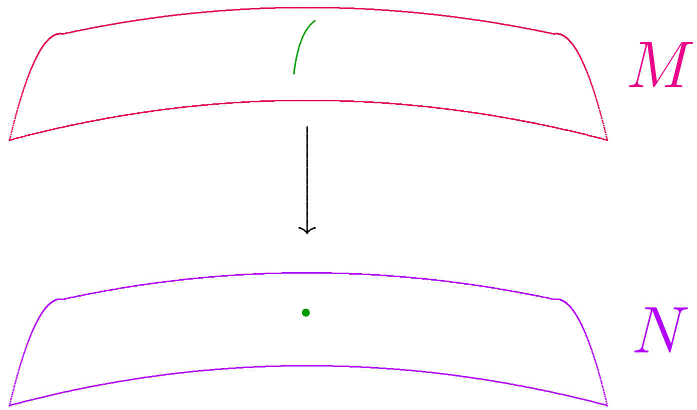

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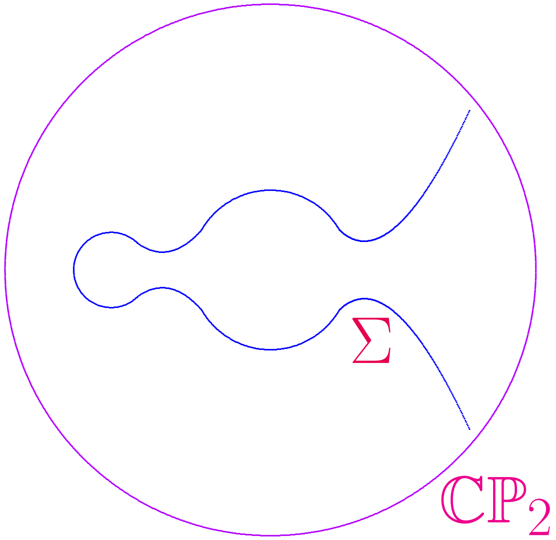

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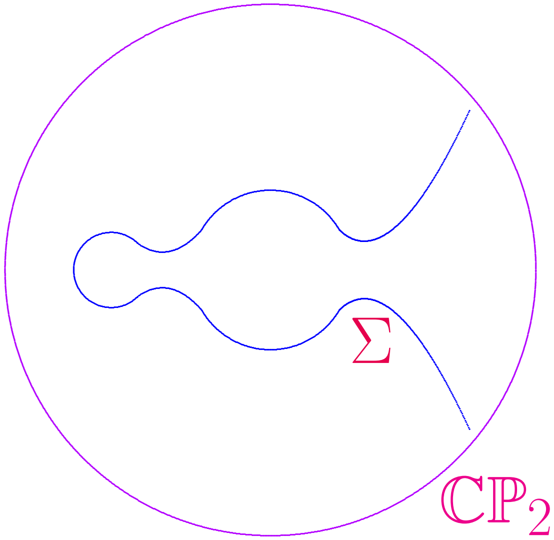

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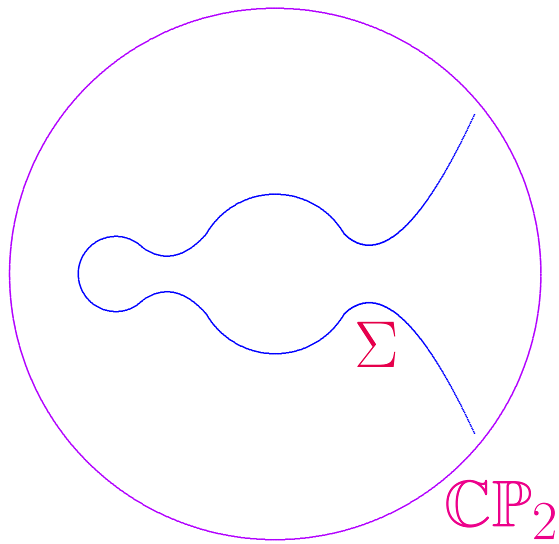

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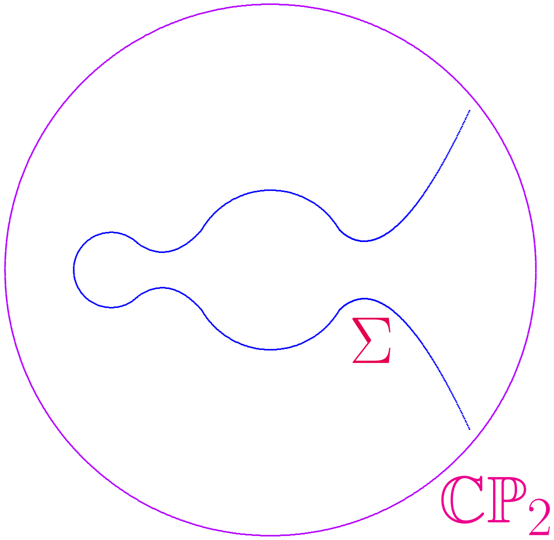

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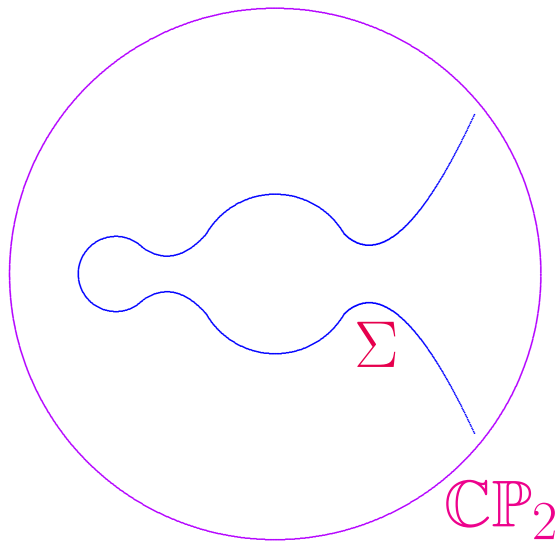

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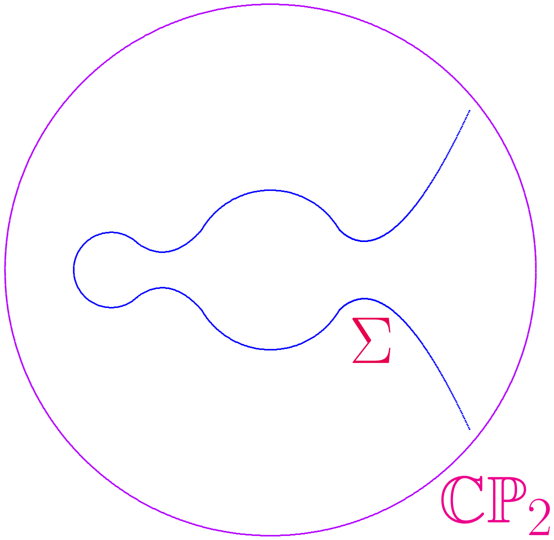

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Theorem $\Longrightarrow$ unique Einstein metric, up to scale.

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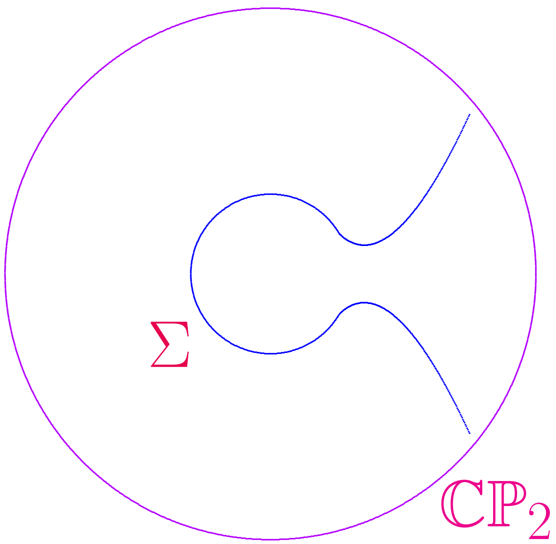

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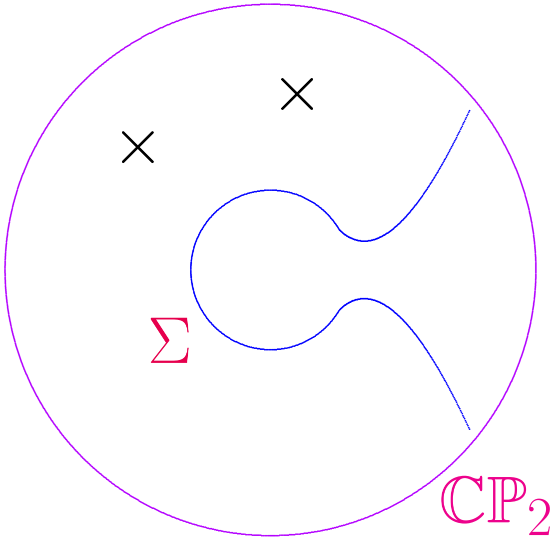

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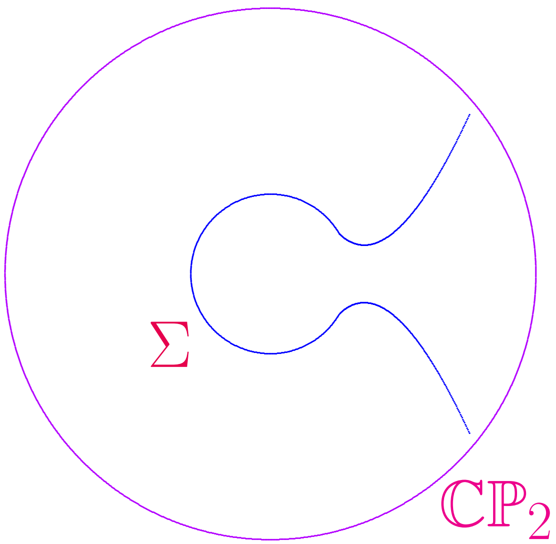

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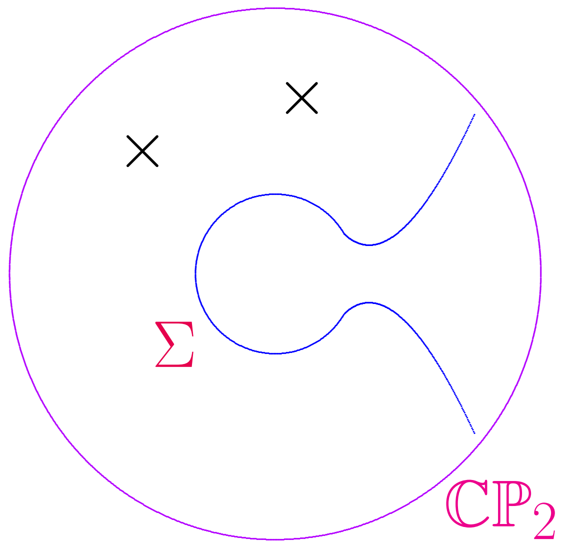

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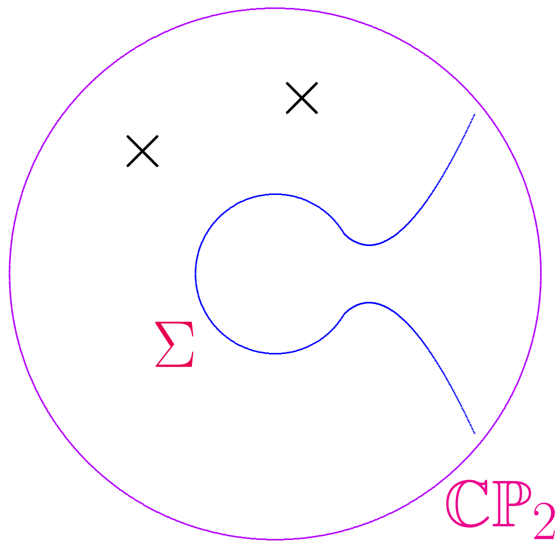

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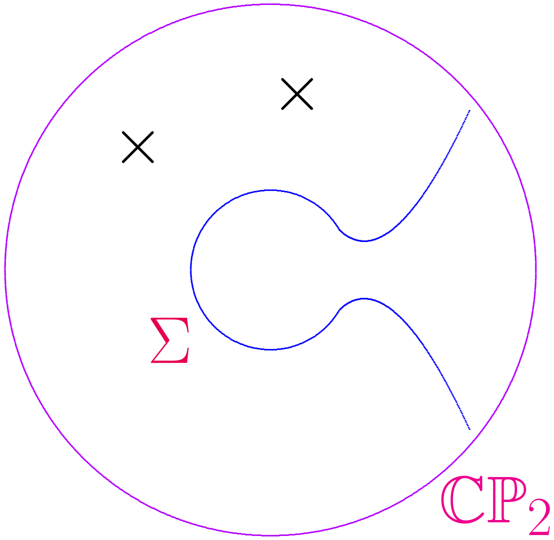

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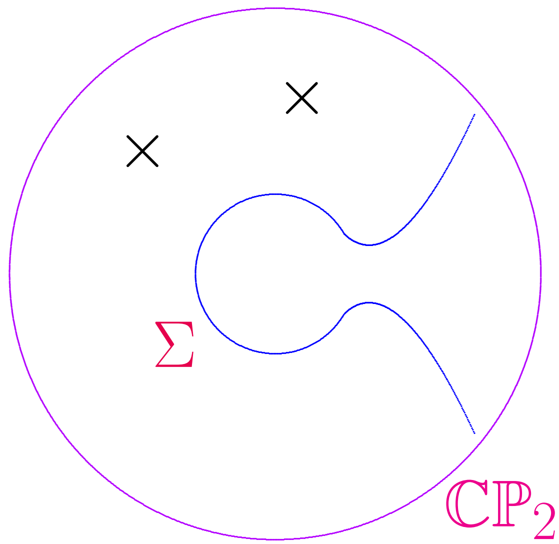

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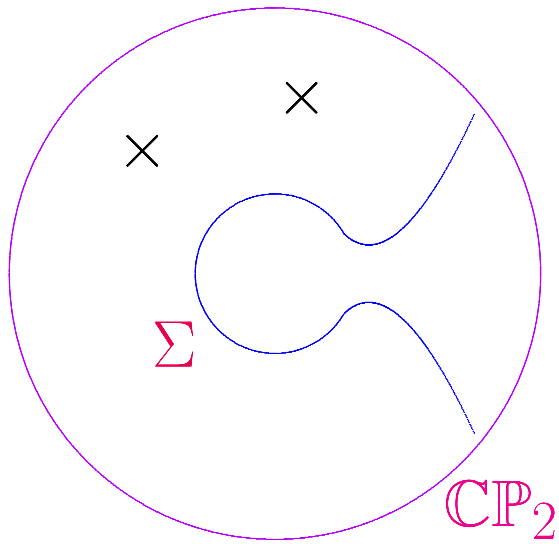

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For small β, Theorem $\Longrightarrow$ (M, Σ) never Einstein.

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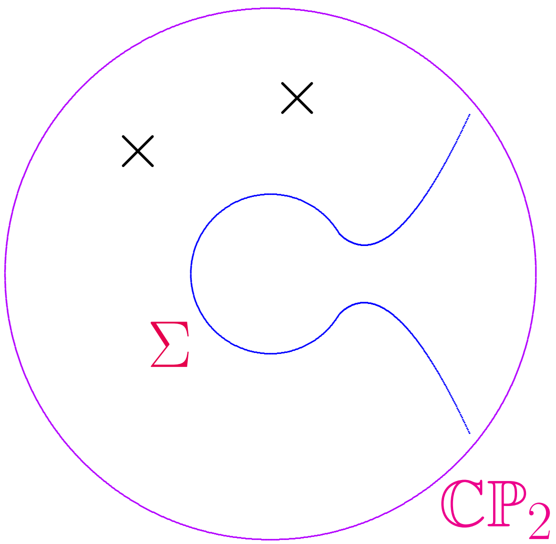

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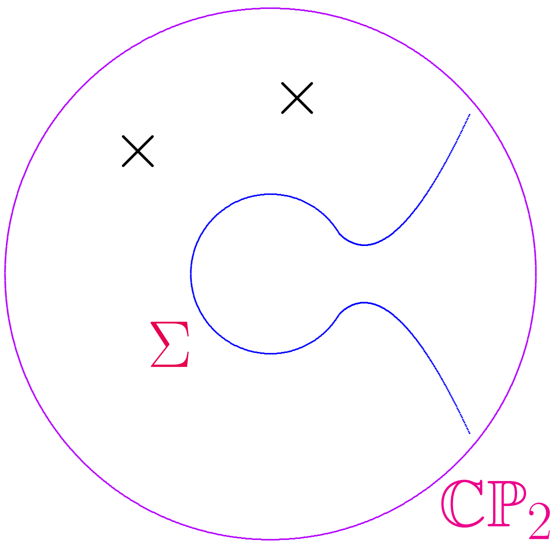

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Berman, Li-Sun, . . .

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Nothing analogous known in other dimensions.

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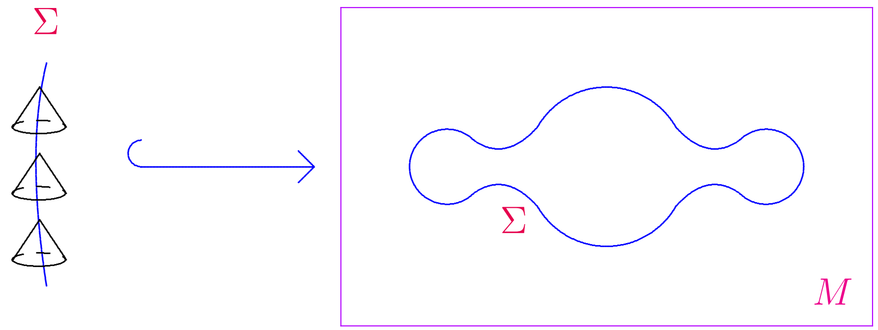

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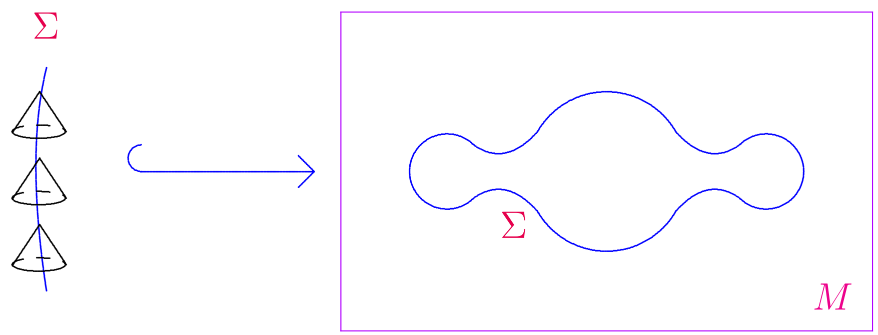
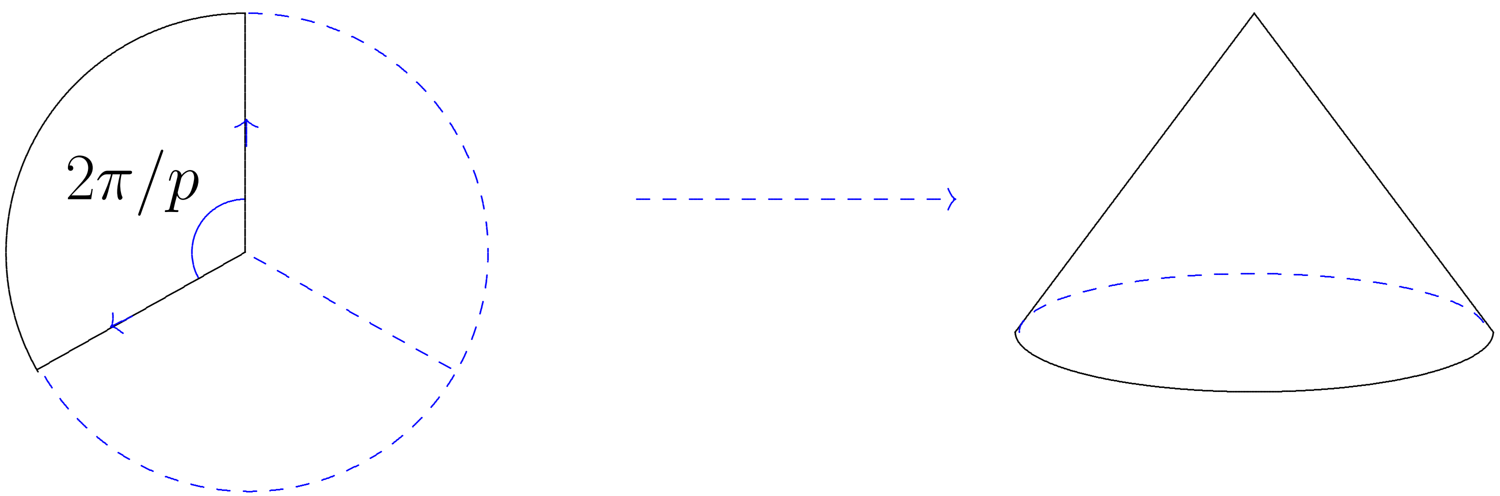


Transverse Picture:



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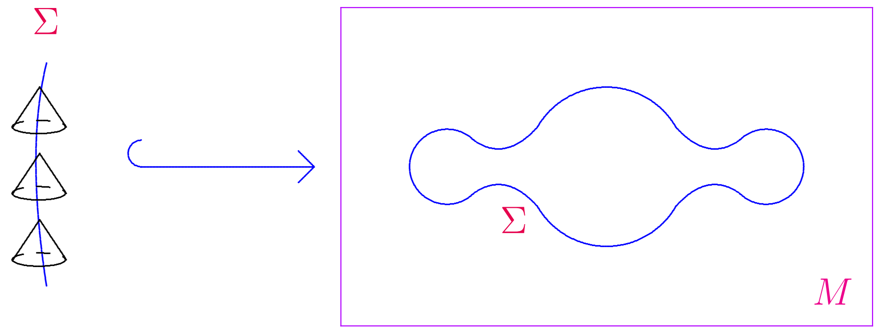

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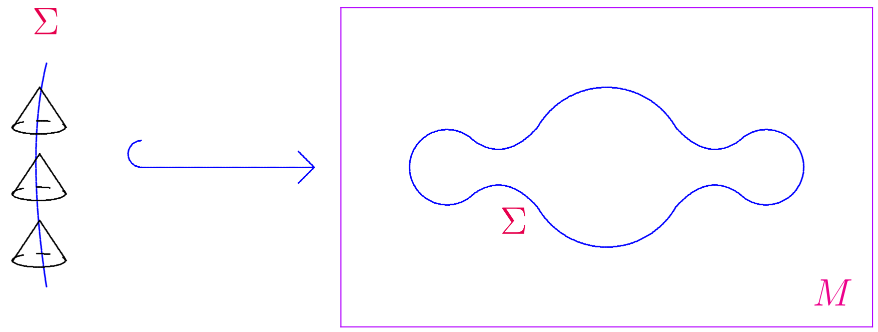

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Once again, 4-dimensional phenomenon.

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3 times better than Hitchin-Thorpe obstruction!

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Symplectic form ω has nothing to do with metric! Inequality guarantees that SW invariant is non-zero. Automatic if $b_+(M) > 1$.

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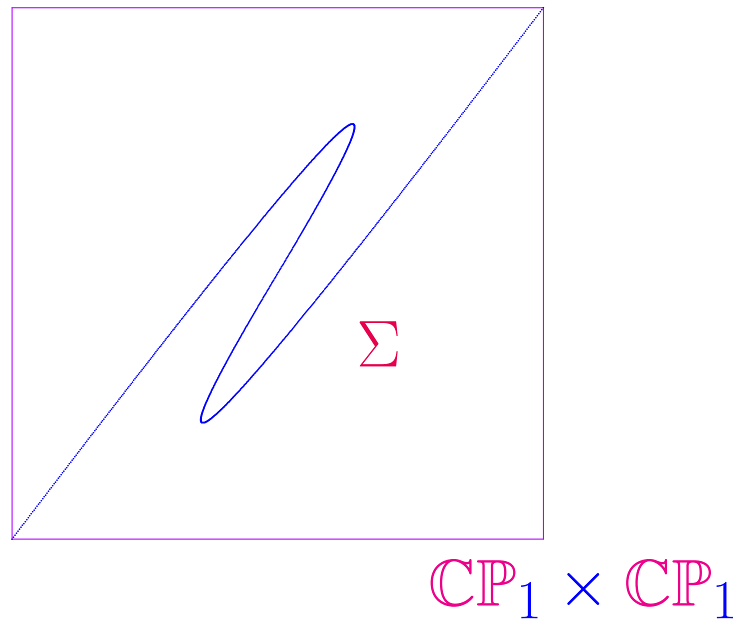

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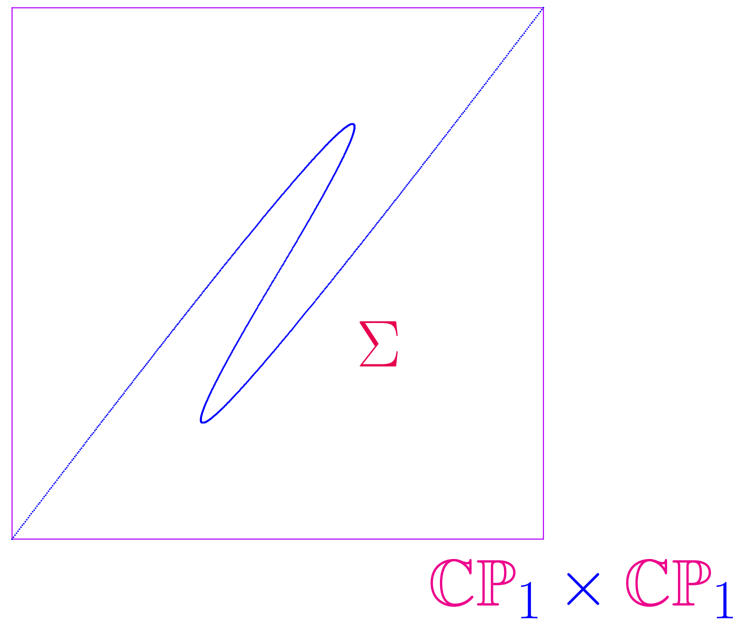

Chen-Donaldson-Sun:

$(\mathbb{CP}_1 \times \mathbb{CP}_1, \Sigma)$ admits Kähler-Einstein metrics of all cone angles $2\pi\beta$, $\beta \in (0,1]$.

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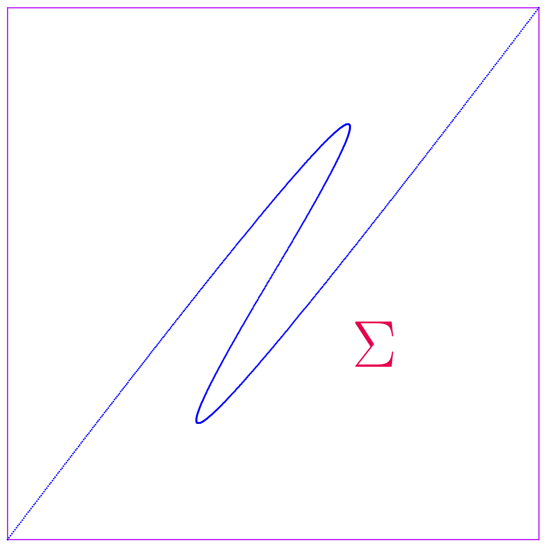

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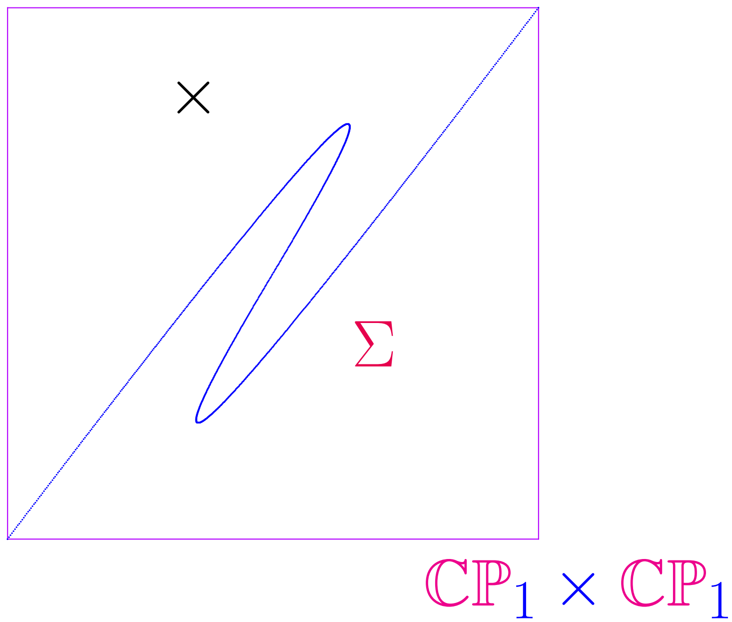

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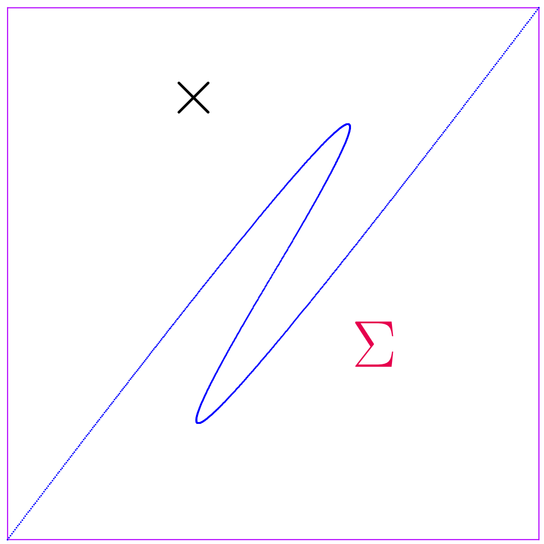

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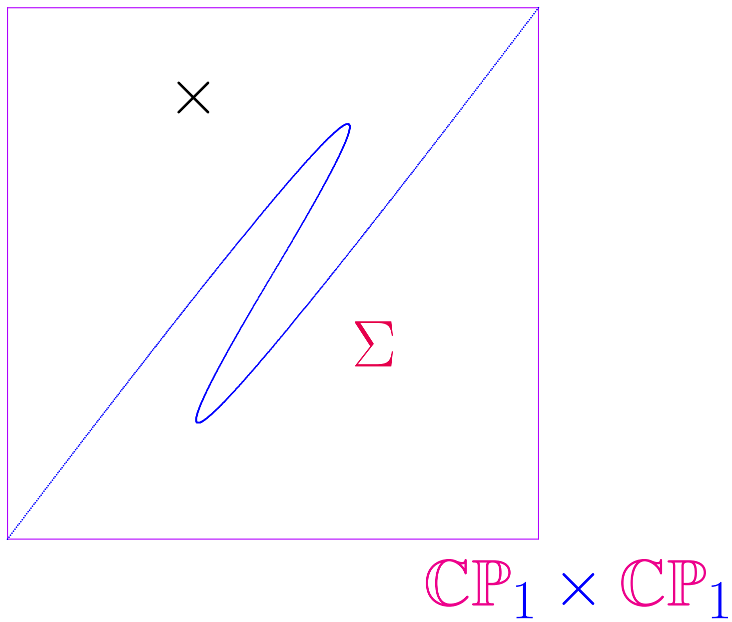

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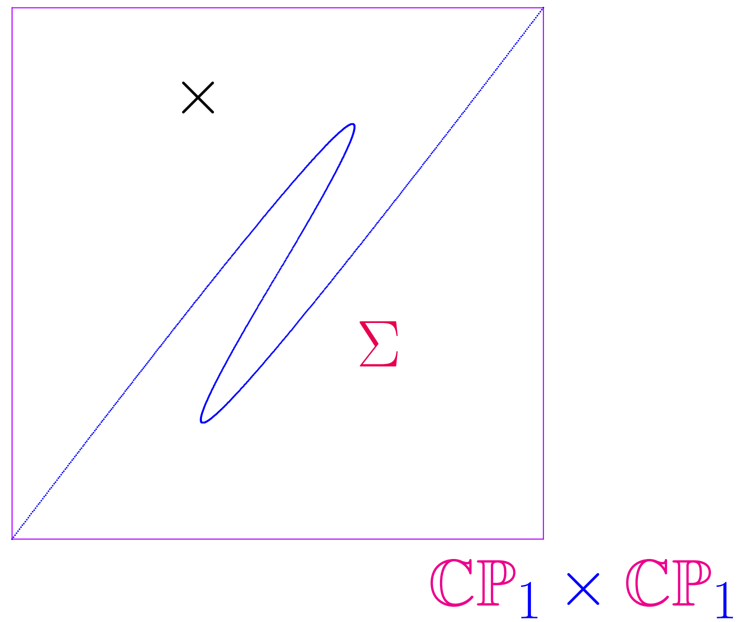

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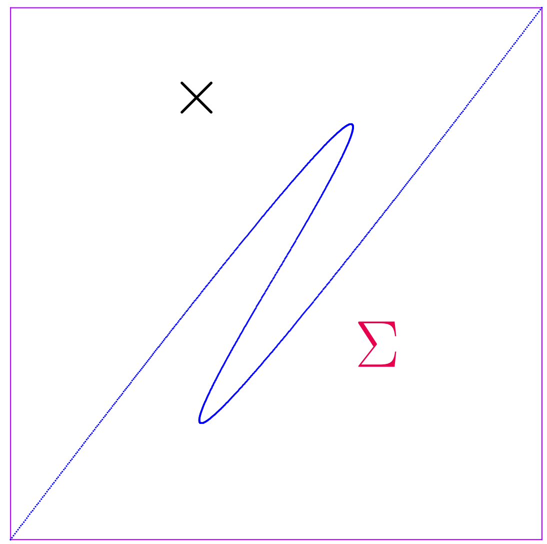

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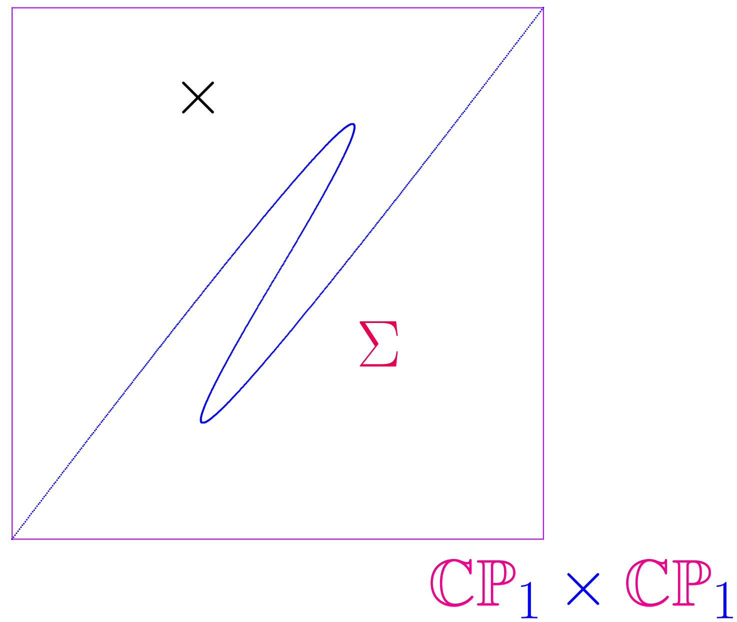

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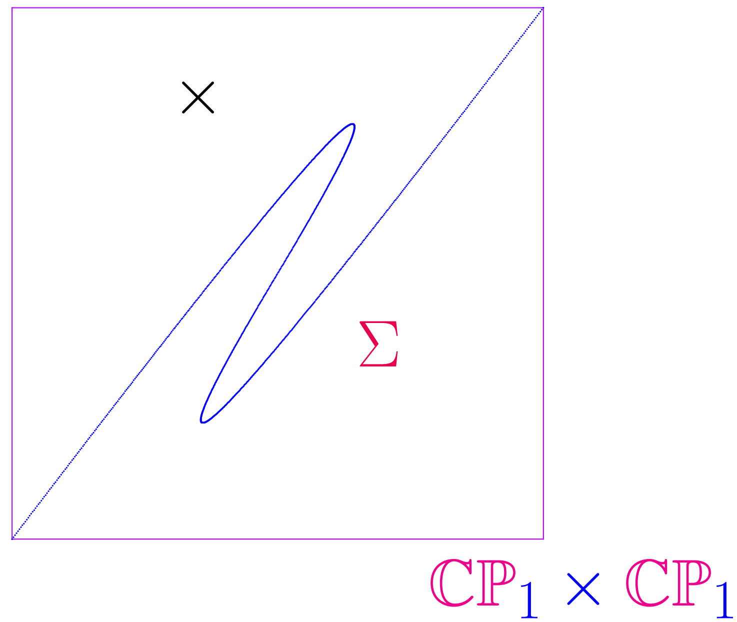

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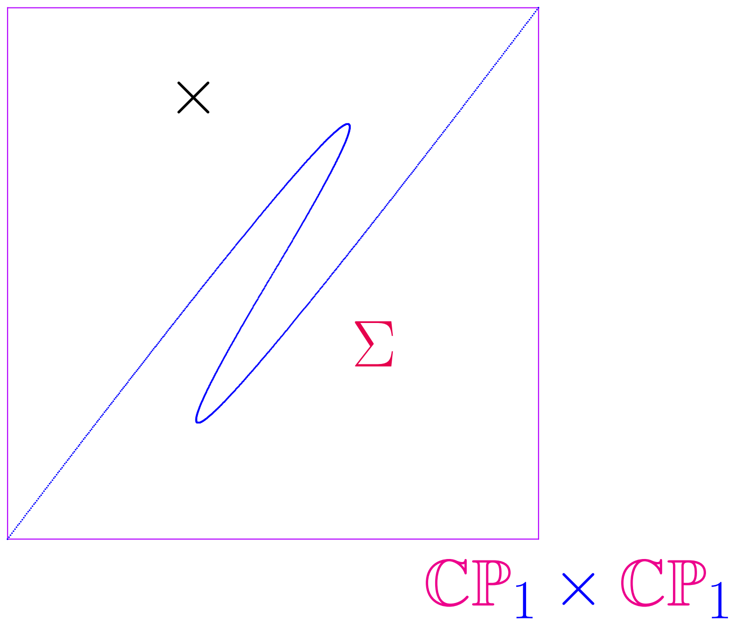

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$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W_+ &= \text{self-dual Weyl curvature} \\ W_- &= \text{anti-self-dual Weyl curvature} \end{aligned}$$

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What about edge-cone metrics?

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Conjecture: Same estimates hold for general β.

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Work in progress: Seiberg-Witten for general β.