On the Markov and Lagrange spectra and their dynamical generalizations

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The classical Markov and Lagrange spectra are sets of real numbers related to diophantine approximations, and can be defined as follows.

The classical Lagrange spectrum is the set $L = \{k(\alpha), \alpha \in \mathbb{R} \setminus \mathbb{Q}\}$, where, for a given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $k(\alpha) = \sup(\{k > 0; |\alpha - p/q| < 1/kq^2$ has infinitely many rational solutions $p/q\})$. The classical Markov spectrum is the set
$M = \{(\inf_{(x,y)\in\mathbb{Z}^2\setminus(0,0)} |f(x,y)|)^{-1};\ f(x,y) = ax^2 + bxy + cy^2; a, b, c \in \mathbb{R}, b^2 - 4ac = 1\}$. Both L and M have interesting geometrical properties. For instance, they are closed subsets of $\mathbb{R}$ such that $[6, +\infty) \subset L \subset M$ and, given any $\beta \in [0, 1]$, there is $t \in [3, \sqrt{12})$ such that the Hausdorff dimentions of $L \cap (-\infty, t)$ and of $M \cap (-\infty, t)$ are both equal to β. These results are related to the geometry of arithmetic sums of regular Cantor sets.

The sets M and L can be characterized as sets of maximum heights and asymptotic maximum heights, respectively, of geodesics in a modular surface. They also can be defined in terms of symbolic dynamics. Inspired by these characterizations, we may associate to a dynamical system together with a real function generalizations of the Markov and Lagrange spectra as follows:

Given a map $\psi : X \to X$ and a function $f : X \to \mathbb{R}$, we define the associated dynamical Markov and Lagrange spectra as $M(f, \psi) = \{\sup f(\psi^n(x)), n \in \mathbb{N}, x \in X\}$ and
$L(f, \psi) = \{\limsup_{n\to\infty} f(\psi^n(x)), x \in X\}$, respectively.

Given a flow $(\varphi^t)_{t\in\mathbb{R}}$ in a manifold X, we define the associated dynamical Markov and Lagrange spectra as $M(f, (\varphi^t)) = \{\sup f(\varphi^t(x)), t \in \mathbb{R}, x \in X\}$ and
$L(f, (\varphi^t)) = \{\limsup_{t\to\infty} f(\varphi^t(x)), x \in X\}$, respectively.

We will discuss the results about the classical spectra, and also more recent results obtained in collaboration with S. Romaña and A. Cerqueira about geometrical properties of dynamical Markov and Lagrange spectra associated to hyperbolic diffeomorphisms and flows.