Dynamical behavior of ergodic measure along the weak direction

Second Palis-Balzan International Symposium on Dynamical Systems

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June 2013

Partially hyperbolic diffeomorphisms

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- (Flow type): f is the perturbation of the time-one map of an Anosov flow.
- (Skew product type) f is the perturbation of a skew product map over $T^2 \times S^1$: $f((x,\theta)) = (g(x), h_x(\theta))$.
- (Derived from Anosov type) $f \in \mathrm{Diff}^1(T^3)$ is in the isotopic class of a linear Anosov diffeomorphism A_0, which has three real eigenvalues with different norms.

Dynamical behavior along the center direction

The dynamics of volume in the center direction are more complicated than in the strong direction. There are the following three cases:

(1) (Fubini's nightmare): there is a full volume subset, which intersects each center leaf with finitely many points.

(2) (Weak absolute continuation): The disintegration of volume along the center direction is absolutely continuous respect, but **not** equivalent to the Lebesgue measure along the center leaf.

(3) (singular but not atomic case) The disintegration of volume along the center direction is not atomic, and has no any absolutely continuous property.

Two examples for Fubini's nightmare

Example 1 (Katok):

$f : S^1 \times I \to S^1 \times I : f(x,t) = (h_t(x), t)$ where

- h_t is uniformly expanding;
- $h_t(0) = 0$ for any $t \in I$ and $Dh_s(0) \neq Dh_t(0)$ for any $s \neq t \in I$.

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Example 2 (Ruelle, Wilkinson):

Let f be a C^2 skew type of partially hyperbolic, volume preserving diffeomorphism. Assume that f is ergodic, and $\lambda^c(f) > 0$.

See also the results of Avila, Viana and Wilkinson in the case the center exponent is vanishing.

Fubini's nightmare

Fix a diffeomorphism f of skew type or flow type. For $\alpha > 0$, denote

$$\Gamma_\alpha(f) = \{x \in M : \liminf \frac{1}{n} \log |Df^n|_{E^c(x)}| > \alpha.\}$$

Proposition (Viana, Y)

There is δ such that for every $x \in \Gamma_\alpha(f)$ and every neighborhood U of x, $\liminf \frac{1}{n} \sum length(f^i(U)) > \delta$.

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Theorem (Viana, Y)

There is $N(\alpha)$ such that $\Gamma_\alpha(f)$ intersects each center leaf in at most N points.

Example for weak absolute continuiation

Consider the Kan's example, $f_0 : S^1 \times [0,1] \to S^1 \times [0,1]$:

$$f_0(x,t) = (3x, t + \cos(2\pi x)(\frac{t}{32})(1-t)).$$

Let f be a C^2 small perturbation of f which preserves $S^1 \times \{0,1\}$ and the fixed vertical line $0 \times [0,1]$, and $\partial_x f(0,0) \neq \partial_x f(0,1)$.

Lemma (Viana, Y)

The disintegration of the volume along the center foliation of f is absolutely continuous respect to the Lebesgue measure on the leaf, but not equivalent to the Lebesgue measure. (i.e.) for any A in a center leaf, we have

- $Leb_c(A) = 0 \Rightarrow m_c(A) = 0$;
- $m_c(A) = 0 \not\Rightarrow Leb_c(A) = 0$.

Derived from Anosov diffeomorphisms

Let $A \in \mathrm{Diff}^1(T^3)$ be a linear Anosov diffeomorphism with exponents $\lambda_1 > \lambda_2 > 0 > \lambda_3$, f be a partially hyperbolic diffeomorphism in the same isotopic class. π_f the Franks semi-conjugation between f and A.

Proposition

- f is dynamically coherent.
- π_f maps center leaves to center leaves.
- there is $K_f > 0$ such that $\pi_f^{-1}(x)$ is a center segment with length bounded by K_f.

Derived Anosov diffeomorphism

Proposition

If f is volume preserving with $\lambda^c(f) > \lambda_2$, then $\mathcal{F}^c$ has no absolutely continuis property.

Example (Varão)

There is a volume preserving f (Anosov) such that the disintegration of the volume along $\mathcal{F}^c$ is singular but not atomic.

Example (Ponce, Tahzibi and Varão)

There is a volume preserving diffeomorphism f with metric entropy λ_1 such that, the disintegration of volume along $\mathcal{F}^c$ is atomic.

Main results

Theorem 1 (Y):

Let μ be any ergodic measure of f with $h_\mu(f) > \lambda_1$, if Γ is a subset with $\mu(\Gamma) = 1$, then Γ intersects almost every center leaf in uncountably many points.

Main results

Theorem 1 (Y):

Let μ be any ergodic measure of f with $h_\mu(f) > \lambda_1$, if Γ is a subset with $\mu(\Gamma) = 1$, then Γ intersects almost every center leaf in uncountably many points.

Theorem 2(Y):

The Franks semi-conjugation $\pi_f : T^3 \to T^3$ induces a bijective map between the ergodic measures with metric entropy larger than λ_1.

Step I: reduce to the linear case

Since $(\pi_f)_*$ preserves metric entropy, we only need to show the following lemma:

Lemma

Let μ be an ergodic measure of A with metric entropy larger than λ_2, then for any subset Γ with $\mu(\Gamma) = 1$, it intersects almost every center leaf in uncountable number of points.

Idea of Proof:
Consider a Markov partition for the center foliation of A. We will show that the disintegration measures have positive dimension.

Dimensional theory

Recall the standard Ledrappier-Young's dimensional theory

- $h_1 = \lambda_1 \gamma_1$
- $h_2 = \lambda_1 \gamma_1 + \lambda_2 \bar{\gamma}_2$
- $\gamma_1 \leq 1$, $\bar{\gamma}_2 \leq 1$.

where γ_1 is the dimension of the disintegration along the strong unstable direction, and γ_2 is the dimension of the projection measure on the center unstable direciton.

Application 1:

If $h_\mu > \lambda_2$, then $\gamma_1 > 0$. Hence, every full μ measure subset intersects the typical strong unstable leaf in a uncountable number of points.

Dimensional theory along the center direction

We need note that $\bar{\gamma}_2$ **cannot** provide the information along the center direction, since it is only about a projection measure.

Example

See the Shub, Wilkinson's example: f is partially hyperbolic, volume preserving with 1-D center, and $\lambda^c > 0$.
By the Pesin Formula, $\gamma_1 + \bar{\gamma}_2 = 2$, hence $\gamma_1 = \bar{\gamma}_2 = 1$
But every center leaf only contains finitely many generic points.

So we need to study γ_2.

Dimensional theory along the center direction

Lemma

$\lambda_1 \geq \lambda_1 \bar{\gamma}_1 \geq h_\mu - h_\mu^c$.

Finally, if there is a μ full measure subset which intersects every center leaf in countably many points, then $h_\mu^c = 0$. So $h_\mu \leq \lambda_1$.

Merci, mes amis!