Hyperbolic Dynamics on Heisenberg Nilmanifold

Yi SHI

Peking University
Institute de Mathématiques de Bourgogne

IHP, June 13, 2013

Consider the 3-dimensional Heisenberg group

$$H \triangleq \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

with the matrix operation

$$(x, y, z) \cdot (a, b, c) = (x + a, y + b, z + c + xb).$$

Heisenberg Nilmanifold

Heisenberg Nilmanifold $\mathcal{H} = H/\Gamma$, where

$$\Gamma = \{(x, y, z) \in H : x, y, z \in \mathbb{Z}\}$$

is the integer lattice.

Heisenberg Group

Consider the 3-dimensional Heisenberg group

$$H \triangleq \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \ x, y, z \in \mathbb{R} \right\}.$$

with the matrix operation

$$(x, y, z) \cdot (a, b, c) = (x + a, y + b, z + c + xb).$$

Heisenberg Nilmanifold

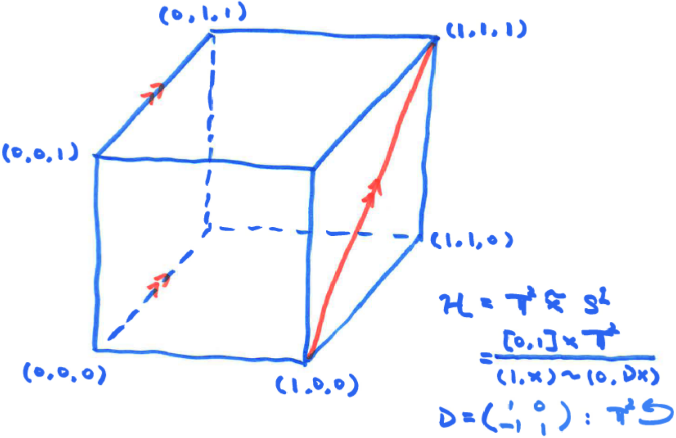
Heisenberg Nilmanifold $\mathcal{H} = H/\Gamma$, where

$$\Gamma = \{(x, y, z) \in H : \ x, y, z \in \mathbb{Z}\}$$

is the integer lattice.

Heisenberg Nilmanifold

Look at a fundamental domain of $\mathcal{H}$, then we can see that $\mathcal{H}$ is an S^1-bundle over $\mathbb{T}^2$ with Euler number 1.



Partially Hyperbolic Automorphisms

Automorphism $\widetilde{f} : H \longrightarrow H$:

$$\widetilde{f}(g_1) \cdot \widetilde{f}(g_2) = \widetilde{f}(g_1 \cdot g_2), \qquad \forall g_1, g_2 \in H$$

If $\widetilde{f}(\Gamma) = \Gamma$, then it induce an automorphism:

$$f : \mathcal{H} \longrightarrow \mathcal{H}.$$

Remark

If f is a partially hyperbolic automorphism on $\mathcal{H}$, then it is a lift of an linear Anosov diffeomorphism on $\mathbb{T}^2$ and the S^1-bundle tangent to the central space.

Partially Hyperbolic Automorphisms

Automorphism $\widetilde{f} : H \longrightarrow H$:

$$\widetilde{f}(g_1) \cdot \widetilde{f}(g_2) = \widetilde{f}(g_1 \cdot g_2), \qquad \forall g_1, g_2 \in H$$

If $\widetilde{f}(\Gamma) = \Gamma$, then it induce an automorphism:

$$f : \mathcal{H} \longrightarrow \mathcal{H}.$$

Remark

If f is a partially hyperbolic automorphism on $\mathcal{H}$, then it is a lift of an linear Anosov diffeomorphism on $\mathbb{T}^2$ and the S^1-bundle tangent to the central space.

Example

Consider the cat map on $\mathbb{T}^2$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

one of its lift on $\mathcal{H}$ is:

$$f_A : (x, y, z) \ \mapsto \ (2x + y, x + y, z + f(x, y))$$

Here

$$f(x, y) = x^2 + xy + \frac{1}{2}y^2 - x - \frac{1}{2}y.$$

Remark

E^c is tangent to the S^1-fibers, and $E^s \oplus E^u$ is a contact plane field, which is not integrable everywhere.

Example

Consider the cat map on $\mathbb{T}^2$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

one of its lift on $\mathcal{H}$ is:

$$f_A : (x, y, z) \;\mapsto\; (2x + y, x + y, z + f(x, y))$$

Here

$$f(x, y) = x^2 + xy + \frac{1}{2}y^2 - x - \frac{1}{2}y.$$

Remark

E^c is tangent to the S^1-fibers, and $E^s \oplus E^u$ is a contact plane field, which is not integrable everywhere.

Contact Plane Fields

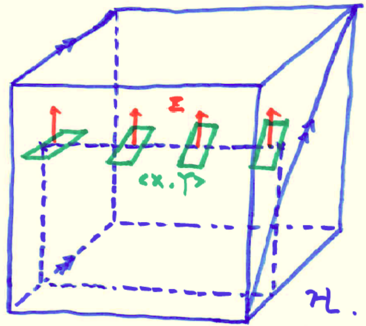
Denote the canonical vector fields

$$X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad Z = \frac{\partial}{\partial z},$$

then

$$E^s \oplus E^u = \langle X - \frac{1}{2}Z \ , \ Y - \frac{1}{2}Z \rangle.$$

is a contact plane field.



Partially Hyperbolic Diffeomorphisms

- All the $\mathcal{C}^2$ partially hyperbolic volume preserving diffeomorphisms on $\mathcal{H}$ is ergodic. [HHU], 2008.

- Every partially hyperbolic diffeomorphisms on $\mathcal{H}$ is leaf conjugate to some automorphism. [H],[HP], 2013.

Main Result

Theorem. For any partially hyperbolic automorphism

$$f_A : \mathcal{H} \longrightarrow \mathcal{H},$$

there exists a sequence of $\mathcal{C}^\infty$ diffeomorphisms $\{f_n\}$, s.t.

- $f_n \longrightarrow f_A$ in $\mathcal{C}^1$ topology.
- f_n is structural stable.
- $\Omega(f_n)$ consists of one attractor and one repeller.

Remark

All these partially hyperbolic automorphisms on $\mathcal{H}$ is stably ergodic, but not robustly transitive.

Main Result

Theorem. For any partially hyperbolic automorphism

$$f_A : \mathcal{H} \longrightarrow \mathcal{H},$$

there exists a sequence of $\mathcal{C}^\infty$ diffeomorphisms $\{f_n\}$, s.t.

- $f_n \longrightarrow f_A$ in $\mathcal{C}^1$ topology.
- f_n is structural stable.
- $\Omega(f_n)$ consists of one attractor and one repeller.

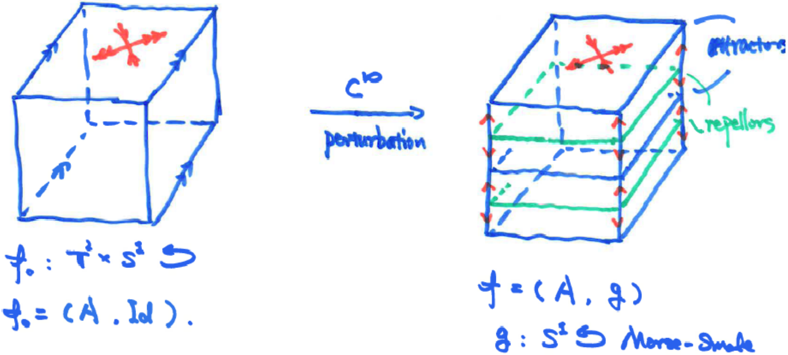
Remark

All these partially hyperbolic automorphisms on $\mathcal{H}$ is stably ergodic, but not robustly transitive.

Naive Example

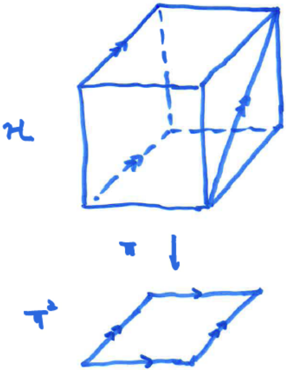
Consider $A \times \mathrm{Id} : \mathbb{T}^2 \times S^1 \longrightarrow \mathbb{T}^2 \times S^1$.

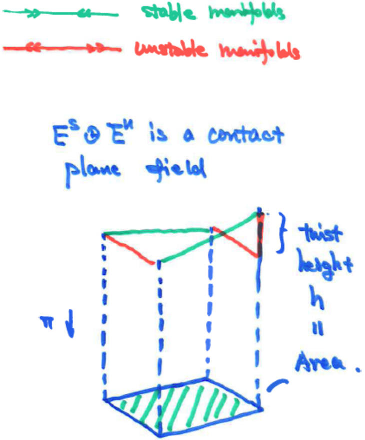
$$E^s \oplus E^u \text{ integrable} \implies A \times f_{\mathcal{MS}}|_{S^1} \xrightarrow{\mathcal{C}^\infty} A \times \mathrm{Id}.$$



Contact Structure

Milnor-Wood Inequality implies that there does not exists any closed surfaces or foliations transverse to the S^1-fibers in $\mathcal{H}$.



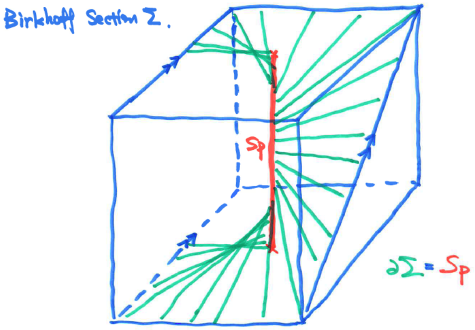


Definition of Birkhoff Sections

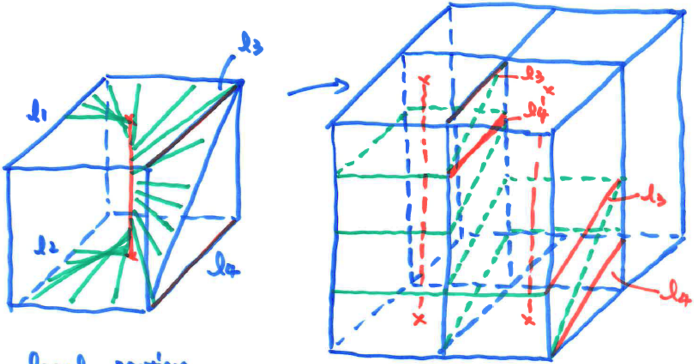
An imbedded surface $\Sigma \hookrightarrow \mathcal{H}$ is a Birkhoff section if:

- $\partial\Sigma = S_{p_1} \cup \cdots \cup S_{p_k}$ consists of finite many S^1-fibers.
- $\mathrm{Int}(\Sigma)$ is transverse to the S^1-fibers of $\mathcal{H}$.

Remark. $\mathrm{Int}(\Sigma)$ is an l-cover of $\mathbb{T}^2 \setminus \{p_1, \cdots, p_k\}$, where $l \leq k$.



Birkhoff Sections



local section.

Birkhoff Section with 4 boundary fibers.

Break Transivity

Idea:

Choose two **parallel** Birkhoff sections to be the candidates of attractor and repeller.

That is if $f_A(\Sigma) \approx \Sigma$ and $f_A(\Sigma') \approx \Sigma'$, then

- far from boundary, Σ central contracting, Σ' central expanding, just like $\mathbb{T}^3$;
- close to boundary, apply the central DA-construction from C.Bonatti and N.Guelman.

Difficulty:

How could a Birkhoff section Σ is "invariant" by f_A?

or

When its image $f_A(\Sigma)$ is isotopic to Σ along the fibers?

Break Transivity

Idea:

Choose two **parallel** Birkhoff sections to be the candidates of attractor and repeller.

That is if $f_A(\Sigma) \approx \Sigma$ and $f_A(\Sigma') \approx \Sigma'$, then

- far from boundary, Σ central contracting, Σ' central expanding, just like $\mathbb{T}^3$;
- close to boundary, apply the central DA-construction from C.Bonatti and N.Guelman.

Difficulty:

How could a Birkhoff section Σ is "invariant" by f_A?

or

When its image $f_A(\Sigma)$ is isotopic to Σ along the fibers?

Boundary Property

For a Birkhoff section Σ_n with boundary

$$\partial\Sigma_n = \pi^{-1}(\{(\frac{k}{m2^n}, \frac{l}{m2^n}) : 0 \leq k, l \leq m2^n - 1\}).$$

Here $m = |\det(A - I)|$, and the twist number of each fiber is positive.

Then

- $\partial f_A(\Sigma_n) = \partial\Sigma_n$.
- Close to each boundary fiber, $f_A(\Sigma_n)$ locally isotopic to Σ.

Boundary Property

For a Birkhoff section Σ_n with boundary

$$\partial\Sigma_n = \pi^{-1}(\{(\frac{k}{m2^n}, \frac{l}{m2^n}) : 0 \leq k, l \leq m2^n - 1\}).$$

Here $m = |\det(A - I)|$, and the twist number of each fiber is positive.

Then

- $\partial f_A(\Sigma_n) = \partial\Sigma_n$.
- Close to each boundary fiber, $f_A(\Sigma_n)$ locally isotopic to Σ.

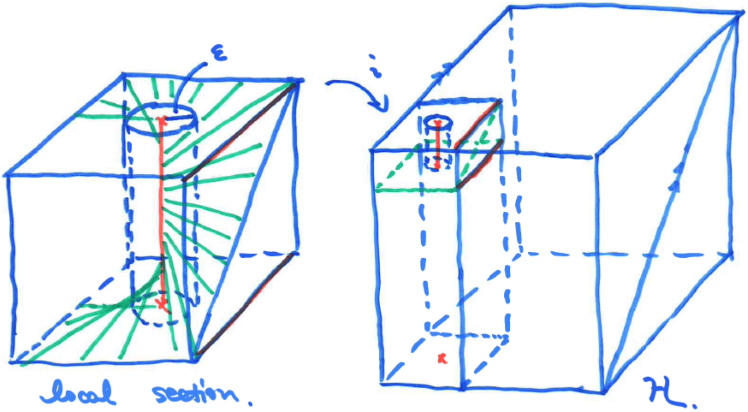
Isotopic Class of Birkhoff Sections

Theorem

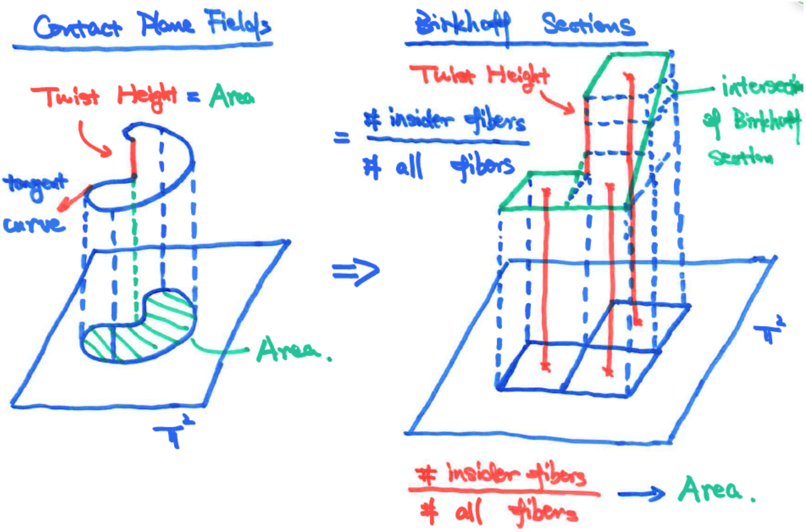
With the boundary property as above, $\exists$ Σ_n isotopic to $f_A(\Sigma_n)$ along the S^1-fibers, and the isotopy class is unique.

Isotopic Class of Birkhoff Sections

Moreover, we can choose Σ_n, such that "far" from the boundary, $f_A(\Sigma_n)$ is $\mathcal{C}^1$ close to Σ_n.



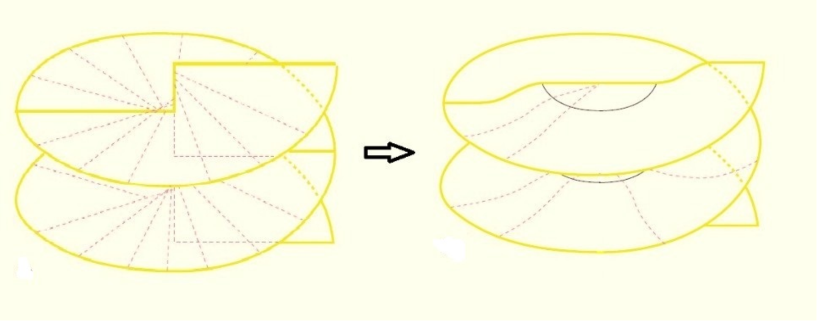
Contact Structure and Birkhoff Section



Central DA-Construction

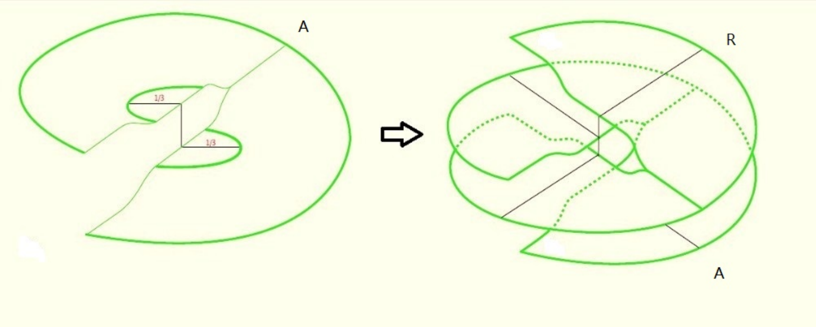
Close to the boundary fiber, we applying the central DA-construction due to C.Bonatti and N.Guelman, which tear the Birkhoff section to be branched:

- the candidate of attractor along unstable direction;
- the candidate of repeller along stable direction.



Central DA-Construction

Perturb f_A to get f_n preserve these two branched surfaces "invariant", and in the central direction one is contracting to get the attractor, the other is expanding to get the repeller.



Thank You !