Ergodic study of some foliations

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Motivations

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- There exists a notion of *geometric entropy* for foliations (Ghys, Langevin, Walczak 1988). But no notion of measure entropy.
- Idea: find a notion of Gibbs measure for the geodesic flow tangent to the leaves which detects the geometric entropy.

The dynamics

S=closed orientable surface of genus ≥ 2.
We're interested in representations:

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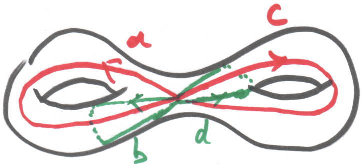

$$\pi_1(S) = \langle a, b, c, d | [a, b][c, d] = 1 \rangle$$

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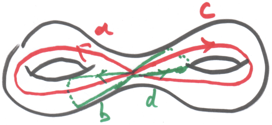
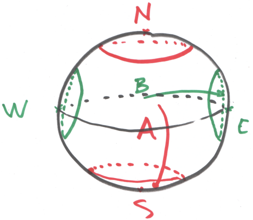
Example: Schottky representation



$$\rho : \pi_1(S) \to PSL_2(\mathbb{C})$$

$$a \mapsto A, b \mapsto Id, c \mapsto B, d \mapsto Id,$$



Group actions parametrized by a Riemannian metric I

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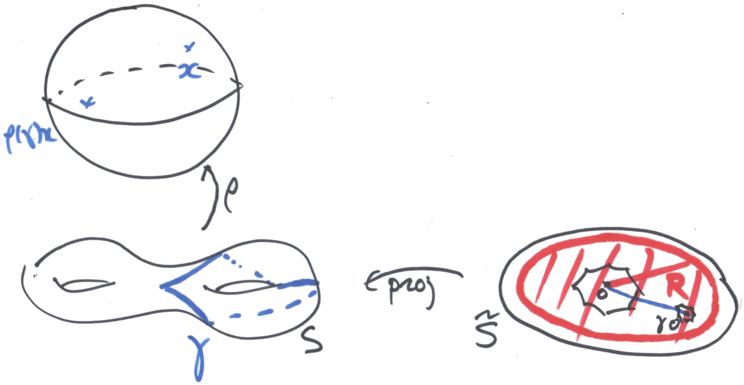
- (S, g) Riemannian metric with negative curvature (not necessarilly constant).
- $\widetilde{S}$ is diffeomorphic to a disc.
- Action $\pi_1(S) \curvearrowright \widetilde{S}$ by isometries.
- Distance on $\pi_1(S)$: $d(\gamma_1, \gamma_2) = \mathrm{dist}(\gamma_1 o, \gamma_2 o)$ for any choice of a base point $o \in \widetilde{S}$.

Group actions parametrized by a Riemannian metric II



Group actions parametrized by a Riemannian metric III

Theorem (A.)

Let $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$ s.t no measure on $\mathbb{CP}^1$ is invariant by all $\rho(\gamma)$. Then $\exists \nu$ on $\mathbb{CP}^1$ s.t $\forall x \in \mathbb{CP}^1$:

$$\frac{1}{|B_R|} \sum_{\gamma \in B_R} \delta_{\rho(\gamma)x} \underset{R \to \infty}{\to} \nu,$$

where $B_R = \{\gamma | d(\gamma, Id) \leq R\}$.

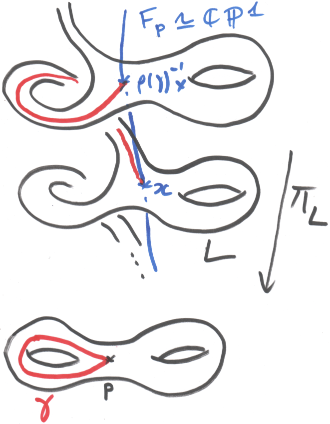
Foliated bundles

Suspension of a representation $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$ gives:

- a fiber bundle $\Pi : M \to S$ with $\mathbb{CP}^1$-fibers,
- a foliation $\mathcal{F}$ transverse to the fibers,
- holonomy representation $\gamma \mapsto Hol_\gamma = \rho(\gamma)^{-1}$.

Holonomy representation

All leaves are covers of S:



Parametrization: Lift the Riemannian metric: $\Pi_{|L} : L \to S$ local isometry.

Unit tangent bundles

- $T^1\mathcal{F} =$ unit vectors tangent to the leaves.
- $D\Pi : T^1\mathcal{F} \to T^1S$ is a $\mathbb{CP}^1$ bundle,
- foliated by the T^1L,
- same holonomy group.

Foliated geodesic flow

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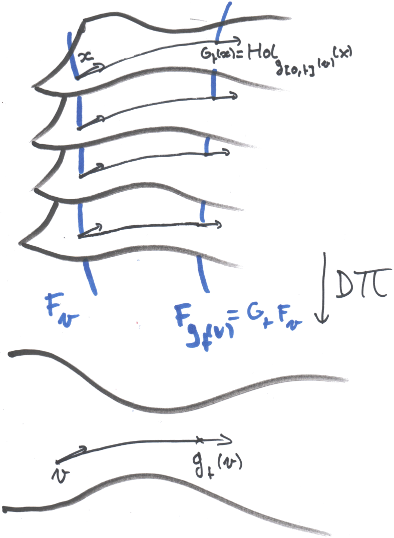
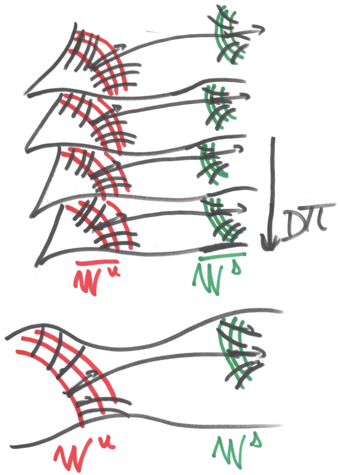
Locally constant projective cocycle: work of Bonatti, Gómez-Mont and Viana on Lyapunov exponents (2003).

Foliated geodesic flow



Foliated hyperbolicity

Negative curvature inside the leaves: *foliated hyperbolicity* (Bonatti, Gómez-Mont and Martínez).



Gibbs u-states

Definition

A Gibbs u-state for G_t is a probability measure in $T^1\mathcal{F}$ s.t:

- *μ is G_t-invariant,*
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The projection on M has Lebesgue disintegration in the leaves of $\mathcal{F}$.

Harmonic measures

Definition (Garnett, 1983)

Harmonic measure for $\mathcal{F}$: proba m on M such that:

- *m has Lebesgue disintegration in the leaves of $\mathcal{F}$*
- *the local densities are harmonic functions.*

Harmonic measures describe the behaviour of Brownian paths along the leaves.

Harmonic class on horocycles

- $(\eta_x)_{x\in\widetilde{S}}$: hitting measures in $S(\infty)$ of Brownian paths.
- Well defined harmonic class: not atomic, charges the open sets.
- Poisson kernel $k(x, y; \xi) = d\eta_y/d\eta_x(\xi)$, log-Hölder in ξ.
- Project harmonic class on horocycles.

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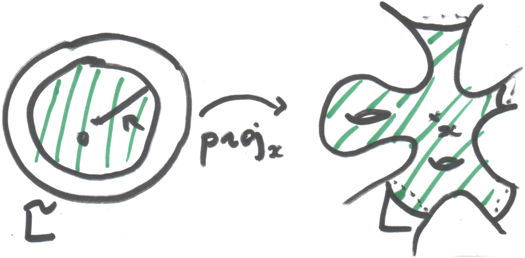
Key tool: Use Poisson integral representation to lift harmonic densities.

Large discs

$\mathrm{proj}_x : (\widetilde{L_x}, o) \to (L_x, x)$: Riemannian universal cover.



$$\nu_{x,R} = \mathrm{proj}_x * \left[\frac{Area_{|D(o,R)}}{Area(D(o,R))} \right]$$

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- Translation along large circles $\simeq$ horocycle flow
- Accumulation points of large discs are invariant by the horocycle flow.
- Associated Gibbs state: Margulis measure.

Unified treatment

	Kernel	**Cond. measures on $\overline{\mathcal{W}}^u$**
u-Gibbs	$k_u(o,z;\xi) = \lim \frac{\mathrm{Jac}^u \Phi_{-T}(v_{\xi,z})}{\mathrm{Jac}^u \Phi_{-T}(v_{\xi,o})}$	Lebesgue
Harmonic measure	Poisson kernel	Harmonic
Large discs	$k_0(o,z;\xi) = \exp(-h\beta_\xi(o,z))$	Margulis

Unique ergodicities

Theorem (A.)

(S, g) closed Riem. surface with negative curvature.
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- *There exists a unique harmonic measure for $\mathcal{F}$;*
- *There is a unique Gibbs u-state for G_t;*
- *There exists a unique accumulation point of large discs.*

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- *There exists a unique harmonic measure for* $\mathcal{F}$*;*
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Moreover, when the curvature is variable, in the case of fuchsian and quasifuchsian representations, these measures are pairwise singular.

Unified proof

Consequence of:

Theorem (A.)

- *(S, g) closed Riem. surface with negative curvature.* $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$
- *No measure on $\mathbb{CP}^1$ is invariant by all $\rho(\gamma)$.*
- *$(M, \mathcal{F})$: suspended foliation.*
- *$F : T^1S \to \mathbb{R}$ Hölder function*
- *$\overline{F} : T^1\mathcal{F} \to \mathbb{R}$ lifted function.*

$\Longrightarrow$ *G_t admits a unique Gibbs measure for the potential $\overline{F}$.*

Thank you!!