SRB measures for Almost Axiom-A diffeomorphisms

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Improves result [Lep 04].

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G_μ of generic points for a f-invariant ergodic probability measure on M μ,

$$\forall \phi \quad \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \phi \circ f^k(x) = \int \phi \, d\mu. \tag{1}$$

Definition

μ is said to be physical if $Leb_M(G_\mu) > 0$.

Usually, physical measures are constructed as *SRB-measures*.

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μ is said to be SRB if its disintegration along the unstable foliation is equivalent to Lebesgue on theses leaves. ⋆

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To construct SRB-measure, one need at least

- Some hyperbolicity to define the stable and unstable foliations.
- That Leb^u sees this hyperbolicity.⋆

There are many ways to degenerate uniform hyperbolicity, thus no general theory for construction of SRB-measures. For *Uniformly Hyperbolic* diffeos, SRB-measures are usually obtained as u-Gibbs states. For *Non-Uniformly Hyperbolic* diffeos, the tools do not exist.

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We are interested by perturbations of Axiom-A such that

- The angles are not perturbed.
- The expansions/contractions properties are perturbed to create a parabolic fixed point. ⋆

Definition

Given $f \in Diff^{1+}(M)$, $\Omega \subset M$ a compact f-invariant set. We say that f is *Almost Axiom A on* Ω if there exists an open set $U \supset \Omega$ such that:

1. $\forall x \in U$ exists $T_xM = E^u(x) \oplus E^s(x)$ splitting with Hölder continuous sub-bundles;
2. there exist continuous nonnegative real functions $x \mapsto k^u(x)$ and $x \mapsto k^s(x)$
   1. $\|df(x)v\|_{f(x)} \leq e^{-k^s(x)}\|v\|_x, \quad \forall v \in E^s(x)$,
   2. $\|df(x)v\|_{f(x)} \geq e^{k^u(x)}\|v\|_x, \quad \forall v \in E^u(x)$;
3. the *exceptional set*, $S = \{x \in U,\ k^u(x) = k^s(x) = 0\}$, satisfies $f(S) = S$.

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$$\liminf_{n\to+\infty} \frac{1}{n} \log \|df^{-n}(x)_{|E^u(x)}\| \leq -\lambda, \quad \liminf_{n\to+\infty} \frac{1}{n} \log \|df^{n}(x)_{|E^s(x)}\| \leq -\lambda.$$

Theorem

Let Λ be an (ε_0, λ)-regular set. If there exists some point $x_0 \in \Lambda$ such that $Leb_{D^u_{\varepsilon_0}(x_0)}(D^u_{\varepsilon_0}(x_0) \cap \Lambda) > 0$, then f has a probability SRB measure.

(ε_0, λ)-regular set= invariant set of points with ε_0 long stable and unstable leaves.

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(ε_0, λ)-regular set$=$ invariant set of points with ε_0 long stable and unstable leaves.

Improvements :

- $\liminf$ instead of $\limsup$.
- Prove the measure is finite.

Discussion on hypotheses:

- Optimal because if exists SRB-measure, assumptions are consequence of the existence (Pesin Theory). ⋆
- Weakest to check.

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Some technical problem: existence of foliations ⋆

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A point $x \in \Omega$ is called a *point of integration* of the hyperbolic splitting if there exist $\varepsilon > 0$ and C^1-disks $D^u_\varepsilon(x)$ and $D^s_\varepsilon(x)$ of size ε centered at x such that $T_y D^i_\varepsilon(x) = E^i(y)$ for all $y \in D^i_\varepsilon(x)$ and $i = u, s$.

Theorem

Every λ-hyperbolic point of bounded type is a point of integration of the hyperbolic splitting.

Bounded type =

$$\lim_{k\to+\infty} \frac{1}{s_k} \log \|df^{-s_k}(x)_{|E^u(x)}\| \leq -\lambda, \quad \lim_{k\to+\infty} \frac{1}{t_k} \log \|df^{t_k}(x)_{|E^s(x)}\| \leq -\lambda$$

$\limsup_{k\to+\infty} \frac{s_{k+1}}{s_k} < +\infty, \ \limsup_{k\to+\infty} \frac{t_{k+1}}{t_k} < +\infty$

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1. Construct Markov rectangles away from critical zone.
2. Induce on this region to construct u-Gibbs state for the induction.
3. Show the return time is integrable.

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Step 1: Markov rectangles

Already done in [Lep04]. $\limsup \to \liminf$ does not change construction.
Main idea= shadowing lemma. Shadowing with jumps far away of critical zone. $\star$
Countably many rectangles but some intersect only finitely many others.
Cut them as in Bowen. Get $R_1, \ldots, R_k$

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Step2: induction in $\cup R_i$

Definition

Given $0 < r < 1$, we say that n is a *r-hyperbolic time* for x if for every $1 \leq k \leq n$

$$\prod_{i=n-k+1}^{n} \|df^{-1}_{|E^u(f^i(x))}\| \leq r^k.$$

Lemma

Hyperbolicity yields existence of hyperbolic times. lim inf *yields positive frequency.*

Proposition

Possible to construct R_i's and choose r such that for any hyperbolic time n, $f^n(x) \in \cup R_i$. ⋆

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Hyperbolicity yields positive frequency for hyperbolic times. $\tau =$ first hyperbolic time.

$$\int \tau \, d\mu = +\infty \iff \lim_{n\to+\infty} \frac{\tau^n}{n} = +\infty \iff \lim_{n\to+\infty} \frac{n}{\tau^n} = 0.$$

$\Longrightarrow$ 0 density for return times= 0-density for hyperbolic times.
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