On the location of the maximum of a continuous stochastic process

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Uniqueness of the location

Let \((X(s), \ s \in K)\) be a continuous stochastic process. Define

\[ M = M(X) := \max_{s \in K} X(s), \]

and

\[ \text{arg max}(X) := \{ s \in K : X(s) = M \} \]

\((K = [a, b] \text{ or } \mathbb{R})\). In many situations we do expect that

\[ \text{arg max}(X) \overset{a.s.}{=} \{ Z \}. \]

How to prove it?
Theorem

For $\lambda \in \mathbb{R}$ let

$$m(\lambda) := \mathbb{E}(M^\lambda),$$

where

$$M^\lambda = M(X^\lambda) \text{ and } X^\lambda(s) := X(s) + \lambda s.$$

$m(\lambda)$ is differentiable at $\lambda = 0$ iff

$$\arg \max(X) \overset{a.s.}{=} \{Z\}.$$  

In this case we have that

$$\mathbb{E}Z = m'(0).$$
Lemma

(This is an analytical statement, we only need to assume that $X$ is a continuous function.) Let

$$Z_1 := \inf \arg \max (X),$$

and

$$Z_2 := \sup \arg \max (X).$$

Then

$$\lim_{\lambda \to 0^-} \frac{M^\lambda - M}{\lambda} = Z_1 \quad \text{and} \quad \lim_{\lambda \to 0^+} \frac{M^\lambda - M}{\lambda} = Z_2.$$
Proof of the Theorem

Notice that
\[ \arg \max (X) \overset{a.s.}{=} \{Z\}, \]
iff
\[ Z_1 \overset{a.s.}{=} Z_2. \]
Since \( Z_1 \leq Z_2 \), this holds iff
\[ \mathbb{E} Z_1 = \mathbb{E} Z_2, \]
which holds iff \( m(\lambda) \) is differentiable at \( \lambda = 0 \) (previous lemma).
Johansson (2003) showed that transversal fluctuations of last-passage percolation are described by the location of the maximum of a Airy process minus a parabola, under the assumption of uniqueness of the location.
Application

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- Uniqueness was proved by Corwin and Hammond (2011), and more recently by Flores, Quastel and Remenik (2012). Both proofs were strongly based on very particular aspects of the Airy process.
Application

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- The previous theorem provides us a very simple and direct proof (which holds under weaker hypothesis).
Corollary

Let \((A(s), s \in \mathbb{R})\) be a continuous and stationary process, and take

\[ X(s) := A(s) - s^2. \]

Then the location of the maximum of \(X\) is a.s. unique and

\[ \mathbb{E}Z = 0. \]

(The Airy process is stationary and has continuous paths.)
Notice that

\[ A(s) - s^2 + \lambda s = A(s) - (s - \frac{\lambda}{2})^2 + \frac{\lambda^2}{4}, \]

By taking expectations of the max and using stationarity, we have that

\[ m(\lambda) = m(0) + \frac{\lambda^2}{4}, \]

and hence

\[ m'(\lambda) = \frac{\lambda}{2}. \]
More Applications

- \( X(s) = B(s) - s^2 \), \( B \) two-sided Brownian motion on \( \mathbb{R} \).
  Then

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- In the previous case it is also possible to add a parabolic drift to get the Groeneboom-Janson relation
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  \text{Var}Z = \frac{\mathbb{E}M}{3}.
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- \( X(s) = B(s) + f(s) \), \( B \) standard Brownian motion on \( [0, t] \). Then
  \[
  \mathbb{E} Z = m'(0) = \text{Cov}(M, B(t)).
  \]