Beyond expansiveness

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Second Palis-Balzan International Symposium on Dynamical Systems

Institut Henri Poincaré - Paris-June-2013

Introduction

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A classical result establishes that every hyperbolic f-invariant and compact subset $\Lambda \subset M$ is expansive.

Expansiveness

First, let us define $\Gamma_\epsilon(x)$, the dynamical ball at x.

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f is expansive if $\exists\, \alpha > 0$ such that $\Gamma_\alpha(x) = \{x\} \, \forall \, x \in X$

$$\Updownarrow$$

$x, y \in X$, $x \neq y$, $\exists\, n \in \mathbb{Z}$ such that $\mathrm{dist}(\mathrm{f^n(x)}, \mathrm{f^n(y)}) > \alpha$.

Dynamics of f $\times$ dynamics of Df

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Usually one cannot expect that a sole notion on the underlying dynamics can guarantee any interesting feature on the infinitesimal level. Hence we ask for a robust property valid in a whole neighborhood of $f \in \mathrm{Diff^r(M)}$, $r \geq 1$.

Expansiveness versus robustness

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For 3-maniflods, we proved:

PPV Generically, $H(p)$ robustly C^1-expansive $\Longrightarrow$ $H(p)$ is hyperbolic.

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$E \times F$ is a dominated splitting if $\exists C > 0,\ 0 < \lambda < 1$ s.t.

$$\frac{\|Df^n|E(x)\|}{\|Df^{-n}|F(f^n(x))\|} \leq C\lambda^n \ \forall x \in \Lambda,\ n \geq 0.$$

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Next we introduce h-expansiveness.

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Let K compact , $x \in K$, and $\Gamma_\epsilon(x)$ be the dynamic ball.

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Buzzi: any $\mathcal{C}^\infty$ diffeo on compact manifold is asymptotically h-expansive.

Robust h-expansiveness: $\exists$ a C^1-neighborhood $\mathcal{U}$ of f s.t. $g \in \mathcal{U}$ then g is h-expansive.

Remark

There are examples of C^∞ diffeomorphisms (even analytic) on S^2 that are not entropy expansive: generalized pseudo Anosov.

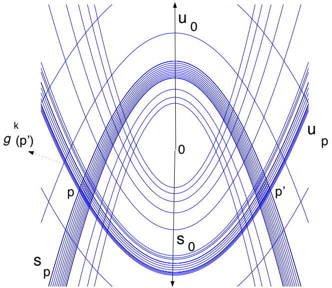

Figure : The singularity of a pseudo-Anosov

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K has a Dom. Splitting $\Longrightarrow f/K$ is h-expansive.

$H(p)$ robust h-expansive $\Longrightarrow K$ has a Dom. Splitting.

The n-dimensional case

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When $H(p)$ isolated $\Longrightarrow E$ is unif. contracting + G unif. expanding.

Reciprocally, for Λ invariant :

DFPV: $T\Lambda = E^s \times \cdots F^i \cdots \times E^u \Rightarrow \Lambda$ is h-expansive.

Measure expansiveness

(X, d): a metric space, $f : X \to X$ a homeo and μ a non-atomic probability on X (not necessarily f-invariant), and $\Gamma_\alpha(x)$ the dynamical ball at x.

f is μ-expansive if $\exists\, \alpha > 0$ such that $\mu(\Gamma_\alpha(x)) = 0$ for all $x \in X$.

The main result

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$\exists$ residual $\mathcal{G} \subset \mathrm{Diff}^1\mathrm{M}$ s.t. all $f \in \mathcal{G}$ is μ-expansive $\forall \mu \ll m$.

$f \in \mathcal{HT}(M) \Longrightarrow \exists g_n \to f$; g_n is not μ-expansive, $\mu \ll m$.

Remark Note we are dealing with $\mu \ll m$. Next I will comment about some results by Sakai, Sumi and Yamamoto.

Fixing notations

Let us fix some notations:

$\mathcal{E}$: the set of expansive diffeos.

$\mathcal{M}(M)$: Borel probability measures with weak topology

$\mathcal{A}(M)$: the set of atomic measures.

$\mathcal{M}_f(M)$: the set of invariant measures.

$\mathcal{M}_f^e(M)$: the set of invariant ergodic measures.

$\mathcal{PE} = \{f; f \text{ is } \mu - \text{expansive} \forall \mu \in \mathcal{M}(M) \setminus \mathcal{A}(M)\}$

$\mathcal{IE} = \{f; f \text{ is } \mu - \text{expansive} \forall \mu \in \mathcal{M}_f(M) \setminus \mathcal{A}(M)\}$

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$$T_{\mathcal{C}}M = E^s \oplus E^c_1 \oplus \cdots \oplus E^c_k \oplus E^u \,.$$

   The minimal stable dimension of the periodic orbits of $H(p)$ is $\dim(E^s)$ or $\dim(E^s) + 1$.

Fake foliations: BW-Theorem

Theorem Let $f : M \to M$, difeo, Λ a f- invariant p.h., s.t.

$$T_\Lambda M = E^s \oplus E_1^c \oplus \cdots \oplus E_k^c \oplus E^u,$$

$E^{cs,i}$ and $E^{cu,i}$ as before, $\tilde{E}^{cs,i}$ and $\tilde{E}^{cu,i}$ extensions to $\mathcal{V}(\Lambda)$. Then for $\epsilon > 0, \exists\ R > r > r_1 > 0$ s. t., $\forall\ p \in \Lambda$, $B(p, r)$ is foliated by $\widehat{W}^{u(s)}(p)$, $\widehat{W}^{cs(cu),i}(p), 1 \leq i \leq k$, s.t.

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Fake foliations

Given $j \in \{1, \ldots, k\}$, using Theorem above, we consider a small r and the submanifold

$$\widetilde{W}^{cs,j}(x) = \bigcup_{z \in \gamma_j(x)} \widehat{W}_x^{cs,j-1}(z,r).$$

This submanifold has dimension $s + j$ and is transverse to $\widehat{W}_x^{cu,j+1}(z)$ for all z close to x.

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Note that $\widetilde{W}^{cs,1}(x)$ is foliated by stable manifolds (recall that $\widehat{W}_x^{cs,0}(z) \subset W^s(z)$).

Idea proof of the main result

We combine the 3 ingridients:

(1) Theorem (CSY) (to obtain the residual subset).

(2) Fake foliations (that behave "almost" hyperbolic ones)

(3) Angles between unitary vectors in the cone fields $\mathcal{C}(E^{cs,j})$ and $\mathcal{C}(E^{cu,j+1})$ are uniformly bounded away from zero

If $\Gamma_\delta(x) \neq \{x\}$, (2) and (3) above allow us to "project" $\Gamma_\delta(x)$ along these foliations obtaining $\Gamma_\delta(x) \subset \widehat{W}^{cs,k}(x)$, for any f in the residual given by (1). As $\dim(\widehat{W}^{cs,k}(x)) < \dim(M)$ we obtain $\mu(\Gamma_\delta(x)) = 0$. Thus we prove:

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Theorem Let μ be probability on M, $\mu \ll m$, and let $f \in \mathcal{G}$ where $\mathcal{G}$ is as in Theorem (CSY). Then $\exists\ \delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$ for all $x \in M$.

Surface diffeomorphisms in $\mathcal{HT}$

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Before the proof, let us recall some facts proved elsewhere.

Horseshoes with positive Lebesgue measure

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Note that it is crucial to work in the C^1-topology : Bowen, proved that C^2 diffeomorphisms have no horseshoes with positive volume.

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We use the construction of R-Y to prove that arbitrarily near a diffeomorphism exhibiting a homoclinic tangency there is one which is not measure-expansive. First, the following holds :

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Lemma B Given a C^1 diffeomorphism $f_1 : M \to M$ with a flat homoclinic tangency there is a C^1 near diffeomorphism f_2 presenting a sequence of horseshoes $\widehat{\Lambda}_n$ such that for all $k \in \mathbb{Z}$: diam$(f^k(\widehat{\Lambda}_n) < r_n$ with $r_n \to 0$ when $n \to \infty$.

A snake-tangency

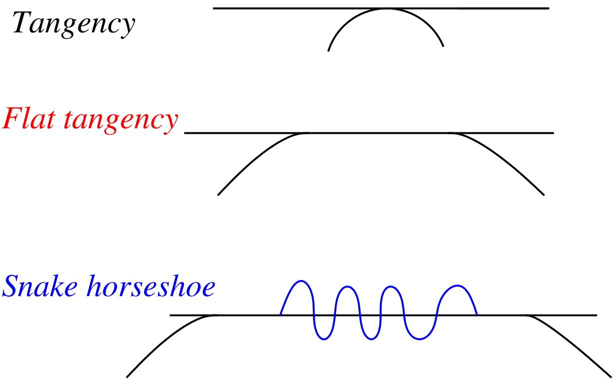

Figure : From tangency to flat-tangency-to snake-tangency

A family of fat horseshoes

Proposition Let $f_2 : M \to M$ as in the thesis of Lemma B. There is a C^1-diffeomorphism $F : M \to M$ arbitrarily near f_2 presenting a sequence of horseshoes Λ_n such that the Lebesgue measure $\mu(\Lambda_n) > 0$ and $\mathrm{diam}(\Lambda_n) < 2r_n$, where r_n is as in Lemma B.

Proof

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Since the support of the perturbation needed to fatten the horseshoe $\widehat{\Lambda}_n$ is contained in a box $B_n \supset \widehat{\Lambda}_n$ such that $\lim_{n\to\infty} \text{diam}(B_n) = 0$ (see Section 3, RY), it can be taken disjoint from the support of the previous perturbations needed to fatten $\widehat{\Lambda}_j$ for $j = 1, \ldots, n-1$ (see Sections 2 and 4 of R-Y). From this it follows that F is C^1- close to f_2 and has the desired sequence of horseshoes Λ_n with $m(\Lambda_n) > 0,$ all n.

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Moreover, the construction of Λ_n gives that the diam(Λ_n) is about the same of that of $\widehat{\Lambda}_n$, so that we can assure that diam$(\Lambda_n) < 2r_n$ from diam$(\widehat{\Lambda}_n) < r_n$.

A snake-horseshoe

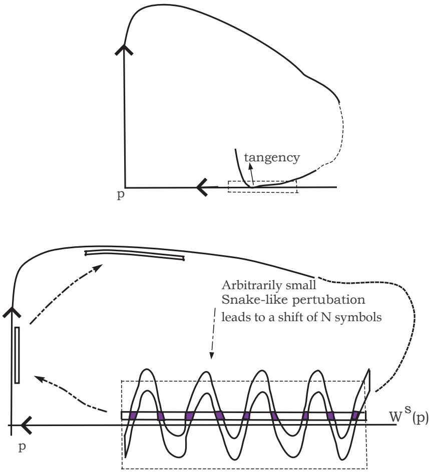

Figure : A snake-horseshoe

Consequence

Theorem Let M be a smooth compact surface. Given a C^1-diffeomorphism $f : M \to M \in \mathcal{HT}$ it is C^1-approximated by a diffeomorphism $F : M \to M$ such that F is not measure expansive with respect to any $\nu \ll m$.

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Proof Let $F : M \to M$ be the C^1 diffeomorphism constructed as in Proposition above. Then for every horseshoe Λ_n associated to F there is a hyperbolic periodic point $p_n \in \Lambda_n$ such that $\mu(\Gamma_{2r_n}(p_n)) \geq \mu(\Lambda_n)) > 0$ where $\mu \ll m$ and $f^*\mu = \mu$. Since $r_n \to 0$ when $n \to \infty$ the proof follows.

THANK YOU !