Diffeos Without Maximal Entropy Measure

Jérôme BUZZI (CNRS & Université Paris-Sud)

June 11, 2013

Palis-Balzan International Symposium on

Dynamical Systems

at Institut Henri Poincaré, Paris, France

Basic Definitions

$T : X \to X$ self-map of compact metric space
$\mathrm{Prob}_{\mathrm{erg}}(T) = \{$ergodic Borel probability meas.$\} +$ weak $\star$ topology

Definition

$B_T(x, \epsilon, n) := \{y \in X : \forall 0 \le k < n \; d(T^k y, T^k x) < \epsilon\}$
$r(\epsilon, n, T, Y) := \min\{\#C : \quad \bigcup_{x \in C} B_T(x, \epsilon, n) \supset Y\}$
$r(\epsilon, n, T, \mu) := \min\{\#C : \mu(\bigcup_{x \in C} B_T(x, \epsilon, n)) > 1/2\}$

Definition

$h_{\mathrm{top}}(T) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(\epsilon, n, T, X)$
$h(T, \mu) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(\epsilon, n, T, \mu)$

The Existence Problem

Theorem (Variational Principle, Goodman 1970)

$$h_{\text{top}}(T) = \sup_{\mu\in\text{Prob}_{\text{erg}}(T)} h(T,\mu)$$

Definition

$$\text{mme}(T) := \{\mu \in \text{Prob}_{\text{erg}}(T) : h(T,\mu) = h_{\text{top}}(T)\}$$

- Describe most points, e.g.:

  asympt. h-exp. $\Longrightarrow \lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\sum_{x\in E_{\epsilon,n}}\delta_{T^k x} \in \overline{<\text{mme}(f)>}$
- Often describe periodic points:
  equidistribution, precise counts (Bowen, Margulis,...)
- Are the main invariants:
  almost Borel classification of Markov shifts by M. Hochman.

Problem: When is $\text{mme}(f) \neq \emptyset$**?**

Existence through smoothness and/or hyperbolicity

Classic strategy:

expansivity $\implies \mu \mapsto h(f,\mu)$ usc $\implies \text{mme}(f) \neq \emptyset$

Through *hyperbolicity:*

Theorem (Parry, Sinai)

$\forall$ *Axiom A diffeomorphism* $\text{mme}(f) \neq \emptyset$

PROOF: (historically by coding)
hyperbolicity $\implies$ expansivity □

Through *smoothness:*

Theorem (Newhouse (1989))

$\forall$ C^∞ *self-map of a compact manifold:* $\text{mme}(f) \neq \emptyset$

PROOF: (historically using Pesin theory, see B 1997)
$C^\infty \implies$ asymptotic entropy-expansiveness (Yomdin, B) □

Existence through smoothness and/or hyperbolicity

Through *combination* of smoothness and hyperbolicity:

$\lambda(f) := \inf_{n\geq 1} \log \sup_x |(f^n)'(x)|$

Theorem (B.-Ruette (2007); Burguet (2012))

$\forall 1 \leq r < \infty\ \forall f : [0,1] \to [0,1]$ *if:*

f is C^r and

$h_{\text{top}}(f) > (1/r)\lambda(f)$

then $\text{mme}(f) \neq \emptyset$

Previous counter-examples to existence

Strategy 1:

Sequence of almost m.m.e.'s $\to$ on zero entropy (e.g. a fixed point)

Theorem (Misiurewicz 1973)

$\forall r < \infty, \forall M^{\geq 4}, \exists f \in \mathrm{Diff}^r(M)$ *with:* $\mathrm{mme}(f) = \emptyset$

Theorem (B 1998)

$\exists$ C^r *interval map with* $h_{\mathrm{top}}(f) \leq \frac{1}{r}\lambda(f)$ *with:* $\mathrm{mme}(f) = \emptyset$

Strategy 2:

Conjugacy to a subsystem whereas m.m.e. is unique with full supp

Theorem (Ruette 2002)

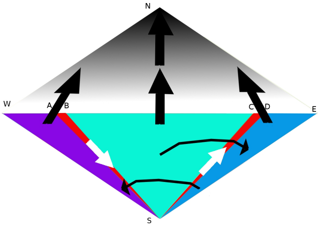
$\exists$ *mixing* C^r *interval map with* $h_{\mathrm{top}}(f) \leq \frac{1}{r}\lambda(f)$ *with:* $\mathrm{mme}(f) = \emptyset$

Piecewise affine continuous maps

$d = 1$:piecewise monotone mapshave mme$(f) \neq \emptyset$

Theorem (B 2009)

$\exists$ C^0 self-map of $[0,1]^2$ which is piecewise affine w/o m.m.e.



PROOF:

$h_{\text{top}}(f) = \log 2$

$h(f,\mu) \approx \log 2 \implies f$ like $(\theta,\rho) \mapsto (1-2|\theta|, \rho e^{\text{sign}(\theta)})$

$h(f,\mu) = \log 2 \implies (f,\mu)$ Bernoulli in θ and random wak in ρ

$\implies$ a.e. point falls into the trap: no such μ $\square$

Diffeomorphisms with finite smoothness

Theorem (B, ETDS 2013)

Let $1 \leq r < \infty$ *be any* **finite** *smoothness*

There exist C^r *diffeos on the unit* 2*-disk with:*

$$\text{mme}(f) = \emptyset \text{ and } f = Id \text{ near } \partial\mathbb{D}^2$$

Answer question by Misiurewicz 1973

Corollary

The same holds in:

- $\text{Diff}^r(M)$ *for any manifold* M^d *with* $d \geq 2$
- $\text{PH}^r(\mathbb{T}^d)$ *for each* $d \geq 2$

Strategy

CONSTRUCTION:
$\Omega(f) =$ Homoclinic loop at dissipative fixed saddle $\cup$\{fixed points\}
Entropy via snake horseshoes H_n with $h_{\mathrm{top}}(H_n) \nearrow \log \Lambda / r$

ANALYSIS:
H_n's are included in a larger homoclinic class!
"New" idea: Lyapunov exponents:
$\forall \mu$ non-periodic $(\lambda(\mu) < h_{\mathrm{top}}(f))$
$\implies \mathrm{mme}(f) = \emptyset$
□

Horseshoe creation (Newhouse)

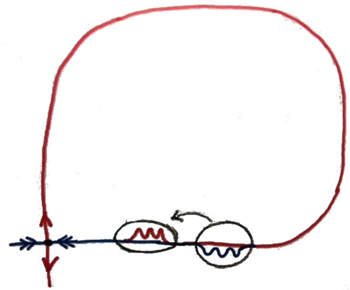
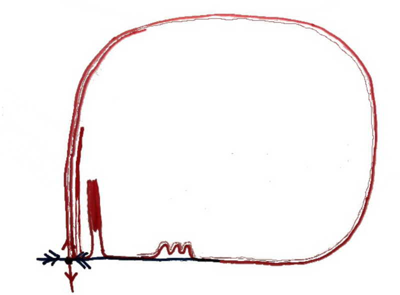


Figure: Create a single horseshoe

Horseshoe creation (Newhouse)



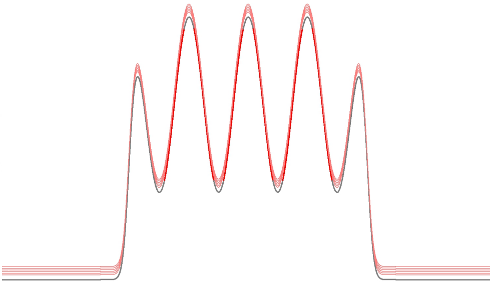


Figure: Horseshoe created by folding of $W^u_{loc}(p)$

Repeat $\rightsquigarrow$ sequence of horseshoes with entropy $\uparrow \frac{1}{r}\log\Lambda$

Basic properties

In coordinates at fixed point p,

$$f'(p) \equiv \begin{pmatrix} \Lambda & 0 \\ 0 & K^{-1} \end{pmatrix} \text{ with } 1 < \Lambda << K.$$

Perturbation around $[a, b] \times \{0\}$ with $K^{-1} b < a$:

- composition with:
  $$g : (x, y) \mapsto \big(x, y + \alpha(x, y)\Lambda^{-T}\,(2 + \sin(\pi N(x - a)))\big)$$
- C^r norm $\approx N^r/\Lambda^T \rightsquigarrow N \leq \Lambda^{T/r}$
- $h_{\text{top}}(f|H) = \log N/T \rightsquigarrow h_{\text{top}}(f|H) \leq \frac{1}{r} \log \Lambda$

Sequence of such horseshoes with $h_{\text{top}}(f|H_n) \uparrow \frac{1}{r} \log \Lambda$:
$\rightsquigarrow h_{\text{top}}(f) \geq \frac{1}{r} \log \Lambda$

Using Lyapunov exponent: example of the C^1 case

Take $\phi : M \to [0,1]$ zero only near p

Replace f with $f \circ \begin{pmatrix} e^{-\phi(t)} & 0 \\ 0 & 1 \end{pmatrix}$

Observe:

- Ruelle's inequality: $h(f,\mu) \leq \lambda^u(\mu) < \log \Lambda$ for any $\mu \neq \delta_p$
- $h(f,\delta_p) = 0$
- $h_{\text{top}}(f) = \log \Lambda$
- $\text{mme}(f) = \emptyset$

The C^r case

Lyapunov exponent:

$$\lambda_f(\mu) := \| \sup_{v \in T_x^1 M} \limsup_{n\to\infty} \frac{1}{n} \log \|(f^n)'.v\| \|_{L^\infty(\mu)}$$

Theorem

Let $1 \leq r < \infty$ and $f \in \mathrm{Diff}^\infty(\mathbb{D}^2)$ with simple homoclinic loop at strongly dissipative fixed saddle p and slow transition.
Then:
$\exists \tilde{f} \to f$ in C^r such that:

- $h_{\mathrm{top}}(\tilde{f}) = \frac{1}{r} \log \Lambda$
- *for $\mu \in \mathrm{Prob}_{aper}(\tilde{f})$ (ergodic, aperiodic)*

$$\lambda_{\tilde{f}}(\mu) < \frac{1}{r} \log \Lambda$$

Exponent estimate

$(x, v) \in TM \setminus \{0\}$
(Lower) Lyapunov exponent of v:
$\lambda(v) := \liminf_{n\to\infty} \frac{1}{n} \log \|(f^n)'.v\|$

Corner where $f(x_1, x_2) = (K^{-1}x, \Lambda y)$ before perturbations:
$C = p + [0, 1]^2$

Proposition

For any $v \in T^1M$*, inside the loop:* $\lambda(v) < \frac{1}{r} \log \Lambda - \chi \cdot \phi_C(x)$*,*
with $\phi_C(x) =$ *positive frequency of visits to the corner*

Step 0: Subdivision of the orbit

Positive orbits are cut by:

– the entry times t_n, ie, $f^{t_n}(x) \in [K^{-1}, 1] \times [0, 1]$ and

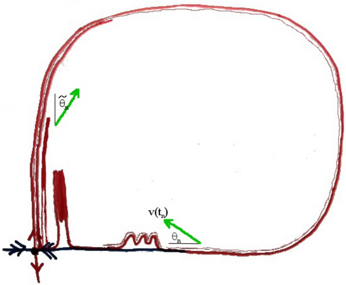
– the exit times s_n from $C \equiv [0, 1]^2$:

$$0 = t_0 < s_1 < t_2 < s_2 < t_3 < s_3 < \dots$$

Define the images and the angles:

$v(k) = (f^k)'(v)$, $x(k) := f^k(x)$

$v(t_n)_2 = \theta_n \cdot v(t_n)_1$, $v(s_n)_1 = \tilde{\theta}_n \cdot v(s_n)_2$



Step 1: Estimates before the perturbation

Compute for any $v \in T^1M$ inside the loop:

- $1 = (x(s_n))_2 = \Lambda^{\tau_n}(x(t_n))_2$
  $(x(t_n))_2 = (x(s_{n-1}))_1 = K^{-\tau_{n-1}}(x(t_{n-1})_1) = K^{-\tau_{n-1}}$
  $\implies \Lambda^{\tau_n} K^{-\tau_{n-1}} = 1$

Conclusions:

- $\tau_n = \eta\tau_n$ $(\eta := \frac{\log K}{\log \Lambda} >> 1)$
- $(x(t_n))_2 = \Lambda^{-\tau_n} = (x(t_{n-1})_2)^\eta$
- orbits converge to the loop and $\lambda(v) = -\log K + \mathcal{O}(1/\log K)$ (with oscillations)

Homoclinic tangency:

previously expanded mapped into the contracting

Step 2: Expansion by block

Needed:

- perturbations do not rotate vectors too much: $|\partial_1 \tilde{f}_2(x,y)| \le |\tilde{f}_2(x,y)|^{1-1/r}$
- τ_n are eventually big

Lemma

If $\theta_{n-1} \le \Lambda^{-(1-1/r)\tau_n}$*:* $|v(t_n)| \le (\log K)\Lambda^{\tau_n/r}|v(t_{n-1})|$
else: $|v(t_n)| \le K^C |v(t_{n-2})|$

Proof of the proposition from the lemma.

• Slow down the flow to absorb constants and create a loss in expansion proportional to n

• Divide $[t_0, t_n]$ into $[t_{i-1}, t_i]$ or $[t_{i-2}, t_i]$ according to above condition

□

On the Main Theorem

Uncontrolled dynamics

The horseshoes are homoclinically related!

Optimal Entropy

Using Newhouse's and Yomdin's theory: $\limsup_{\tilde{f}\to_{C^r} f} h_{\text{top}}(\tilde{f}) = \frac{1}{r}\log\Lambda$

ie, as much entropy as possible

Fragile proof

For $r > 1$ we use a very special situation

$$\lim_{n\to\infty}\sup_{\mu}\lambda_{\tilde{f}_n}(\mu) = \frac{1}{r}\log\Lambda < \lim_{\tilde{f}\to_{C^\infty}}\sup_{\mu}\lambda_{\tilde{f}}(\mu) = \log\Lambda$$

(suprema over ergodic, aperiodic measures)

Is it big?

For $1 \leq r < \infty$, let

$$\text{NoMax}^r(M) := \{f \in \text{Diff}^r(M) : \text{mme}(f) = \emptyset\}$$

Question

Is $\text{NoMax}^r(M)$ *a big subset of* $\text{Diff}^r(M)$*?*
Is it locally generic? locally dense?

Remarks

- $\text{NoMax}^r(M)$ has empty interior: C^∞ diffeos have m.m.e
- B-Fisher (to appear) gives a non-empty C^1-open set of diffeos with generically no symbolic extension but always a m.m.e. Hence:

- (i) $\mu \mapsto h(f, \mu)$ usc $\iff$ $\exists$ principal symbolic extension; (ii) both implies $\text{mme}(f) \neq \emptyset$, but there is no converse
- $\text{NoMax}^1(M)$ is not dense away from hyperbolic systems

Finally

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- J. Buzzi, *C^r surface diffeomorphisms with no maximum entropy measure,* Ergod. Th. Dynam. Syst., to appear.
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Thank you!