Disquisitiones Mathematicae

Matheus' Weblog
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Finiteness of algebraically primitive closed SL(2,R)-orbits in moduli spaces

Today I gave a talk at the Second Palis-Balzan conference on Dynamical Systems (http://www.impa.br/opencms/pt/eventos/store/evento_1305) (held at Institut Henri Poincaré (http://www.ihp.fr/), Paris). In fact, I was not supposed to talk in this conference: as I'm serving as a local organizer (together with Sylvain Crovisier (http://www.math.u-psud.fr/%7Ecrovisie/)), I was planning to give to others the opportunity to speak. However, Jacob Palis insisted that everyone must talk (the local organizers included), and, since he is the main organizer of this conference, I could not refuse his invitation.

Anyhow, my talk concerned a joint work with Alex Wright (http://math.uchicago.edu/%7Eamwright/) (currently a PhD student at U. of Chicago under the supervision of Alex Eskin (http://math.uchicago.edu/%7Eeskin/)) about the finiteness of algebraically primitive closed $SL(2,\mathbb{R})$-orbits on moduli spaces (of Abelian differentials).

Below the fold, I will transcript my lecture notes for this talk.

1. Preliminaries and statement of main result

The data of a Riemann surface M of genus $g\geq 1$ and a (non-trivial) Abelian differential (i.e., holomorphic 1-form) ω on M determines a structure of *translation surface* on M, that is, by local integration of ω *outside* the (finite) set Σ of its zeroes, we obtain an atlas on $M-\Sigma$ such that all changes of coordinates are given by translation on $\mathbb{C}\simeq\mathbb{R}^2$. In particular, after cutting M along the edges of a triangulation $\mathcal{T}$ such that the set Σ of zeroes of ω is exactly the set of vertices of $\mathcal{T}$, we can think of (M,ω) as a finite collection of triangles T_l on the plane $\mathbb{R}^2$ whose sides are glued by translations.

The set of translation structures on a surface of genus $g\geq 1$ is naturally *stratified* by the subsets obtained by *fixing* the multiplicities of the zeroes of Σ. Note that, for each $g\geq 1$, the number of strata is precisely the number of partitions of $2g-2$: indeed, by Riemann-Hurwitz theorem, the sum $\sum_{s=1}^{\sigma} k_s$ of the orders k_s of the zeroes of a non-trivial Abelian differential ω is always $2g-2$.

In this language, given a partition $(k_1,\dots,k_\sigma)$ of $2g-2$, the corresponding *stratum* $H(k_1,\dots,k_\sigma)$ of the *moduli space* of *unit area* Abelian differentials / translation structures is the set of pairs (M,ω) of Riemann surfaces and Abelian differentials with unit area (that is, the total area of the triangles T_l mentioned above is one) and zeroes of orders k_s *modulo* biholomorphisms. In terms of translation structures, $H(k_1,\dots,k_s)$ is the set of translation structures with conical singularities at vertices of the triangles T_l with a total angle of $2\pi(k_s+1)$ and the total area of the triangles T_l is one *modulo* cutting and pasting by translations.

For more informations on translation structures and their moduli spaces, the reader might wish to consult these posts (http://matheuscmss.wordpress.com/2010/09/02/lyapunov-spectrum-of-kontsevich-zorich-cocycle-on-the-hodge-bundle-of-square-tiled-cyclic-covers-i/) here (http://matheuscmss.wordpress.com/2010/11/02/lyapunov-spectrum-of-kontsevich-zorich-cocycle-on-the-hodge-bundle-of-square-tiled-cyclic-covers-ii/).

From the description of $H(k_1, \dots, k_\sigma)$ in terms of translation structures (and triangles glued by translations), it is clear that $H(k_1, \dots, k_\sigma)$ has a natural $SL(2,\mathbb{R})$-action.

This action of $SL(2,\mathbb{R})$ on $H(k_1, \dots, k_\sigma)$ is useful in the investigation of deviations of ergodic averages of interval exchange transformations, translation flows and billiards in rational polygons: for instance, it was recently applied by Delecroix-Hubert-Lelièvre (http://arxiv.org/abs/1107.1810) to confirm a conjecture of the physicists Hardy-Weber on the rates of diffusions of typical trajectories in typical realizations of Ehrenfest wind-tree model of Lorentz gases.

In particular, a lot of attention was given to the question of understanding the closure of $SL(2,\mathbb{R})$-orbits and/or ergodic $SL(2,\mathbb{R})$-invariant probability measures. Here, it was widely believed among experts that a Ratner-like result (http://en.wikipedia.org/wiki/Ratner%27s_theorems) should be true for the $SL(2,\mathbb{R})$-action on $H(k_1, \dots, k_\sigma)$ because, philosophically speaking, $H(k_1, \dots, k_\sigma)$ is "very close to a homogenous space" (despite its non-homogeneity). In this direction, Eskin-Mirzakhani (http://arxiv.org/abs/1302.3320) showed the following profound result:

> **Theorem 1 (Eskin-Mirzakhani)** *Any ergodic $SL(2,\mathbb{R})$-invariant probability measure μ on any stratum $H(k_1, \dots, k_\sigma)$ is affine, i.e., it is a density (Lebesgue) measure on certain affine subspaces (in period coordinates) associated to its support.*

Of course, this theorem gives *hope* for a general *classification* result of all ergodic $SL(2,\mathbb{R})$-invariant probability measures on all strata $H(k_1, \dots, k_\sigma)$.

On the other hand, a *complete* classification of such measures is only available in genus 2 (i.e., for the strata $H(2)$ and $H(1,1)$) thanks to the works of K. Calta (http://www.ams.org/mathscinet-getitem?mr=2083470) and C. McMullen (http://www.ams.org/mathscinet-getitem?mr=2299738) in 2004. However, as it was pointed out by C. McMullen himself, his methods do *not* to genus $g\geq 3$.

Nevertheless, some partial progress was made recently for certain types of ergodic $SL(2,\mathbb{R})$-invariant probability measures such as the ones supported in a (*single*) *closed* $SL(2,\mathbb{R})$-orbits — also known as *Teichmüller curves*.

More concretely, M. Möller and his collaborators considered *algebraically primitive* closed $SL(2,\mathbb{R})$-orbits of genus $g\geq 3$, that is, Teichmüller curves such that the field generated by the traces of the stabilizers in $SL(2,\mathbb{R})$ of any of its points has degree g over $\mathbb{Q}$. Here, they showed that there are only *finitely many* algebraically primitive Teichmüller curves in the following connected components of stratum:

- the hyperelliptic component $H(g-1,g-1)^{hyp}$ consisting of translation surfaces in $H(g-1,g-1)$ with an hyperelliptic involution exchanging the two zeroes (cf. this paper of M. Möller in 2008 (http://www.ams.org/mathscinet-getitem?mr=2401216));
- $H(3,1)$ (cf. this paper of M. Bainbridge and M. Möller in 2011 (http://www.ams.org/mathscinet-getitem?mr=2910796));
- the hyperelliptic component $H(4)^{hyp}$ of $H(4)$ (this is part of a work in progress by Bainbridge, Habegger and Möller).

> **Remark 1** *In genus 2, it follows from the works of Calta and McMullen that there are infinitely many algebraically primitive Teichmüller curves in $H(2)$ and only one algebraically primitive Teichmüller curve in $H(1,1)$.*

This scenario motivates the following question: *Given a connected component $\mathcal{C}$ of a stratum $H(k_1,\dots,k_\sigma)$ in genus $g\geq 3$, are there only finitely many algebraically primitive Teichmüller curves inside $\mathcal{C}$?*

Of course, a positive answer to it hints that, *a priori*, it could be a good idea to start this classification by listing $SL(2,\mathbb{R})$-invariant probability measures supported on algebraically primitive Teichmüller curves (as there would be only finitely many of them per stratum).

In this post, we want to discuss the following modest contribution to this question:

> **Theorem 2 (C. M. and A. Wright)** *Let $\mathcal{C}$ be a connected component of a (minimal) stratum $H(2g-2)$ in prime genus $g\geq 3$. Then, there are only finitely many algebraically primitive Teichmüller curves in $\mathcal{C}$.*

Before giving the proof of Theorem 2 above, let us make two quick comments.

First, the arguments we use to prove our finiteness result are *dynamical* in nature (based on the analysis of the so-called Kontsevich-Zorich cocycle), while the methods of Möller and his collaborators were more algebro-geometrical (based on the behavior of algebraically primitive Teichmüller curves near the boundary of Deligne-Mumford's compactification of moduli spaces).

Interestingly enough, these two methods seem to be "complementary" in the sense that our dynamical method is particularly adapted to the study of algebraically primitive Teichmüller curves in *minimal* strata $H(2g-2)$, while the algebro-geometrical method of Möller and his collaborators is better adapted to *non-minimal* strata (and this explains why the result of Bainbridge-Habegger-Möller mentioned above was obtained only very recently compared to the other results).

Secondly, A. Wright and I also have partial results for other (non-minimal) strata, e.g., we can show that algebraically primitive Teichmüller curves can't form a dense subset of a given connected component of a stratum in genus $g\geq 3$, but we will restrict our discussion to Theorem 2 because its proof already contain all essential ideas of our forthcoming paper.

2. Proof of Theorem 2

The basic idea is very simple. Let us fix $g\geq 3$ *prime* and let us suppose by contradiction that there are infinitely many algebraically primitive Teichmüller curves C_n in some connected component $\mathcal{C}$ of the minimal stratum $H(2g-2)$.

A recent equidistribution result of A. Eskin, M. Mirzakhani and A. Mohammadi (http://arxiv.org/abs/1305.3015) implies that C_n will equidistribute *inside* the support $\mathcal{M}$ of an ergodic $SL(2,\mathbb{R})$-invariant probability measure. Actually, for our current purposes, we do not the full strength of this equidistribution result: it is sufficient to know that C_n eventually becomes dense inside $\mathcal{M}$ in the sense that there exists an integer $n_0\in\mathbb{N}$ such that $C_n\subset\mathcal{M}$ for all $n\geq n_0$ and the subset $\bigcup\limits_{n\geq n_0} C_n$ is dense in $\mathcal{M}$.

Now, we claim that $\mathcal{M}$ is *equal* to the whole connected component $\mathcal{C}$ of $H(2g-2)$, that is, the sequence of algebraically primitive Teichmüller curves C_n becomes dense in $\mathcal{C}$. In fact, as we mentioned earlier, by the results of A. Eskin and M. Mirzakhani (cf. Theorem 1 above), $\mathcal{M}$ is represented by affine subspaces in period coordinates. As it was shown by A. Wright in this paper here (http://arxiv.org/abs/1210.4806), these affine subspaces representing $\mathcal{M}$ are defined over some field $k(\mathcal{M})$ of degree $1\leq deg(\mathcal{M})\leq g$ over $\mathbb{Q}$, and, moreover, since $\mathcal{M}\subset H(2g-2)$, one also has the inequality

$$deg(\mathcal{M}) \cdot (\dim(\mathcal{M}) + 1) \leq 4g \qquad (1)$$

On the other hand, it is possible to check that the affine subspaces representing a Teichmüller curve (closed $SL(2,\mathbb{R})$-orbit) D are defined over a field $k(D)$ whose degree over $\mathbb{Q}$ coincide with the degree of the field generated by the traces of the stabilizers of D in $SL(2,\mathbb{R})$.

In particular, it follows (from the definition of algebraic primitivity) that the affine subspaces representing the algebraically primitive Teichmüller curves C_n are defined over some fields whose degrees over $\mathbb{Q}$ are g. Because $C_n \subset \mathcal{M}$ for all $n \geq n_0$, we deduce that the degree $deg(\mathcal{M})$ of $k(\mathcal{M})$ *divides* g.

Since g is prime (by hypothesis), we have that $deg(\mathcal{M}) = [k(\mathcal{M}):\mathbb{Q}] = 1$ or g. We affirm that $k(\mathcal{M}) = \mathbb{Q}$: indeed, if this is not true, then $deg(\mathcal{M}) = g$; by inserting this into the inequality (1), we would have $dim(\mathcal{M}) \leq 3$; since $\mathcal{M}$ consists of entire $SL(2,\mathbb{R})$-orbits (and $SL(2,\mathbb{R})$ has dimension 3), we get that $dim(\mathcal{M}) = 3$; however, this is impossible because $\mathcal{M}$ contains infinitely many closed $SL(2,\mathbb{R})$-orbits, namely, C_n for $n \geq n_0$.

Once we know that $k(\mathcal{M}) = \mathbb{Q}$, i.e., $deg(\mathcal{M}) = 1$, we affirm that $dim(\mathcal{M}) = 4g-1$, that is, the inequality (1) is an equality. In fact, let us consider the subset $\mathbb{R}\cdot\mathcal{M}$ consisting of Abelian differentials belonging to $\mathcal{M}$ after normalization of its total area and let us denote by $T_x(\mathbb{R}\cdot\mathcal{M})$ at some point $x \in C_{n0} \subset \mathcal{M}$. By definition, $T_x(\mathbb{R}\cdot\mathcal{M})$ contains the 4-dimensional subspace $T_x(\mathbb{R}\cdot C_{n0})$. Since $T_x(\mathbb{R}\cdot\mathcal{M})$ is defined over $k(\mathcal{M}) = \mathbb{Q}$ and $T_x(\mathbb{R}\cdot C_{n0})$ is defined over a field of degree g over $\mathbb{Q}$, we see that $T_x(\mathbb{R}\cdot\mathcal{M})$ must contain all g Galois conjugates $T_x(\mathbb{R}\cdot C_{n0})$, that is, the linear subspace $T_x(\mathbb{R}\cdot\mathcal{M})$ has dimension $\geq 4g$, and, *a fortiori*, $dim(\mathcal{M}) \geq 4g-1$. By combining this with inequality (1), we deduce that $dim(\mathcal{M}) = 4g-1$.

At this point, we can complete the proof of our claim that $\mathcal{M} = \mathcal{C} \subset H(2g-2)$ by simply counting dimensions: using the period coordinates on $H(2g-2)$, we can identify the tangent space of any point $x = (M,\omega) \in \mathbb{R}\cdot H(2g-2)$ with $H^1(M,\mathbb{R}) \oplus iH^1(M,\mathbb{R})$, a vector space of real dimension $4g$; hence $\mathbb{R}\cdot H(2g-2)$ has dimension $4g$ and, *a fortiori*, $dim(\mathcal{C}) = dim(H(2g-2)) = 4g-1$.

In summary, we just proved the following statement:

> **Proposition 3** *If there are infinitely many algebraically primitive Teichmüller curves C_n in some connected component $\mathcal{C}$ of a minimal stratum $H(2g-2)$ with $g \geq 2$ prime, then they must form a dense subset of $\mathcal{C}$, that is,*
>
> $$\overline{\bigcup_{n\in\mathbb{N}} C_n} = \mathcal{C}$$

Our plan is to contradict this denseness property of C_n along the following lines.

Let us consider the so-called Kontsevich-Zorich (KZ) cocycle (http://matheuscmss.wordpress.com/2011/02/24/lyapunov-spectrum-of-the-kontsevich-zorich-cocycle-on-the-hodge-bundle-over-square-tiled-cyclic-covers-iii/) over $\mathcal{C} \subset H(2g-2)$, that is, we use the $SL(2,\mathbb{R})$-action to move around $\mathcal{C} \subset H(2g-2)$, and every time that we cut and paste a translation structure $g\cdot(M,\omega)$ obtained from $(M,\omega) \in \mathcal{C}$ by applying $g \in SL(2,\mathbb{R})$ to get a shape "close" to (M,ω), we keep track of how the homology cycles on (M,ω) were changed. In practice, by selecting an appropriate symplectic basis of $H_1(M,\mathbb{R})$ (with respect to the usual intersection form on homology), this amounts to "attach" a symplectic matrix $A_{g,(M,\omega)} \in Sp(2g,\mathbb{Z})$ to each $g \in SL(2,\mathbb{R})$ and $(M,\omega) \in \mathcal{C}$.

Generally speaking, the Kontsevich-Zorich cocycle preserves a family of 2-dimensional symplectic plane $P_{(M,\omega)} \subset H_1(M,\mathbb{R})$ (called *tautological planes*). Since this cocycle is symplectic, the symplectic orthogonals $P^{\dagger}_{(M,\omega)}$ of the planes $P_{(M,\omega)}$ are also preserved, so that the Kontsevich-Zorich cocycle along any $SL(2,\mathbb{R})$-orbit decomposes into a 2×2-block corresponding to its restriction to $P_{(M,\omega)}$ and a $(2g-2)\times(2g-2)$-block corresponding to its restriction to $P^{\dagger}_{(M,\omega)}$. For later use, we will call *restricted Kontsevich-Zorich cocycle* the restriction of the Kontsevich-Zorich cocycle to $P^{\dagger}_{(M,\omega)}$. By definition, the restricted KZ cocycle is a $Sp(2g-2,\mathbb{R})$-cocycle over $SL(2,\mathbb{R})$-orbits.

As it turns out, the restricted KZ cocycle tends to behave poorly or richly depending on the $SL(2,\mathbb{R})$ -orbit we look at.

For example, the restricted KZ cocycle over an algebraically primitive Teichmüller curve D in a stratum of genus g decomposes into $g-1$ blocks consisting of its restrictions to the $g-1$ non-trivial Galois conjugates of the tautological planes (because these tautological planes are defined over a field of degree g over $\mathbb{Q}$ when the Teichmüller curve D is algebraically primitive). In particular, the matrices of the restricted KZ cocycle over D belong to a "poor" subgroup isomorphic to $SL(2,\mathbb{R})^{g-1}$ of the symplectic group $Sp(2g-2,\mathbb{R})$.

On the other hand, it is expected that the restricted KZ cocycle over "most" $SL(2,\mathbb{R})$-orbits (in any connected component of any [not necessarily minimal] stratum) has a "rich" behavior: for instance, the simplicity result of Avila-Viana (http://www.ams.org/mathscinet-getitem?mr=2316268) (as well as some numerical experiments) gives hope that the monoid of matrices of the restricted KZ cocycle over a typical $SL(2,\mathbb{R})$-orbit has full Zariski closure $Sp(2g-2,\mathbb{R})$.

So, if some family of algebraically primitive Teichmüller curves C_n becomes dense in some connected component $\mathcal{C}$ of some stratum in genus $g\geq 3$, we would have to two opposite properties fighting one against the other: on one hand, the restricted KZ cocycle over the dense subset $\bigcup\limits_{n\in\mathbb{N}} C_n$ has a very rigid structure, namely, they decompose into $(g-1)$ blocks of sizes 2×2; on the other hand, the typical $SL(2,\mathbb{R})$-orbit has a "rich" restricted KZ cocycle, say Zariski dense in $Sp(2g-2,\mathbb{R})$, and thus it can not be have a very rigid structure.

Therefore, we will contradict the denseness property of C_n in $\mathcal{C}$ if we can show that:

- (i) some non-trivial piece of information coming from the very rigid structure of the restricted KZ cocycle over C_n "pass to the limit" in the sense that the restricted KZ cocycle over *all* $SL(2,\mathbb{R})$ -orbits in $\mathcal{C}$ would have some (partial) rigidity property;
- (ii) show that the restricted KZ cocycle over *some* $SL(2,\mathbb{R})$-orbit in $\mathcal{C}$ is rich enough so that no rigidity property can occur (not even partially).

Concerning (i), it is tempting to use the decomposition into $(g-1)$ blocks of sizes 2×2 as the very rigid property that passes through the limit. However, it is simply *not* true that the limit of a decomposition into $(g-1)$ blocks is still a set of $(g-1)$ blocks: indeed, even though the Grassmanian of planes is compact (and hence we can extract limits), some of these blocks might collapse.

Nevertheless, it possible to check that the restricted KZ cocycle over C_n *preserves* a very special (projective) subset of Grassmanian of (symplectic) planes that A. Wright and I call *Hodge-Teichmüller planes* or HT planes for short. More precisely, we say that a plane $P\subset H^1(M,\mathbb{R})$ is a *HT plane* if the images of its complexification $P_{\mathbb{C}} := P\otimes\mathbb{C}\subset H^1(M,\mathbb{C})$ under all matrices $A_{g,(M,\omega)}$ of the KZ cocycle intersect non-trivially the $(1,0)$ and $(0,1)$ subspaces $H^{1,0}(M)$ and $H^{0,1}(M)$ of the Hodge filtration (http://en.wikipedia.org/wiki/Hodge_structure) of $H^1(M,\mathbb{C})$ (into holomorphic and anti-holomorphic forms).

Note that HT planes are really very special: since the usual action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2\equiv\mathbb{C}$ is *not* holomorphic, the complex structure of (M,ω) *changes* when we apply the elements $g\in SL(2,\mathbb{R})$, so that the subspaces $H^{1,0}(M)$ and $H^{0,1}(M)$ move around $H^1(M,\mathbb{C})$ and hence it is not easy for the images of a plane P to keep intersecting the "moving targets" $H^{1,0}$ and $H^{0,1}$.

As it turns out, the (rigid) property of preservation of a non-trivial set of HT planes passes through the limit: indeed, this follows from the facts that the HT planes P were defined in terms of the intersections of $A_{g,(M,\omega)}P$ with $H^{1,0}(M)$ and $H^{0,1}(M)$ and it is known that $H^{1,0}(M)$ and $H^{0,1}(M)$ vary *continuously* with (M,ω).

Also, in the case of the algebraically primitive Teichmüller curves C_n, the $(g-1)$ HT planes (obtained from the non-trivial Galois conjugates of the tautological planes) are *Hodge orthogonal*, that is, they are mutually orthogonal with respect to the Hodge inner product (http://matheuscmss.wordpress.com/2012/10/06/on-sums-of-lyapunov-exponents-of-kontsevich-zorich-cocycle-or-an-informal-preparation-for-pascal-huberts-bourbaki-seminar-talk-on-october-20-2012/) in cohomology. Since the Hodge inner product is also known to depend continuously on (M,ω), we have that the limits of the $(g-1)$ HT planes associated to C_n lead to $(g-1)$ Hodge-orthogonal HT planes.

In other words, we showed the following result (that formalizes item (i) above):

Proposition 4 *Let C_n, $n\in\mathbb{N}$, be a family algebraically primitive Teichmüller curves in some connected component $\mathcal{C}$ of some stratum in genus $g\geq 2$ such that*

$$\overline{\bigcup_{n\in\mathbb{N}} C_n}=\mathcal{C}.$$

Then, the restricted Kontsevich-Zorich cocycle over any $SL(2,\mathbb{R})$-orbit in $\mathcal{C}$ preserves the non-trivial set of at least $(g-1)$ HT planes that are mutually Hodge-orthogonal.

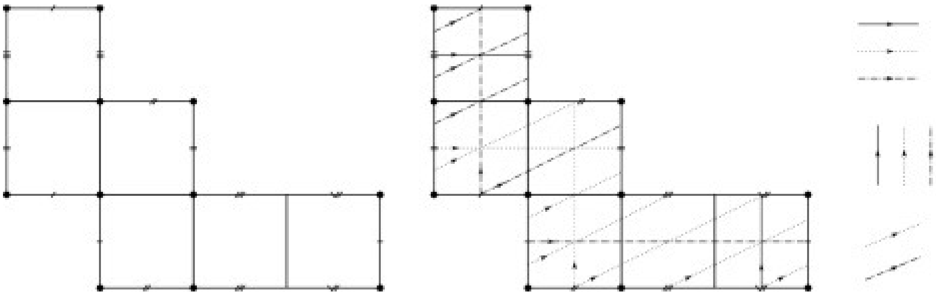
From Propositions 3 and 4, we see that the proof of Theorem 2 is reduced to the following proposition:

Proposition 5 *In any connected component $\mathcal{C}$ of any stratum in genus $g\geq 3$ one can find $SL(2,\mathbb{R})$-orbits such that the restricted KZ cocycle can not have $(g-1)$ Hodge-orthogonal HT planes.*

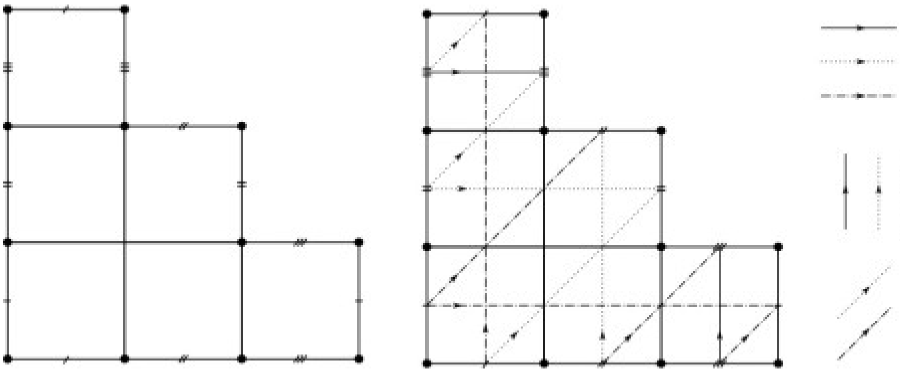
The proof of Proposition 5 is based on the following observations. Since the definition of HT planes is *algebraic* in nature, it is not hard to see that the set of at least $g-1$ HT planes preserved by the restricted KZ cocycle is also preserved by its Zariski closure (in $Sp(2g-2,\mathbb{R})$). Moreover, this set of HT planes is a proper compact subset of the Grassmanian of symplectic planes. Therefore, since $Sp(4,\mathbb{R})$ has no proper compact invariant subset in the Grassmanian of symplectic planes, we have that any $SL(2,\mathbb{R})$-orbit such that the Zariski closure of the restricted KZ cocycle is "at least as rich as $Sp(4,\mathbb{R})$" is likely to do not have $(g-1)$ HT planes. In other words, the proof of Proposition 5 follows *if* we construct on each connected component $\mathcal{C}$ of any stratum in genus $g\geq 3$ some $SL(2,\mathbb{R})$-orbit whose restricted KZ cocycle has a "mild rich Zariski closure".

In our first attempts, A. Wright and I tried to construct such $SL(2,\mathbb{R})$-orbits by taking adequate ramified covers of certain closed $SL(2,\mathbb{R})$-orbits in genus 3. In fact, this idea works to produce nice examples in *some* connected components of *some* minimal strata $H(2g-2)$ (where g runs along some arithmetic progression), but if one wishes to construct examples in *all* connected components of *all* strata in genus $g\geq 3$, then one runs into some nasty combinatorial problems that we do not know how to solve.

For this reason, we decided to change our initial strategy outlined in the previous paragraph into the following one. Firstly, we considered the following two *square-tiled surfaces* (i.e., translation surfaces obtained from finite coverings of the unit torus $\mathbb{R}^2/\mathbb{Z}^2$ that are ramified only at the origin $0 \in \mathbb{R}^2/\mathbb{Z}^2$):



(http://matheuscmss.files.wordpress.com/2013/06/mw-h4hyp1.jpg)
A square-tiled surface in $M_* \in H(4)^{hyp}$.

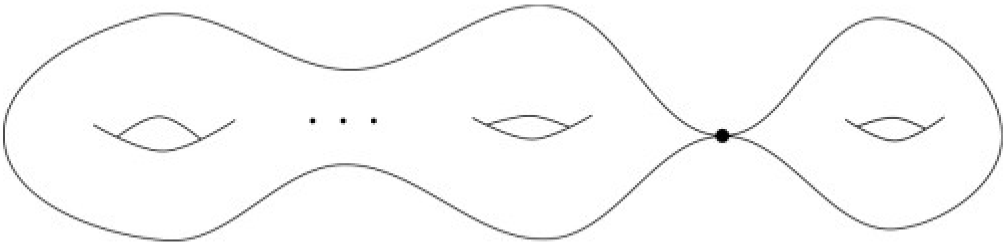


(http://matheuscmss.files.wordpress.com/2013/06/mw-h4odd.jpg)
A square-tiled surface in $M_{**} \in H(4)^{odd}$.

These square-tiled surfaces generate closed $SL(2,\mathbb{R})$-orbits in the two connected components $H(4)^{hyp}$ and $H(4)^{odd}$ of $H(4)$. Moreover, by computing some matrices of the restricted KZ cocycle over these square-tiled surfaces (using appropriate Dehn twists along the directions indicated above), we can show that the Zariski closure of the restricted KZ cocycle is $Sp(4,\mathbb{R})$ in both cases. In particular, from our discussion so far, this proves Proposition 5 for the cases of the two connected components of $H(4)$.

Now, the idea to prove Proposition 5 in the general case of a connected component $\mathcal{C}$ is to find some translation surface (in $\mathcal{C}$) whose restricted KZ cocycle behaves a "little bit" like the corresponding cocycle for M_* or M_{**}. However, we are not quite able to do this and we proceed in a slightly different way.

Formally, we show Proposition 5 by contradiction and induction using the fact that this proposition was already shown to be true for the connected components of $H(4)$. More concretely, assume that all translation surfaces M in some connected component $\mathcal{C}$ of some stratum have $(g-1)$ Hodge-orthogonal HT planes. It was shown by Kontsevich-Zorich (http://www.ams.org/mathscinet-getitem?mr=2000471) (in their classification of connected components of strata) that, by carefully *collapsing zeroes*, and *pinching off the handles* of M like in this picture here



(http://matheuscmss.files.wordpress.com/2013/06/mw-bdry.jpg)

and by deleting the torus component, we reach one of the connected components of $H(4)$.

Furthermore, by a close inspection of Kontsevich-Zorich procedure, we can see that, if the initial translation surface has $(g-1)$ Hodge-orthogonal HT planes and if the pinching procedure is really careful, then each time we pinch off a handle, we kill off at most one of the HT planes. In particular, among the $(g-1)$ Hodge-orthogonal HT planes of the initial translation surfaces, we know that at least two of them survive by the end of the procedure, and, hence, we obtain some translation surface in some of the connected components of $H(4)$ having two Hodge-orthogonal HT planes, a contradiction.

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