Ergodicity and Classification of Partially Hyperbolic Systems

Andy Hammerlindl

UNSW and USyd

June 10, 2013

Partial Hyperbolicity: $f : M \to M$

Tf-invariant splitting $TM = E^u \oplus E^c \oplus E^s$

E^u expanding, E^s contracting, E^c dominated.

In this talk, all diffeomorphisms considered are

Partial Hyperbolicity: $f : M \to M$

Tf-invariant splitting $TM = E^u \oplus E^c \oplus E^s$

E^u expanding, E^s contracting, E^c dominated.

In this talk, all diffeomorphisms considered are

C^2

Partial Hyperbolicity: $f : M \to M$

Tf-invariant splitting $TM = E^u \oplus E^c \oplus E^s$

E^u expanding, E^s contracting, E^c dominated.

In this talk, all diffeomorphisms considered are

- C^2
- partially hyperbolic, and

Partial Hyperbolicity: $f : M \to M$

Tf-invariant splitting $TM = E^u \oplus E^c \oplus E^s$

E^u expanding, E^s contracting, E^c dominated.

In this talk, all diffeomorphisms considered are

- C^2
- partially hyperbolic, and
- preserve a smooth measure.

Partial Hyperbolicity: $f : M \to M$

Tf-invariant splitting $TM = E^u \oplus E^c \oplus E^s$

E^u expanding, E^s contracting, E^c dominated.

In this talk, all diffeomorphisms considered are

- C^2
- partially hyperbolic, and
- preserve a smooth measure.

Pugh-Shub Conjecture.

Partial Hyperbolicity: $f : M \to M$

Tf-invariant splitting $TM = E^u \oplus E^c \oplus E^s$

E^u expanding, E^s contracting, E^c dominated.

In this talk, all diffeomorphisms considered are

- C^2
- partially hyperbolic, and
- preserve a smooth measure.

Pugh-Shub Conjecture.

Stable ergodicity is open and dense.

Pugh-Shub Conjecture 1

Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2

Pugh-Shub Conjecture 3

Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2 Accessibility is open-dense.

Pugh-Shub Conjecture 3

Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2 Accessibility is open-dense.

Pugh-Shub Conjecture 3 Accessibility implies ergodicity.

Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2 Accessibility is open-dense.

Pugh-Shub Conjecture 3 Accessibility implies ergodicity.

$TM = E^u \oplus E^c \oplus E^s$

Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2 Accessibility is open-dense.

Pugh-Shub Conjecture 3 Accessibility implies ergodicity.

$TM = E^u \oplus E^c \oplus E^s$ There are foliations tangent to E^u and E^s.

Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2 Accessibility is open-dense.

Pugh-Shub Conjecture 3 Accessibility implies ergodicity.

$TM = E^u \oplus E^c \oplus E^s$ There are foliations tangent to E^u and E^s.

Accessibility:

Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2 Accessibility is open-dense.

Pugh-Shub Conjecture 3 Accessibility implies ergodicity.

$TM = E^u \oplus E^c \oplus E^s$ There are foliations tangent to E^u and E^s.

Accessibility:

Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2 Accessibility is open-dense.

Pugh-Shub Conjecture 3 Accessibility implies ergodicity.

$TM = E^u \oplus E^c \oplus E^s$ There are foliations tangent to E^u and E^s.

Accessibility:

Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2 Accessibility is open-dense.

Pugh-Shub Conjecture 3 Accessibility implies ergodicity.

$TM = E^u \oplus E^c \oplus E^s$ There are foliations tangent to E^u and E^s.

Accessibility:

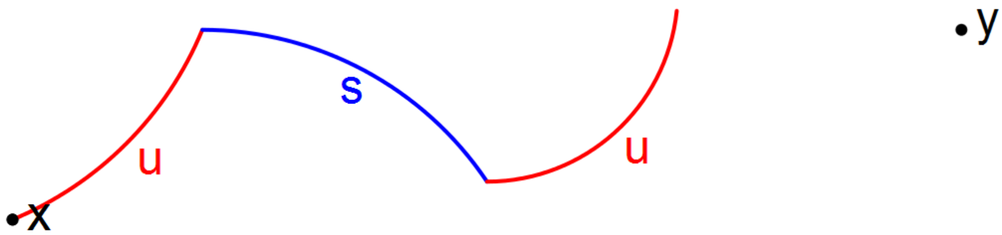
Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2 Accessibility is open-dense.

Pugh-Shub Conjecture 3 Accessibility implies ergodicity.

$TM = E^u \oplus E^c \oplus E^s$ There are foliations tangent to E^u and E^s.

Accessibility:



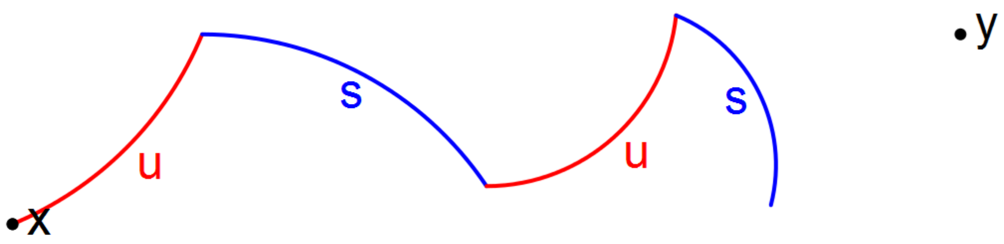
Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2 Accessibility is open-dense.

Pugh-Shub Conjecture 3 Accessibility implies ergodicity.

$TM = E^u \oplus E^c \oplus E^s$ There are foliations tangent to E^u and E^s.

Accessibility:



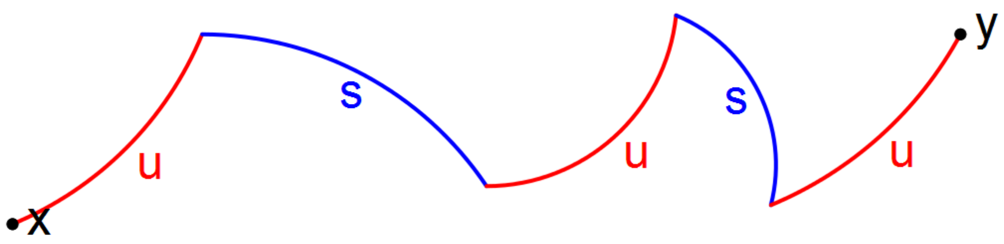
Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2 Accessibility is open-dense.

Pugh-Shub Conjecture 3 Accessibility implies ergodicity.

$TM = E^u \oplus E^c \oplus E^s$ There are foliations tangent to E^u and E^s.

Accessibility:



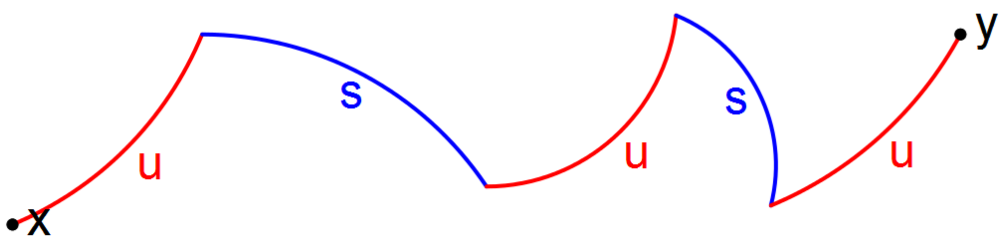
Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2 Accessibility is open-dense.

Pugh-Shub Conjecture 3 Accessibility implies ergodicity.

$TM = E^u \oplus E^c \oplus E^s$ There are foliations tangent to E^u and E^s.

Accessibility:



Conjectures are true when $\dim(E^c) = 1$

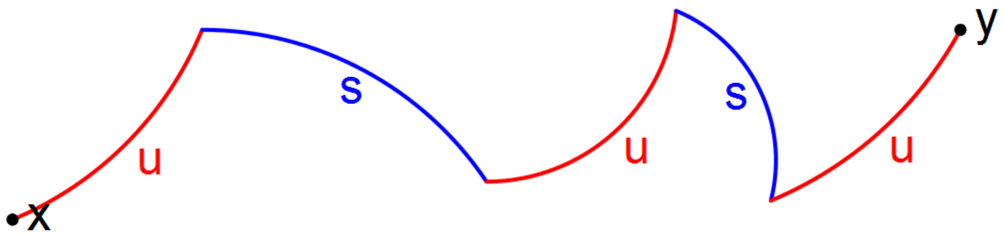
Pugh-Shub Conjecture 1 Stable ergodicity is open-dense.

Pugh-Shub Conjecture 2 Accessibility is open-dense.

Pugh-Shub Conjecture 3 Accessibility implies ergodicity.

$TM = E^u \oplus E^c \oplus E^s$ There are foliations tangent to E^u and E^s.

Accessibility:



Conjectures are true when $\dim(E^c) = 1$

(Rodriguez-Hertz, Rodriguez-Hertz, Ures).

From now on, assume $\dim(E^u) = \dim(E^c) = \dim(E^s) = 1$.

From now on, assume $\dim(E^u) = \dim(E^c) = \dim(E^s) = 1$.

•Accessibility is open-dense.

•Accessibility implies ergodicity.

From now on, assume $\dim(E^u) = \dim(E^c) = \dim(E^s) = 1$.

•Accessibility is open-dense.

•Accessibility implies ergodicity.

Conjecture (RHRHU). If f is **not** ergodic, then:

From now on, assume $\dim(E^u) = \dim(E^c) = \dim(E^s) = 1$.

•Accessibility is open-dense.

•Accessibility implies ergodicity.

Conjecture (RHRHU). If f is **not** ergodic, then:

(1) the manifold M is one of three specific types

From now on, assume $\dim(E^u) = \dim(E^c) = \dim(E^s) = 1$.

•Accessibility is open-dense.

•Accessibility implies ergodicity.

Conjecture (RHRHU). If f is **not** ergodic, then:

(1) the manifold M is one of three specific types

(all of which have solvable fundamental group).

From now on, assume $\dim(E^u) = \dim(E^c) = \dim(E^s) = 1$.

•Accessibility is open-dense.

•Accessibility implies ergodicity.

Conjecture (RHRHU). If f is **not** ergodic, then:

(1) the manifold M is one of three specific types

(all of which have solvable fundamental group).

(2) there is a periodic torus T tangent to $E^u \oplus E^s$.

From now on, assume $\dim(E^u) = \dim(E^c) = \dim(E^s) = 1$.

•Accessibility is open-dense.

•Accessibility implies ergodicity.

Conjecture (RHRHU). If f is **not** ergodic, then:

(1) the manifold M is one of three specific types
(all of which have solvable fundamental group).

(2) there is a periodic torus T tangent to $E^u \oplus E^s$.

They also showed that (2) implies (1).

From now on, assume $\dim(E^u) = \dim(E^c) = \dim(E^s) = 1$.

•Accessibility is open-dense.

•Accessibility implies ergodicity.

Conjecture (RHRHU). If f is **not** ergodic, then:

(1) the manifold M is one of three specific types

(all of which have solvable fundamental group).

(2) there is a periodic torus T tangent to $E^u \oplus E^s$.

They also showed that (2) implies (1).

Question. How does this relate to classification results?

Partially hyperbolic systems on 3-manifolds with solvable fundamental group.

Partially hyperbolic systems on 3-manifolds
with solvable fundamental group.

There are four types of systems (H,Potrie):

Partially hyperbolic systems on 3-manifolds
with solvable fundamental group.

There are four types of systems (H,Potrie):

- Derived-from-Anosov systems

Partially hyperbolic systems on 3-manifolds
with solvable fundamental group.

There are four types of systems (H,Potrie):

- Derived-from-Anosov systems
- skew-products with non-trivial bundles

Partially hyperbolic systems on 3-manifolds

with solvable fundamental group.

There are four types of systems (H,Potrie):

- Derived-from-Anosov systems
- skew-products with non-trivial bundles
- skew-products with trivial bundles

Partially hyperbolic systems on 3-manifolds
with solvable fundamental group.

There are four types of systems (H,Potrie):

- Derived-from-Anosov systems
- skew-products with non-trivial bundles
- skew-products with trivial bundles
- “suspension-like” systems

Partially hyperbolic systems on 3-manifolds

with solvable fundamental group.

There are four types of systems (H,Potrie):

- Derived-from-Anosov systems
- skew-products with non-trivial bundles
- skew-products with trivial bundles
- “suspension-like” systems
- non-dynamically coherent systems
  with attracting or repelling invariant tori

Partially hyperbolic systems on 3-manifolds

with solvable fundamental group.

There are four types of systems (H,Potrie):

- Derived-from-Anosov systems
- skew-products with non-trivial bundles
- skew-products with trivial bundles
- “suspension-like” systems
- non-dynamically coherent systems
  with attracting or repelling invariant tori

Such non-dynamically coherent examples exist [RHRHU],

Partially hyperbolic systems on 3-manifolds

with solvable fundamental group.

There are four types of systems (H,Potrie):

- Derived-from-Anosov systems
- skew-products with non-trivial bundles
- skew-products with trivial bundles
- “suspension-like” systems
- non-dynamically coherent systems
  with attracting or repelling invariant tori

Such non-dynamically coherent examples exist [RHRHU],
but not in the conservative case.

Partially hyperbolic systems on 3-manifolds

with solvable fundamental group.

There are four types of systems (H,Potrie):

- Derived-from-Anosov systems
- skew-products with non-trivial bundles
- skew-products with trivial bundles
- “suspension-like” systems
- non-dynamically coherent systems
  with attracting or repelling invariant tori

Such non-dynamically coherent examples exist [RHRHU],
but not in the conservative case.

Derived-from-Anosov systems

These are partially hyperbolic diffeomorphisms $f : \mathsf{T}^3 \to \mathsf{T}^3$

Derived-from-Anosov systems

These are partially hyperbolic diffeomorphisms $f : \mathsf{T}^3 \to \mathsf{T}^3$ where f is homotopic to a linear Anosov map $A : \mathsf{T}^3 \to \mathsf{T}^3$.

Derived-from-Anosov systems

These are partially hyperbolic diffeomorphisms $f : \mathsf{T}^3 \to \mathsf{T}^3$ where f is homotopic to a linear Anosov map $A : \mathsf{T}^3 \to \mathsf{T}^3$.

As the action f_* on $\;_1(\mathsf{T}^3) \cong \mathsf{Z}^3$ is hyperbolic there cannot be a periodic 2-torus tangent to $E^u \oplus E^s$.

Derived-from-Anosov systems

These are partially hyperbolic diffeomorphisms $f : \mathbb{T}^3 \to \mathbb{T}^3$ where f is homotopic to a linear Anosov map $A : \mathbb{T}^3 \to \mathbb{T}^3$.

As the action f_* on $\;_1(\mathbb{T}^3) \cong \mathbb{Z}^3$ is hyperbolic there cannot be a periodic 2-torus tangent to $E^u \oplus E^s$.

Conjecture. All such systems are ergodic.

Derived-from-Anosov systems

These are partially hyperbolic diffeomorphisms $f : \mathsf{T}^3 \to \mathsf{T}^3$

where f is homotopic to a linear Anosov map $A : \mathsf{T}^3 \to \mathsf{T}^3$.

As the action f_* on $\quad {}_1(\mathsf{T}^3) \cong \mathsf{Z}^3$ is hyperbolic

there cannot be a periodic 2-torus tangent to $E^u \oplus E^s$.

Conjecture. All such systems are ergodic.

Theorem (H,Ures). If f is homotopic to an Anosov map A

Derived-from-Anosov systems

These are partially hyperbolic diffeomorphisms $f : \mathsf{T}^3 \to \mathsf{T}^3$

where f is homotopic to a linear Anosov map $A : \mathsf{T}^3 \to \mathsf{T}^3$.

As the action f_* on $\;_1(\mathsf{T}^3) \cong \mathsf{Z}^3$ is hyperbolic

there cannot be a periodic 2-torus tangent to $E^u \oplus E^s$.

Conjecture. All such systems are ergodic.

Theorem (H,Ures). If f is homotopic to an Anosov map A

and f is **not** accessible,

Derived-from-Anosov systems

These are partially hyperbolic diffeomorphisms $f : \mathsf{T}^3 \to \mathsf{T}^3$

where f is homotopic to a linear Anosov map $A : \mathsf{T}^3 \to \mathsf{T}^3$.

As the action f_* on $\quad{}_1(\mathsf{T}^3) \cong \mathsf{Z}^3$ is hyperbolic

there cannot be a periodic 2-torus tangent to $E^u \oplus E^s$.

Conjecture. All such systems are ergodic.

Theorem (H,Ures). If f is homotopic to an Anosov map A

and f is **not** accessible,

then f is topologically conjugate to A.

Skew-products

M is a 3-manifold with a (continuous) bundle map $p : M \to \mathsf{T}^2$,

Skew-products

M is a 3-manifold with a (continuous) bundle map $p : M \to \mathsf{T}^2$,

the fibers of p are compact center leaves, and

7

Skew-products

M is a 3-manifold with a (continuous) bundle map $p : M \to \mathsf{T}^2$,

the fibers of p are compact center leaves, and

there is a Anosov map $A : \mathsf{T}^2 \to \mathsf{T}^2$ such that

$$p \circ f = A \circ p.$$

Skew-products

M is a 3-manifold with a (continuous) bundle map $p : M \to \mathsf{T}^2$,

the fibers of p are compact center leaves, and

there is a Anosov map $A : \mathsf{T}^2 \to \mathsf{T}^2$ such that

$$p \circ f = A \circ p.$$

Example: f on $\mathsf{T}^3 = \mathsf{T}^2 \times \mathsf{S}^1$, $\quad f(v,t) = (Av, t + \quad).$

Skew-products

M is a 3-manifold with a (continuous) bundle map $p : M \to \mathsf{T}^2$,

the fibers of p are compact center leaves, and

there is a Anosov map $A : \mathsf{T}^2 \to \mathsf{T}^2$ such that

$$p \circ f = A \circ p.$$

Example: f on $\mathsf{T}^3 = \mathsf{T}^2 \times \mathsf{S}^1$, $\quad f(v,t) = (Av, t + \quad)$.

Ergodic if and only if defines an irrational rotation.

Skew-products

M is a 3-manifold with a (continuous) bundle map $p : M \to \mathsf{T}^2$,

the fibers of p are compact center leaves, and

there is a Anosov map $A : \mathsf{T}^2 \to \mathsf{T}^2$ such that

$$p \circ f = A \circ p.$$

Example: f on $\mathsf{T}^3 = \mathsf{T}^2 \times \mathsf{S}^1$, $f(v, t) = (Av, t + \quad)$.

Ergodic if and only if defines an irrational rotation.

The bundle is trivial if and only if M is the 3-torus.

Skew-products

M is a 3-manifold with a (continuous) bundle map $p : M \to \mathsf{T}^2$,

the fibers of p are compact center leaves, and

there is a Anosov map $A : \mathsf{T}^2 \to \mathsf{T}^2$ such that

$$p \circ f = A \circ p.$$

Example: f on $\mathsf{T}^3 = \mathsf{T}^2 \times \mathsf{S}^1$, $f(v,t) = (Av, t + \quad)$.

Ergodic if and only if defines an irrational rotation.

The bundle is trivial if and only if M is the 3-torus.

Theorem (RHRHU). If f is defined on a non-trivial bundle,

Skew-products

M is a 3-manifold with a (continuous) bundle map $p : M \to \mathsf{T}^2$,

the fibers of p are compact center leaves, and

there is a Anosov map $A : \mathsf{T}^2 \to \mathsf{T}^2$ such that

$$p \circ f = A \circ p.$$

Example: f on $\mathsf{T}^3 = \mathsf{T}^2 \times \mathsf{S}^1$, $f(v,t) = (Av, t + \quad)$.

Ergodic if and only if defines an irrational rotation.

The bundle is trivial if and only if M is the 3-torus.

Theorem (RHRHU). If f is defined on a non-trivial bundle, then f is accessible and ergodic.

Partially hyperbolic systems on 3-manifolds
with solvable fundamental group.

There are four types of systems (H,Potrie):

- Derived-from-Anosov systems
- skew-products with non-trivial bundles
- skew-products with trivial bundles
- “suspension-like” systems

Manifolds defined by suspension

If $B : \mathsf{T}^2 \to \mathsf{T}^2$ is a diffeomorphism, it defines a manifold M_B by

$$\mathsf{R}^2 \times \mathsf{R} \Big/ (Bv, t) \sim (v, t+1),$$

Manifolds defined by suspension

If $B : \mathsf{T}^2 \to \mathsf{T}^2$ is a diffeomorphism, it defines a manifold M_B by

$$\mathsf{R}^2 \times \mathsf{R} \Big/ (Bv, t) \sim (v, t+1),$$

and a suspension flow: ${}_t(v, s) = (v, s+t)$.

Manifolds defined by suspension

If $B : \mathsf{T}^2 \to \mathsf{T}^2$ is a diffeomorphism, it defines a manifold M_B by

$$\mathsf{R}^2 \times \mathsf{R} \Big/ (B\,v, t) \sim (v, t+1),$$

and a suspension flow: $\quad {}_t(v, s) = (v, s+t).$

If B is an Anosov diffeomorphism, is an Anosov flow.

Manifolds defined by suspension

If $B : \mathsf{T}^2 \to \mathsf{T}^2$ is a diffeomorphism, it defines a manifold M_B by

$$\mathsf{R}^2 \times \mathsf{R} \Big/ (B\,v, t) \sim (v, t+1),$$

and a suspension flow: ${}_t(v, s) = (v, s+t)$.

If B is an Anosov diffeomorphism, is an Anosov flow.

Then the diffeo ${}_t$ is ergodic if and only if t is irrational.

Manifolds defined by suspension

If $B : \mathsf{T}^2 \to \mathsf{T}^2$ is a diffeomorphism, it defines a manifold M_B by

$$\mathsf{R}^2 \times \mathsf{R} \Big/ (Bv, t) \sim (v, t+1),$$

and a suspension flow: $\quad {}_t(v, s) = (v, s+t)$.

If B is an Anosov diffeomorphism, is an Anosov flow.

Then the diffeo ${}_t$ is ergodic if and only if t is irrational.

Theorem (H, Potrie)**.**

Manifolds defined by suspension

If $B : \mathsf{T}^2 \to \mathsf{T}^2$ is a diffeomorphism, it defines a manifold M_B by

$$\mathsf{R}^2 \times \mathsf{R} \Big/ (Bv, t) \sim (v, t+1),$$

and a suspension flow: $\quad {}_t(v, s) = (v, s+t)$.

If B is an Anosov diffeomorphism, is an Anosov flow.

Then the diffeo ${}_t$ is ergodic if and only if t is irrational.

Theorem (H, Potrie)**.**

If $f : M_B \to M_B$ is partially hyperbolic, and B is Anosov,

Manifolds defined by suspension

If $B : \mathsf{T}^2 \to \mathsf{T}^2$ is a diffeomorphism, it defines a manifold M_B by

$$\mathsf{R}^2 \times \mathsf{R} \big/ (B\,v, t) \sim (v, t+1),$$

and a suspension flow: $\quad {}_t(v, s) = (v, s+t)$.

If B is an Anosov diffeomorphism, is an Anosov flow.

Then the diffeo ${}_t$ is ergodic if and only if t is irrational.

Theorem (H, Potrie).

If $f : M_B \to M_B$ is partially hyperbolic, and B is Anosov,

the center foliation of f is equivalent to the orbits of

Manifolds defined by suspension

If $B : \mathsf{T}^2 \to \mathsf{T}^2$ is a diffeomorphism, it defines a manifold M_B by

$$\mathsf{R}^2 \times \mathsf{R} \Big/ (Bv, t) \sim (v, t+1),$$

and a suspension flow: $\quad {}_t(v, s) = (v, s+t)$.

If B is an Anosov diffeomorphism, is an Anosov flow.

Then the diffeo ${}_t$ is ergodic if and only if t is irrational.

Theorem (H, Potrie)**.**

If $f : M_B \to M_B$ is partially hyperbolic, and B is Anosov, the center foliation of f is equivalent to the orbits of and there is n such that f^n fixes every center leaf.

Theorem. Suppose $f: M \to M$ is C^2, meas. pres. and p.h. where $\dim(M) = 3$ and M has solvable fundamental group. Then at least one of the following holds:

Theorem. Suppose $f : M \to M$ is C^2, meas. pres. and p.h. where $\dim(M) = 3$ and M has solvable fundamental group. Then at least one of the following holds:

- f is accessible and stably ergodic.

Theorem. Suppose $f : M \to M$ is C^2, meas. pres. and p.h. where $\dim(M) = 3$ and M has solvable fundamental group. Then at least one of the following holds:

- f is accessible and stably ergodic.
- f is topologically conjugate to an affine map.

Theorem. Suppose $f : M \to M$ is C^2, meas. pres. and p.h. where $\dim(M) = 3$ and M has solvable fundamental group. Then at least one of the following holds:

- f is accessible and stably ergodic.
- f is topologically conjugate to an affine map.
- There is $n \geq 1$, a continuous map $p : M \to \mathsf{S}^1$, and a closed subset $K \subset \mathsf{S}^1$, such that

Theorem. Suppose $f : M \to M$ is C^2, meas. pres. and p.h. where $\dim(M) = 3$ and M has solvable fundamental group. Then at least one of the following holds:

- f is accessible and stably ergodic.
- f is topologically conjugate to an affine map.
- There is $n \geq 1$, a continuous map $p : M \to \mathsf{S}^1$, and a closed subset $K \subset \mathsf{S}^1$, such that
  - If $t \in K$, then
    $p^{-1}(t)$ is an f^n-invariant torus tangent to $E^u \oplus E^s$

Theorem. Suppose $f : M \to M$ is C^2, meas. pres. and p.h. where $\dim(M) = 3$ and M has solvable fundamental group. Then at least one of the following holds:

- f is accessible and stably ergodic.
- f is topologically conjugate to an affine map.
- There is $n \geq 1$, a continuous map $p : M \to \mathsf{S}^1$, and a closed subset $K \subset \mathsf{S}^1$, such that
  - If $t \in K$, then $p^{-1}(t)$ is an f^n-invariant torus tangent to $E^u \oplus E^s$
  - If U is a connected component of $\mathsf{S}^1 \setminus K$, then $p^{-1}(U)$ is an ergodic component of f^n and is homeomorphic to $\mathsf{T}^2 \times U$.