Livšic theorem for diffeomorphism cocycles

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Hyperbolic dynamics

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- M closed smooth manifold

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- $f\colon M \circlearrowleft C^r$-diffeomorphism

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- E^s and E^u are Df-invariant
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Anosov diffeomorphism

f is **Anosov** if M is hyperbolic set

Cocycles and Coboundaries

G-cocycles

Let G be a topological group. A G-**cocycle** is a map $\Phi\colon M \to G$. We write

$$\Phi^{(n)}(p) := \Phi(f^{n-1}(p))\Phi(f^{n-2}(p)) \cdots \Phi(p)$$

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Periodic Orbit Obstructions (POO)

Necessary condition to be a cocycle:

$$\Phi^{(n)}(p) = u(f^n(p))u(p)^{-1} = id_G, \quad \forall p \in \mathrm{Fix}(f^n)$$

Livšic Theorem ($G = \mathbb{R}$)

Theorem [Livšic, 1971]

1. $f\colon M \circlearrowleft C^1$ transitive Anosov diffeomorphism
2. $G = \mathbb{R}$ and $\Phi\colon M \to \mathbb{R}$ Hölder continuous.

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4. de la Llave, Marco, Moriyon [1986]: f is C^∞, $\Phi \in C^\infty \implies u \in C^\infty$

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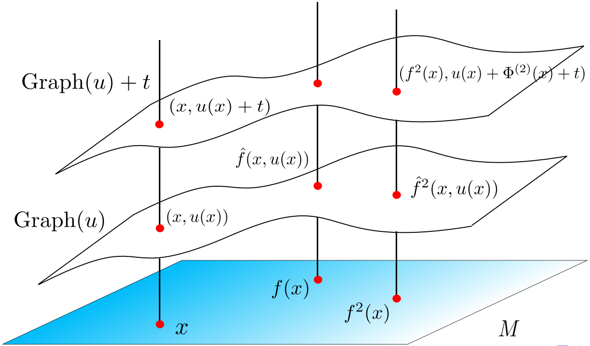

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3. **Question:** Can the function $u\colon \mathcal{O}_f(x) \to \mathbb{R}$ be continuously extended to M?
4. **Answer:** Yes, because of **Anosov Closing Lemma:** $\exists c, \delta_0 > 0$ s.t. for any $x \in M$ and $k \in \mathbb{N}$ s.t. $d(x, f^k(x)) < \delta_0$, $\exists^1 p \in \mathrm{Fix}(f^k)$ and $y \in M$ such that

$$d(f^i(x), f^i(p)) \leqslant c d(x, f^k(x)) \lambda^{\min(i,k-i)},$$

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$\forall i \in \{0, 1, \ldots, k\}$.

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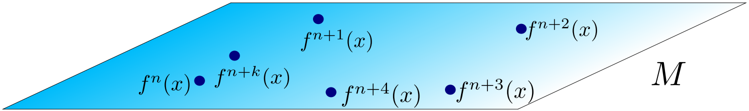

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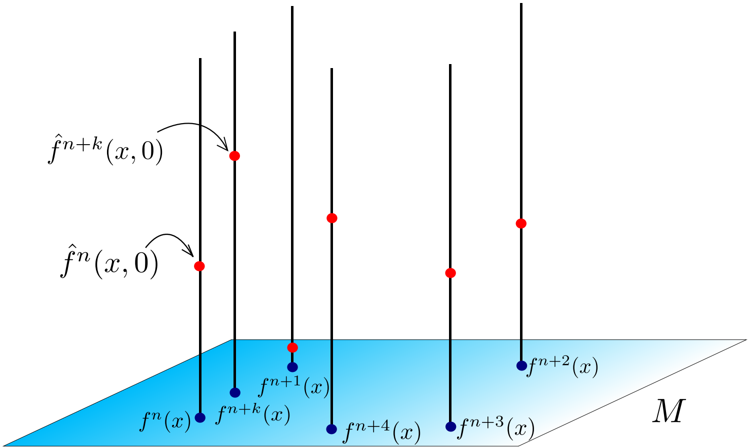

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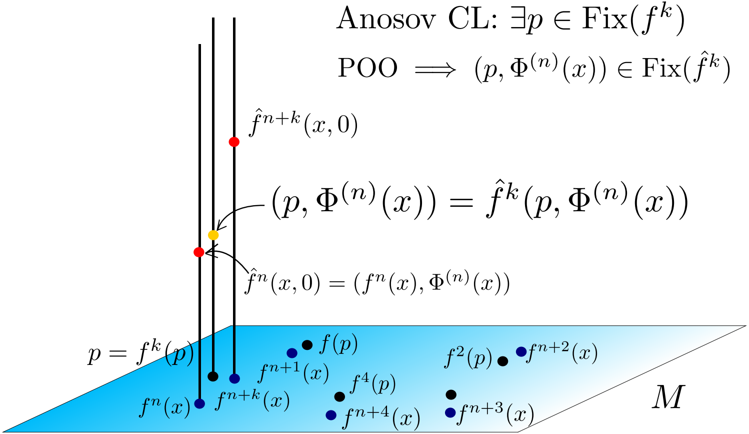

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Theorem [Livšic, 1972] (Perturbative result)

If G is (finite-dimensional) Lie group, $\Phi\colon M \to G$ is Hölder **close enough to constant** id_G (LOCALIZATION) and satisfies POO $\implies$ Φ Hölder coboundary.

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Theorem [Kalinin, 2011]

If $G = \mathrm{GL}(d, \mathbb{R})$, the GLOBAL Livšic thm holds.

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Theorem [Nițică-Török, 1995]

1. $f\colon M \circlearrowleft$ C^1 transitive Anosov diffeo
2. X *nicely embed ed* in $\mathbb{R}^N$
3. $\Phi\colon M \to \mathrm{Diff}^r(X)$ Hölder, with $r \geqslant 4$ s.t.
    1. POO
    2. Localization: $d_{C^r}(\Phi_p^{\pm 1}, id_X)$ small $\forall p \in M$

Then, there exists $u\colon M \to \mathrm{Diff}^{r-2}(X)$ Hölder s.t.

$$\Phi_p = u(f(p) \circ u(p)^{-1}, \quad \forall p \in M.$$

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Theorem [de la Llave-Windsor, 2008, 2010]

1. If $\Phi \in C^{k+\alpha}(M, \mathrm{Diff}^r(X))$ and $u \in C^{k+\alpha}(M, \mathrm{Diff}^1(X))$, then $u \in C^{k+\alpha}(M, \mathrm{Diff}^r(X))$.

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2. Similar to [NT95], $\forall$ closed manifold X and localization with d_{C^1}

Main result

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Theorem [K.-Potrie]

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1. f transitive C^r Anosov diffeomorphism
2. $\Phi\colon M \to G$ a C^r-cocycle satisfying POO

Then $\exists\ u\colon M \to G$ C^r such that

$$\Phi_x = u(f(x)) \circ u(x)^{-1}, \quad \forall x \in M$$

Idea of the proof ($G = \mathrm{Diff}^r(\mathbb{T})$)

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Proposition 1

$\hat{f}$ is absolutely **partially hyperbolic**, with

$$\widehat{\mathcal{W}}^c(x,t) = \{x\} \times \mathbb{T},$$

and every **central Lyapunov exponent** $\lambda^c = 0$

Proof of Proposition 1

1. Suppose $\exists n_k \uparrow \infty$ and $(x_k, t_k) \in M \times \mathbb{T}$ s.t.

$$\left|\partial_t \hat{f}^{n_k}(x_k, t_k)\right| < \lambda^{n_k}$$

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2. Let $\mu \in \mathfrak{M}(\hat{f})$ be an accumulation point of the sequence $(\mu_k)_k$

$$\mu_k := \frac{1}{n_k} \sum_{j=0}^{n_k - 1} \hat{f}_\star^j \delta_{(x_k, t_k)}$$

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6. By classical *cone field arguments*, $\hat{f}$ is absolutely partially hyperbolic

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The closure of a every $\hat{f}$-orbit a is graph over M, i.e. $\forall (x,t) \in M \times \mathbb{T}$, $\exists\, V_{x,t} \colon \overline{\mathcal{O}_f(x)} \to \mathbb{T}$ continuous s.t.

$$\overline{\mathcal{O}_{\hat{f}}(x,t)} = \mathrm{Graph}(V_{x,t})$$

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It suffices to show $\hat{f}^k(x,t)$ is close to (x,t), whenever $f^k(x)$ is close to x

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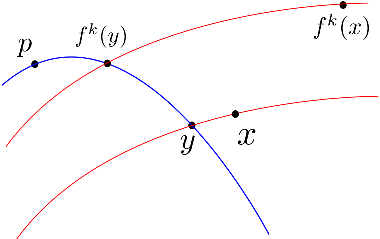

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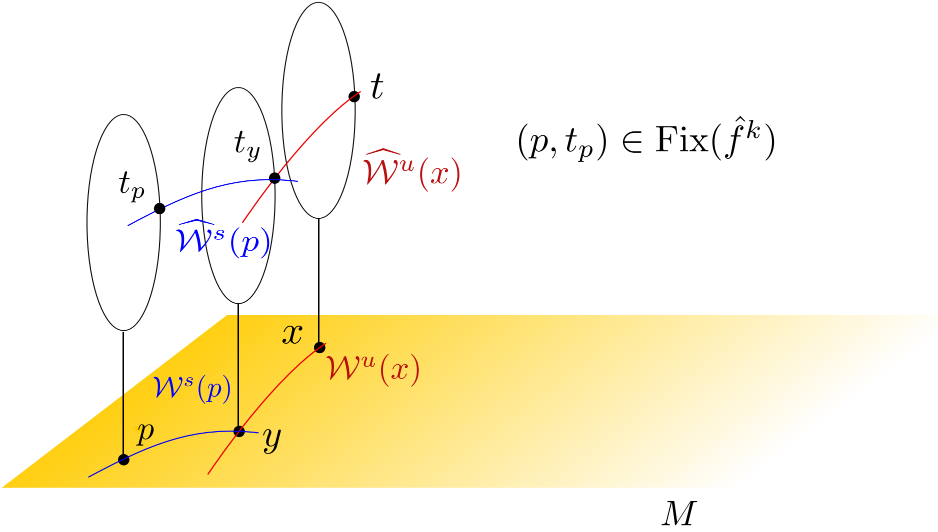

$$(p, t_p) \in \mathrm{Fix}(\hat{f}^k)$$

$\widehat{\mathcal{W}}^{s,u}$-saturation of $\mathrm{Graph}(V_{x,t})$

We fix $x_0 \in M$ with $\overline{\mathcal{O}_f(x_0)} = M$ and define

$$\mathcal{V}_t := \mathrm{Graph}(V_{x_0,t}) \subset M \times \mathbb{T}, \quad \forall t \in \mathbb{T}$$

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Proposition 3

$\forall t_0 \in \mathbb{T}$ and $\forall (x,t) \in \mathcal{V}_{t_0}$

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Corollary of Prop 3

1. $\mathcal{V} = (\mathcal{V}_t)_{t\in\mathbb{T}}$ is a foliation
2. The leaves of $\mathcal{V}$ are C^r

Proof of Proposition 3

It is a straightforward consequence of

Theorem (Topological Invariant Principle) [Avila-Viana,2010]

Let $\hat{f}: M \times X \circlearrowleft$ be a skew-product over $f: M \circlearrowleft$ and $\hat{\mu} \in \mathfrak{M}(\hat{f})$ s.t.:

1. $\hat{f}$ admits s, u-holonomies
2. every fibered Lyapunov exponent of $\hat{\mu}$ vanishes
3. $\hat{\mu}$ project over an f-invariant measure with l.p.s.

Then, the disintegration $(\hat{\mu}_x)_{x \in M}$ of $\hat{\mu}$ along the fibers varies continuously with x and it's **invariant along** s, u**-holonomies**.

Proof of Corollary

It's a straight forward consequence of

Theorem [Journé, 1988]

Let $\Psi\colon M \to \mathbb{R}$ be a continuous function such that it is C^r along $\mathcal{W}^s$ and $\mathcal{W}^u$-leaves. Then, Ψ is C^r.

Defining u

Now we can define the "transfer" function $u\colon M \to \mathrm{Diff}^r(\mathbb{T})$ satisfying

$$\Phi_x = u_{f(x)} \circ u_x^{-1}, \quad \forall x \in M$$

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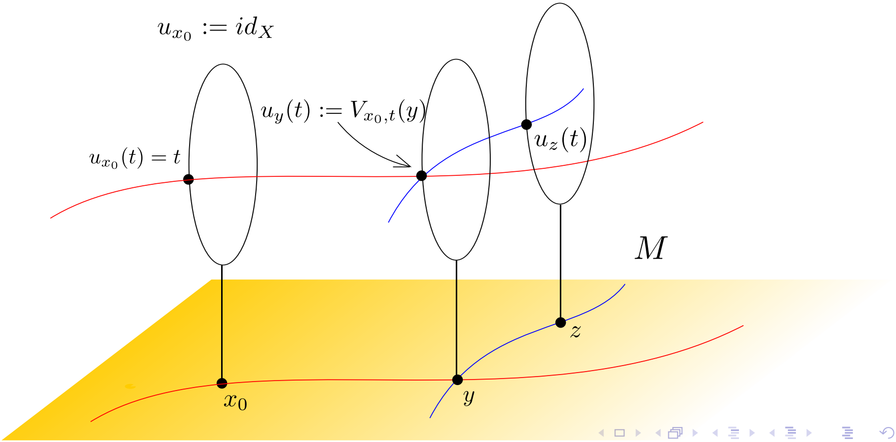

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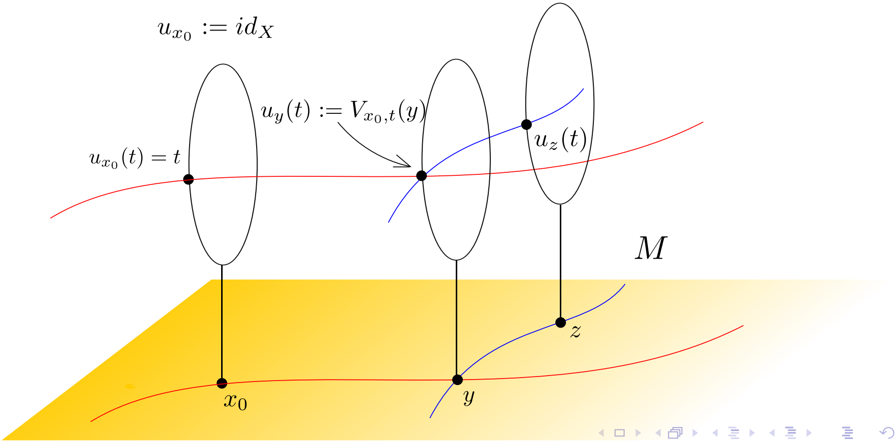

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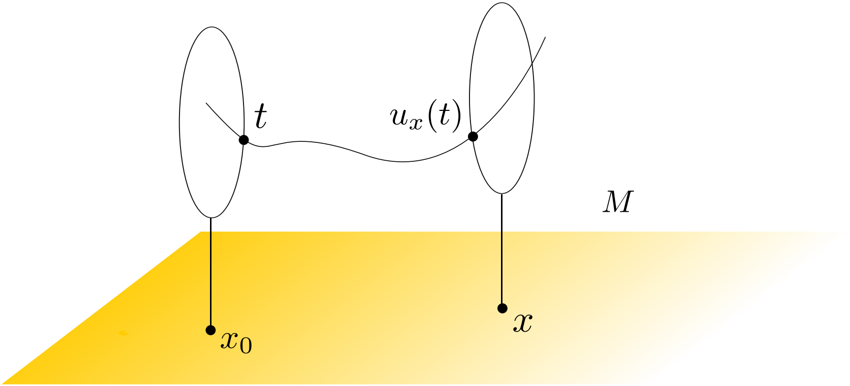

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- Everything but vanishing fibered Lyapunov exponent works in higher dimensions
- For the smoothness of $\mathcal{V}$-holonomies we need Kalinin thm instead of classical Livšic one in higher dim

Merci beaucoup!