Nowhere differentiable functions arising in Dynamical Systems

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1. Introduction

In the early nineteenth century, many mathematicians believed that a continuous function is differentiable at most of its domain. In 1872, Karl Weiertrass presented a function which was everywhere continuous but nowhere differentiable:

$$W(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x),$$

where a is a real number with $0 < a < 1$, b is an odd integer and $ab > 1 + 3\pi/2$.

In 1916, Hardy [4] proved that the function W defined above is continuous and nowhere differentiable if $0 < a < 1$, $ab \geq 1$. The constant b does not need to be an integer.

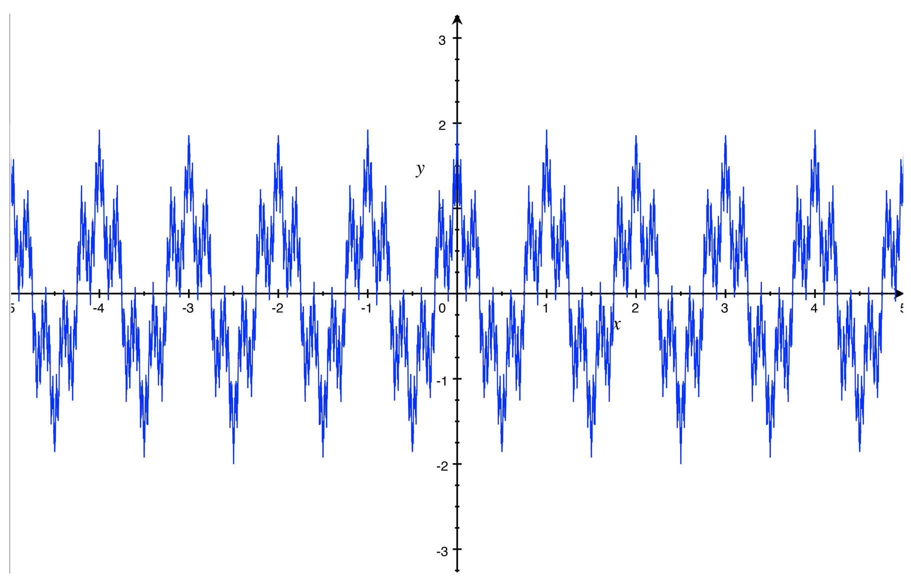

Figure 1: Graph of W(x) with $a = 0.5$ and $b = 3$.

In 1903, Takagi presented a simpler example of a continuous nowhere differentiable function:

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \inf_{m \in \mathbb{Z}} |2^k x - m|.$$

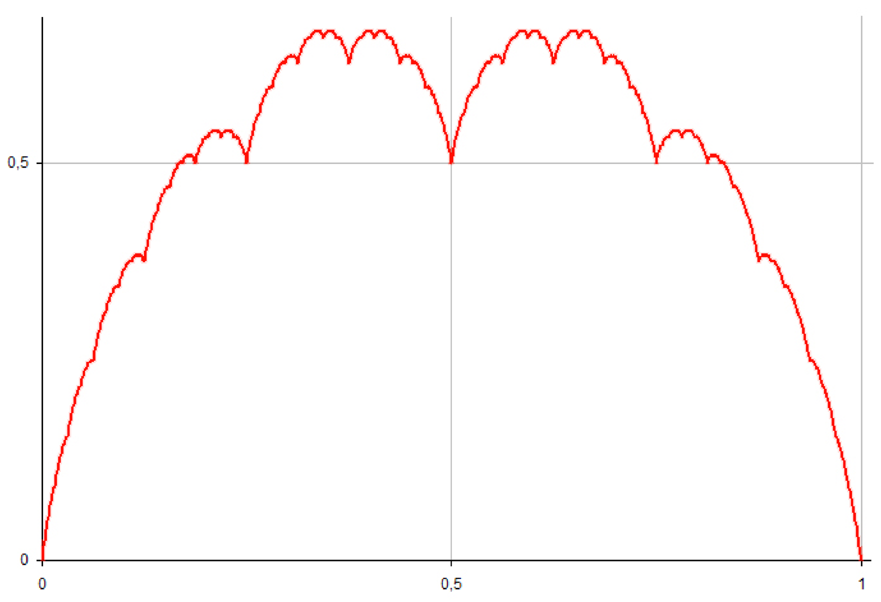

Figure 2: Graph of Takagi function
Available at <http://en.wikipedia.org/wiki/Blancmange_curve>

Gamkrelidze ([2], and [3]), proved a Central Limit Theorem-type result for the modulus of continuity of the Weierstrass and Takagi functions:

$$\lim_{h \to 0} \mu \left\{ x : \frac{W(x+h) - W(x)}{\pi h \sqrt{\frac{1}{2} \log \frac{1}{|h|}}} \leq y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} dx,$$

and

$$\lim_{h \to 0} \mu \left\{ x : \frac{T(x+h) - T(x)}{h \sqrt{\frac{1}{2} \log_2 \frac{1}{|h|}}} \leq y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} dx,$$

where μ denotes Lebesgue measure.

In another work, Heurteaux [5] considered a generalization of the Weierstrass function:

$$F(x) = \sum_{n=0}^{\infty} b^{-n} g(b^n x),$$

where $1 < b < \infty$, g is an almost periodic $C^{1+\varepsilon}$ function. Such function is called a Weiertrass-type function. He proved that ther are only two mutually exclusive cases: Either F is of class $C^{1+\varepsilon}$ or F is nowhere differentiable.

Now we are going to consider a family of maps $t \in (-\delta, \delta) \mapsto f_t \in C^1(S^1)$. If f_0 is an expanding map, then there is δ_0 such that for all $t \in (-\delta_0, \delta_0)$, f_t is also an expanding map and there is a homeomorphism h_t such that $h_t \circ f_0(x) = f_t \circ h_t(x)$. Differentiating this equation with respect to t, we obtain

$$v_t(y) = \alpha_t(f_t(y)) - \partial_x f_t(y) \alpha_t(y),$$

where $\alpha_t(y) := (\partial_t h_t) \circ h_t^{-1}(y_t)$ and $v_t(x) := \partial_t f_t(x)$. Fixing t, we have the *twisted cohomological equation*

$$v(y) = \alpha(f(y)) - Df(y)\alpha(y). \qquad (1)$$

There exists a unique bounded function satis fying 1 and this function is given by

$$\alpha(x) = -\sum_{n=1}^{\infty} \frac{v(f^{n-1}(x))}{Df^n(x)}. \qquad (2)$$

Let $f \in C^{2+\varepsilon}(S^1)$ be an expanding map and $v : S^1 \to \mathbb{R}$ a periodic function of class $C^{1+\varepsilon}$.

2. Some Properties of α

Our goal is to study smoothness properties the function α defined by (2).

We say that a function g is in the Zygmund class if there is $C > 0$ such that for all $x \in \mathbb{R}$:

$$|g(x+h) + g(x-h) - 2g(x)| \leq C|h|.$$

And we can prove the following proposition:

Proposition 1: α is in Zygmund class.

Another result that we can prove is about the differentiability class of α.

Theorem 1: One of the following statements holds:

(i) α is of class $C^{1+\varepsilon}$

(ii) α is nowhere differentiable.

Similar to the Central Limit Theorem-type proved by Gamkrelidze, we prove the following theorem:

Theorem 2: *Supose that α is nowhere differentiable. Then there exists $\sigma > 0$ such that*

$$\lim_{h \to 0} \mu \left\{ x : \frac{\alpha(x+h) - \alpha(x)}{h \sqrt{\log \frac{1}{|h|}}} \leq y \right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2\sigma^2}} dt,$$

where μ is the absolutely continuous invariant measure with respect to the Lebesgue measure.

References

1. Baladi, V. and Smania, D., *Smooth deformations of piecewise expanding unimodal maps*, Discrete and Continuous Dynamical Systems (DCDS-A), 23 (2009), 685 - 703.
2. Gamkrelidze, N. G., *Modulus of continuity of the Weiertrass function*, Mat. Zametki, 36:1 (1984), 35–38.
3. Gamkrelidze, N. G., *On a probabilistic properties of Takagi's function*, J. Math. Kyoto Univ. 30 (1990), no. 2, 227–229.
4. Hardy, G. H., *Weierstrass's non-differentiable function*, Trans. Amer. Math. Soc. 17 (1916), 301–325.
5. Heurteaux, Y., *Weierstrass functions in Zygmund's class*, Trans. Amer. Math. Soc., 355 : 3065-3077, 2003.

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