Killed Brownian Motion with a Prescribed Lifetime Distribution and Models of Default

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Motivation

- Counterparty risk has to be taken into account when pricing a transaction or portfolio, and it is necessary to model the occurrence of default jointly with the behavior of asset values.
- The default time is sometimes modeled as the first passage time of a credit index process below a barrier.
- Hull and White model the default time as the first time a Brownian motion hits a given time-dependent barrier.

Model Description

We adopt a perspective similar to that of Hull and White. Namely, we model the default time as

$$\tau := \inf\left\{t > 0 : \lambda \int_0^t \psi(Y_s - b(s))\, ds > U\right\} \tag{1}$$

where the diffusion Y is some credit index process, U is an independent mean one exponentially distributed random variable, $0 \leq \psi \leq 1$ is a suitably smooth, non-increasing function with $\lim_{x\to-\infty} \psi(x) = 1$ and $\lim_{x\to+\infty} \psi(x) = 0$, and $\lambda > 0$ is a rate parameter, so that

$$\mathbb{P}\{\tau > t\} = \mathbb{E}\left[\exp\left(-\lambda \int_0^t \psi(Y_s - b(s))\, ds\right)\right]. \tag{2}$$

Global Existence and Uniqueness

Theorem *Suppose the following.*

- *The survival function G is twice continuously differentiable with first and second derivatives g and g' and $0 < -g(t) < \lambda G(t)$ for all $t \geq 0$ for some constant $\lambda > 0$.*

- *The initial density f satisfies $\int_{\mathbb{R}} f(x)\,dx = 1$, $f(x) > 0$ for all $x \in \mathbb{R}$, $f \in C^2(\mathbb{R})$, and the functions f, f', f'' are bounded.*

- *The function ψ is non-increasing and belongs to $C^3(\mathbb{R})$, and for some $h > 0$, $\psi(x) = 1$ for $x \leq -h$ and $\psi(x) = 0$ for $x \geq h$.*

Then, there exists a unique continuously differentiable function b such that the following holds

$$G(t) = \mathbb{P}\{\tau > t\} = \mathbb{E}\left[\exp\left(-\lambda \int_0^t \psi(Y_s - b(s))\,ds\right)\right],\ t \geq 0.$$