LECTURE NOTES

FROM RANDOM WALK TRAJECTORIES TO RANDOM INTERLACEMENTS

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Contents

Chapter 1

Introduction

In these notes, we intend to explore some of the recent advances in the study of random walk trajectories. This subject has received a lot of attention in the last decades due to its innumerous applications and theoretical importance. Motivated by a question of H.J. Hilhorst relating random walks and corrosion, A.S-Sznitman introduced in [14] the model of random interlacements. This process describes the asymptotic picture left by a random walk on a finite graph. Besides the importance of random interlacements in answering the original questions posed by H.J. Hilhorst, this subject is interesting on its own, due to its close relations with potential theory, percolation and statistical physics.

The main objective of these notes is to introduce the above topics in a self contained fashion. The basic background on random walk trajectories and percolation will be presented in exercises, but several details of the theory will be worked in full detail to give the reader familiarity with the subject. Random interlacements is currently a very active area of research and some of the techniques discussed here are useful in a broad range of other problems in statistical physics.

We now give a brief overview of the contents of these notes.

1.1 Random walks

The goal of these notes is to introduce the model of random interlacements, explaining how it naturally appears in the study of random walk trajectories and later develop some of its main properties.

Let us first define what is a simple random walk on a graph. For this, fix a graph $G = (V, \mathcal{E})$ with vertices V and edges set $\mathcal{E}$. We are going to consider the random movement of a particle on G prescribed as follows. Let $x \in V$ be a starting vertex, meaning that at time zero our particle is found at x (we denote this fact by $X_0 = x$). In the subsequent time $t = 1$, the particle will choose a random vertex, uniformly among all neighbors of x to jump to, this new position is denoted by X_1. We now continue this procedure inductively, obtaining a random sequence $X_0, X_1, \dots$ that we call a random walk on G starting at x.

This seemingly simple definition has been source of intense research and important applications, such as modeling: the motion of a particles in a gas, variation in stock prices, population dynamics, Internet surfing and even neuron synapses. For each application, one may be interested in considering different graphs, such as a d-dimensional lattice or a network of neurons or websites. Moreover, each application may motivate a different question concerning the random walk behavior, such as: Where is the random walker expected to be at time t? How is the typical shape of the random walk trajectory? How much it typically takes for the walk to visit every site of G?

These and other questions have been intensively studied, providing us with interesting techniques that work in several classes of graphs. Currently there is a great deal of studding material on this subject, see for instance [8], [18], [10] and [13]. Nevertheless, there are still several interesting questions and vast areas of research which are still to be further explored. In these notes, we intend to give a very brief introduction to one of these areas, related to applications of random walks in corrosion of materials. We hope this will be a good opportunity to provide an introduction to random walks, as well as to some other related areas of probability theory.

The original motivation for the problems we discuss in these notes comes from a question posed by M.J. Hilhorst. Consider a d dimensional discrete torus $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$ which will be regarded as a piece of crystalline solid. This set can be made into a graph by adding edges between two points at Euclidean distance one from each other. Fix any given vertex $x \in \mathbb{T}_N^d$ and start a simple random walk $X_0, X_1, \dots$ from x. Imagine now that this random

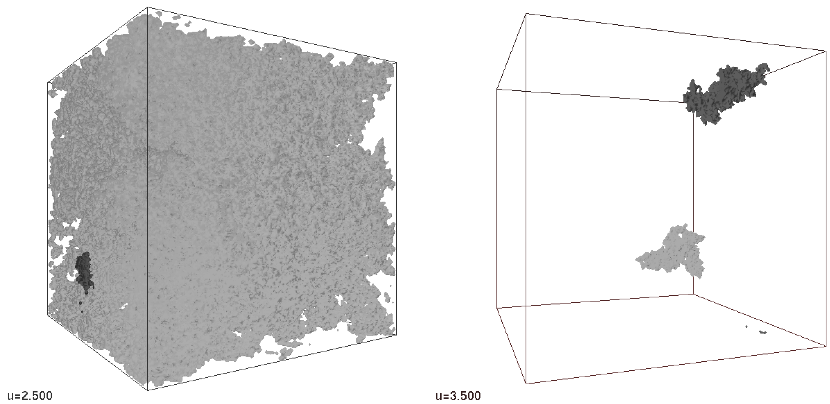

Figure 1.1: A computer simulation of the largest component (light gray) and second largest component (dark gray) of the vacant set left by a random walk on $(\mathbb{Z}/N\mathbb{Z})^3$ after $[uN^3]$ steps, for $N = 200$. The picture on the left-hand side corresponds to $u = 2.5$, the right-hand side to $u = 3.5$.

walk represents a corrosive particle wandering erratically in this crystal, while it marks all visited vertices as 'corroded'. As time runs, we expect the random walk to have deteriorated the crystal so much that only small connected fragments should be left. To be more precise, let us define the vacant set left by the random walk in the torus up to time t

$$\mathcal{V}_N^t = \mathbb{T}_N^d \setminus \{X_0, X_1, \dots, X_t\}. \tag{1.1}$$

Which is nothing more than the set of vertices not visited up to this time.

We will be mainly interested in understanding how the corrosion affects the connectivity properties of $\mathcal{V}_N^t$. More specifically, we will be interested in the size of the largest component $\mathcal{C}_N^t$ of the vacant set. Intuitively, one expects that for short times t, the cluster $\mathcal{C}_N^t$ should be a very distinct and large connected component of $\mathcal{V}_N^t$, while all the other components should be small. On the other hand, for larger times, $\mathcal{C}_N^t$ is expected to be just one of the various small fragments left in $\mathcal{V}_N^t$, see Figure 1.1. If this intuition is correct, one would like to be able to establish the existence of these two distinct phases, as well as to understand how the transition between them occurs. In these notes we will explain a little further these questions, emphasizing the theory of random interlacements that has evolved from it.

In Chapter 2 we are going to restrict our analysis to the case $d \geq 3$, which differs considerably from the cases $d = 1, 2$. In this context, we will define what we call the 'local-picture' left by the random walk on $\mathbb{T}_N^d$. Suppose that N is large and that we are only interested in what happens in a small box $A \subset \mathbb{T}_N^d$. It is clear that as t grows, the random walk will visit A several times, leaving a 'texture' of visited and unvisited sites inside this box.

What we will do in Chapter 2 is to split the random walk trajectory into what we call 'excursions' which correspond to the successive visits to A. Using some classical results from random walk theory, we will establish two key facts about these excursions:

- the successive excursions to A are roughly independent from each other,
- the first visited point in A by each excursion has a limiting distribution (as N grows), which we call this the normalized equilibrium distribution on A.

Starting from these two properties of the random walk excursions, we can define a measure on $\{0, 1\}^A$, which is the candidate for the asymptotic distribution of $\mathbf{1}\{\mathcal{V}_N^t \cap A\}$ (for growing N and $t = t(N)$). This limiting measure is what we called the local picture.

Of course one can map the local picture process (in the box A) to some isomorphic copy A of A in $\mathbb{Z}^d$. This seemingly trivial step reveals an important property of the local picture measure, namely, the compatibility. Let us informally describe what we mean with that. Suppose that we had chosen two boxes $A \subset A'$ in $\mathbb{T}_N^d$ and obtained the local picture for both at the same time (by letting N grow). Then, their corresponding local pictures in $\mathsf{A} \subset \mathsf{A}' \subset \mathbb{Z}^d$ would be consistent, in the sense that the restriction of the local picture in A' to A would have the same law as the local picture in A. This compatibility allows us to extend this distribution to a process in the whole lattice $\mathbb{Z}^d$, which we call random interlacements.

1.2 Random interlacements

As we have informally described, random interlacements will represent the infinite analog of the local picture, defined to study the trace left by a random walk on the tours. The description given in the previous section (derived from the compatibility of the local pictures) is abstract and therefore not very convenient. In Chapter 3, we are going to give a more constructive definition of random interlacements, that provides a way to perform calculations and prove some of its properties.

In short, the construction of random interlacements is governed by a Poisson point process of random walk trajectories. Intuitively speaking, the trajectory appearing in this Poisson soup correspond to excursions of the random walk in the torus. In Theorem 3.1 we prove the existence of a measure ν on the space of doubly-infinite random walk trajectories on $\mathbb{Z}^d$ modulo time-shift, see (3.10). The above mentioned Poisson point process will have intensity measure $u\nu$, where u is a positive real number, used to control the amount of the trajectories entering the picture. As we increase u, more and more trajectories appear in this random soup (in a similar way as more excursions appear as increase t for the random walk on the torus).

After having defined the random interlacements measure, we will obtain some of its main properties. For instance, we compare the law of random interlacements in $\{0, 1\}^{\mathbb{Z}^d}$ with the law obtained by independently assigning 0's and 1's to each vertex of $\mathbb{Z}^d$, the so-called Bernoulli site percolation. This comparison helps determining which of the techniques that have already been developed for Bernoulli percolation have chance to work in the random interlacements setting. As some of the techniques for the independent case may not be directly applicable for random interlacements, we will need to adapt or develop new techniques that are robust enough to deal with its dependence. The development of new techniques are a reason on its own to study random interlacements, besides the relation it has with the local picture left by a random walk on the torus. Nevertheless, the recent developments in the random interlacements have indeed been useful to better understand the original questions concerning $\mathcal{V}_N^t$ and $\mathcal{C}_N^t$, see [17].

1.3 Organization of these notes

We would like to precise the scope and structure of these notes. We do not want to present a comprehensive reference of what is currently known about random interlacements. Instead, we intend to favor a more motivated and self-contained exposition, with more detailed proofs of basic facts that should give the reader familiarity with the tools needed to work on this subject. The results presented here are not the most precise currently available, instead they were chosen in a way to balance between simplicity and relevance. Some of the details and requisites of the lectures are going to be left as exercises, presented in the end of these notes. Only in Chapter 5 we intend to give a more informal overview of another interesting direction of research related to random interlacements.

These notes are organized as follows. In Chapter 2 we give an overview of the basic properties of random walks on the torus, obtaining in the end the description of the so-called local picture that we mentioned above. Chapter 3 is separated in two different sections, the first being devoted to the construction of random interlacements and the second establishing some of the main properties of this process. In Chapter 4, we prove a result related to the existence of a phase transition for random interlacements on high dimensions. The main purpose of Chapter 4 is to illustrate the use of a very important technique in various problems in probability theory, namely multi-scale renormalization. Finally, in Chapter 5, we study the trace left by a random walk on a random regular graph, mentioning some relations of this with random interlacements on regular trees.

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Chapter 2

Random walk on the torus

In this chapter we discuss some properties of random walk on a discrete torus. The results obtained below will motivate the definition of the so-called local picture, which is the main ingredient in the construction of random interlacements in Chapter 3.

2.1 Notation

We consider, for $N \geq 1$ the discrete torus $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$. This can be regarded as a graph, with an edge connecting two vertices if and only if their Euclidean distance is one.

As mentioned in the introduction, we will be interested in the random walk on $\mathbb{T}_N^d$ and for this, let us denote by π the uniform distribution on $\mathbb{T}_N^d$. Denote by P the law of a simple random walk starting with distribution π and write $(X_n)_{n\geq 0}$ for the canonical coordinate maps of the walk. For technical reasons that will be explained later, we actually consider the so called lazy random walk which with probability one half stays put and otherwise jumps to a uniformly chosen neighbor. The law of a random walk starting at a specified point $x \in \mathbb{T}_N^d$ is denoted by P_x. We note that the index N has been omitted from the notations π, P, P_x and X_n. This will be done in other situations throughout the text hopping that the context will clarify the omission.

We observe that the uniform measure π is reversible for the random walk X_n, i.e. the probability of jumping from x to y is symmetric with respect to x and y.

For $k \geq 0$, we introduce the canonical shift operator θ_k in the space of trajectories, which is characterized by $X_n \circ \theta_k = X_{n+k}$ for every $n \geq 1$. Analogously, we can define θ_T, where T is a random time.

In the study of a simple random walk on a finite graph, it is useful to consider its adjacency matrix $C(x, y)$ (where x and y are vertices of $\mathbb{T}_N^d$) given by

$$C(x,y) = \begin{cases} 1/2 & \text{if } x = y, \\ 1/4d, & \text{if } x \text{ and } y \text{ are neighbors in } \mathbb{T}_N^d \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

It is not difficult to prove (see Exercise 5.12) that

$$\begin{gathered} C(\cdot,\cdot) \text{ has only positive eigenvalues } 1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N^d} > 0 \\ \text{and that the so called spectral gap } \Lambda_N = \lambda_1 - \lambda_2 \geq cN^{-2}. \end{gathered} \tag{2.2}$$

Moreover, a simple calculation leads to

$$\sup_{x,y \in \mathbb{T}_N^d} |P_x[X_n = y] - \pi(y)| \leq e^{-\Lambda_N n}, \text{ for all } n \geq 0, \tag{2.3}$$

see Exercise 5.13.

We define the regeneration time r_N associated to the simple random walk on $\mathbb{T}_N^d$ by $r_N = \lambda_N^{-1} \log^2 N$. To justify the name regeneration time, let us observe by (2.2) and (2.3) that

$$\sup_{x \in \mathbb{T}_N^d} \| P_x[X_{r_N} = \cdot] - \pi(\cdot) \|_{\mathrm{TV}} \leq e^{-\log^2 N}, \tag{2.4}$$

which decays fast to zero as N tends to infinity. This means that after time r_N the distribution of the random walk position is very close to uniform.

Let us also define the simple (lazy) random walk on the infinite lattice $\mathbb{Z}^d$ where edges again connect points within Euclidean distance one. The law of this random walk starting at some point $x \in \mathbb{Z}^d$ is denoted by $P_x^{\mathbb{Z}^d}$ and if no confusion may arise, we write simply P_x.

We introduce the entrance and hitting times H_A and $\tilde{H}_A$ of a set A of vertices in $\mathbb{T}_N^d$ (or in $\mathbb{Z}^d$) by

$$H_A = \inf\{t \geq 0 : X_t \in A\}, \tag{2.5}$$

$$\tilde{H}_A = \inf\{t \geq 1 : X_t \in A\}. \tag{2.6}$$

Throughout this notes, we will suppose that the dimension d is greater or equal to three, implying that

$$\text{the random walk on } \mathbb{Z}^d \text{ is transient,} \tag{2.7}$$

see Exercise 5.14.

Fix now a finite set $\mathsf{A} \subset \mathbb{Z}^d$ (usually we will denote subsets of $\mathbb{Z}^d$ by $\mathsf{A}, \mathsf{B}, \dots$). Due to the transience of the random walk, we can define the equilibrium measure (e_A) and capacity ($\text{cap}(\mathsf{A})$) of A by

$$e_\mathsf{A}(x) = \mathbf{1}_{x \in \mathsf{A}} P_x[\tilde{H}_\mathsf{A} = \infty], \text{ for } x \in \mathbb{Z}^d, \tag{2.8}$$

$$\text{cap}(\mathsf{A}) = e_\mathsf{A}(\mathbb{Z}^d). \tag{2.9}$$

Note that $\text{cap}(\mathsf{A})$ normalizes the measure e_A into a probability distribution.

2.2 Local entrance point

We are going to be interested in the local picture left by the random walk on $\mathbb{T}^d_N$. To make clear what we mean by local picture, we first consider a box $\mathsf{A} \subset \mathbb{Z}^d$ centered at the origin. For each N larger than the diameter of A, one can find an copy A_N of this box inside $\mathbb{T}^d_N$. The type of question we are going to be interested concerns the intersection of the random walk trajectory (run up to time n) with the set A_N, in other words $\{X_0, X_1, \dots, X_n\} \cap A_N$. As N gets large, the boxes A_N get much smaller compared to the whole torus T^d_N, explaining the use of the terminology 'local picture'.

As soon as N is strictly larger than the diameter of the box A, we can find an isomorphism $\phi_N : A_N \to \mathsf{A}$ between the box A and its copy of it in the infinite lattice. Again we observe that the subindices N in the notation ϕ_N and A_N may be dropped to avoid a clumsy notation.

Observe that

$$\pi(A) \text{ converges to zero as } N \text{ tends to infinity.} \tag{2.10}$$

The first question we attempt to answer concerns the distribution of the point where the random walk enters the box A. We study this by splitting the random walk trajectory into successive excursions to A . To make this more precise, consider a sequence of boxes A'_N centered at the origin in $\mathbb{Z}^d$ and having diameter $N^{1/2}$ (the specific value $1/2$ is not particular important, any value strictly between zero and one would work for our purposes here).

Note that for N large enough A'_N contains A and $N^{1/2} \leq N$. Therefore, we can extend the isomorphism ϕ_N defined above to $\phi_N : \mathsf{A}'_N \to A'_N \subset \mathbb{T}^d_N$, where A'_N is a copy of A'_N inside $\mathbb{T}^d_N$.

Lemma 2.1. ($d \geq 3$) *For A' and A' as above, there exists a constant $\delta > 0$ such that*

$$\sup_{x \in \mathbb{T}^d_N \setminus A'} P_x[H_A \leq r_N] \leq N^{-\delta}, \text{ ‘regeneration happens before } H_A\text{’} \tag{2.11}$$

$$\sup_{x \in \mathbb{Z}^d \setminus \mathsf{A}'_N} \mathbb{P}^{\mathbb{Z}^d}_x[H_{\mathsf{A}} < \infty] \leq N^{-\delta}, \text{ ‘escape to infinity before hitting } \phi(A)\text{’} \tag{2.12}$$

Proof of Lemma 2.1. The bound (2.12) follows from [9], Proposition 1.5.10, (see p. 36). We now prove (2.11) and for this, let Π be the canonical projection from $\mathbb{Z}^d$ onto $\mathbb{T}^d_N$. Given an x in $\mathbb{T}^d_N \setminus A'$, we can bound $P_x\left[H_A \leq r_N\right]$ by

$$P_{\phi(x)}\left[H_{B^c(\phi(x), N\log^2 N)} \leq r_N\right] + P_{\phi(x)}\left[H_{\Pi^{-1}(A)\cap B(\phi(x), N\log^2 N)} < \infty\right]. \tag{2.13}$$

By some concentration inequality, (see for instance Section 3.5 of [11]),

$$P_{\phi(x)}\left[H_{B^c(\phi(x), N\log^2 N)} \leq r_N\right] \leq c\exp\left(-c(N\log^2 N)^2/r_N\right) \leq ce^{-c\log^2 N}, \tag{2.14}$$

see Exercise 5.15 for more details. The set $\Pi^{-1}(A) \cap B(\phi(x), N\log^2 N)$ is contained in a union of no more than $c\log^c N$ translated copies of the ball A. By choice of x, $\phi(x)$ is at distance at least $cN^{1/2}$ from each of these boxes. Hence, using the union bound and again the estimate in [9], Proposition 1.5.10 on the hitting probability, we obtain that

$$P_{\phi(x)}\left[H_{\Pi^{-1}(A)\cap B(\phi(x), N\log^2 N)} < \infty\right] \leq c(\log N)^c N^{-c}.$$

Inserting the last two estimates into (2.13), we have shown (2.11). □

For simplicity of notation, we write A, A', A rather than A_N, A'_N, A_N from now on.

We first derive a consequence of (2.11). The following lemma states that, up to a typically small error, the probability $P_y[X_{H_A} = x]$ does not depend on the starting point $y \in \mathbb{T}^d_N \setminus A'$:

Lemma 2.2.

$$\sup_{\substack{x \in A, \\ y,y' \in \mathbb{T}_N^d \setminus A'}} \Big| P_y[X_{H_A} = x] - P_{y'}[X_{H_A} = x] \Big| \le cN^{-\delta}. \tag{2.15}$$

Proof. We apply the following intuitive argument: it is unlikely that the random walk started at $y \in \mathbb{T}_N^d \setminus A'$ visits the set A before time r_N, and at time r_N the distribution of the random walk is already close to uniform. To make this precise, we first deduce from inequality (2.3) that

$$\begin{aligned} &\sup_{y \in \mathbb{T}_N^d \setminus A'} \Big| E_y\big[P_{X_{r_N}}[X_{H_A} = x]\big] - P[X_{H_A} = x] \Big| \\ &\quad \le \sum_{y' \in \mathbb{T}_N^d} \sup_{y \in \mathbb{T}_N^d \setminus B'} \Big| P_y[X_{r_N} = y'] - \pi(y') \Big| P_{y'}[X_{H_A} = x] \\ &\quad \le cN^d e^{-c\log^2 N} \le e^{-c\log^2 N}. \end{aligned} \tag{2.16}$$

We have, for any $y \in \mathbb{T}_N^d \setminus A'$, by the simple Markov property applied at time r_N and the estimate (2.16),

$$\begin{aligned} P_y[X_{H_A} = x] &\le P_y[X_{H_A} = x, H_A > r_N] + P_y[H_A \le r_N] \\ &\le E_y\big[P_{X_{r_N}}[X_{H_A} = x]\big] + P_y[H_A \le r_N] \\ &\le P[X_{H_A} = x] + e^{-c\log^2 N} + P_y[H_A \le r_N]. \end{aligned} \tag{2.17}$$

With (2.11), we have therefore shown that for any $y \in \mathbb{T}_N^d \setminus A'$,

$$P_y[X_{H_A} = x] - P[X_{H_A} = x] \le N^{-\delta}. \tag{2.18}$$

The other part of (2.15) is proved similarly. Indeed, for any $y \in \mathbb{T}_N^d \setminus A'$, we have by the simple Markov property applied at time r_N,

$$\begin{aligned} P_y[X_{H_A} = x] &\ge P_y[X_{H_A} = x, H_A > r_N] \\ &\ge E_y\big[P_{X_{r_N}}[X_{H_A} = x]\big] - P_y[H_A \le r_N] \\ &\overset{(2.16),(2.11)}{\ge} P[X_{H_A} = x] - N^{-\delta}. \end{aligned} \tag{2.19}$$

Together with (2.18), this proves that

$$\sup_{y \in \mathbb{T}_N^d \setminus A'} \Big| P_y[X_{H_A} = x] - P[X_{H_A} = x] \Big| \le N^{-\delta},$$

from which (2.15) readily follows. □

Given that the distribution of the entrance point of the random walk in A is roughly independent of the starting point (out of A'), we are naturally tempted to estimate such distribution. This is the content of the next lemma, which will play an important role in motivating our main definitions.

Lemma 2.3. *For A and A' as above,*

$$\sup_{x\in A,\, y\in\mathbb{T}_N^d\setminus A'} \left| P_y[X_{H_A}=x] - \frac{e_{\mathsf{A}}(\phi(x))}{\mathrm{cap}(\mathsf{A})} \right| \le N^{-\delta}. \tag{2.20}$$

Note that the entrance law is approximated by the (normalized) exit distribution. This is intimaly related to the reversibility of the random walk.

Proof. Let us fix vertices $x\in A, y\in\mathbb{T}_N^d\setminus A'$. We first define the equilibrium measure of A, with respect to the random walk killed when exiting A' by

$$e_A^{A'}(z) = \mathbf{1}_A(z) P_z[H_{\mathbb{T}_N^d\setminus A'} < \tilde{H}_A], \text{ for any } z\in A.$$

Note that by (2.12) and the strong Markov property applied at time $H_{\mathbb{T}_N^d\setminus A'}$,

$$e_{\mathsf{A}}(\phi(z)) \le e_A^{A'}(z) \le e_{\mathsf{A}}(\phi(z)) + N^{-\delta}, \text{ for any } z\in A. \tag{2.21}$$

In order to make the expression $P_y[X_{H_A}=x]$ appear, we consider the probability that the random walk started at x escapes from A to $\mathbb{T}_N^d\setminus A'$ and then returns to the set A at some point other than x. By reversibility of the random walk with respect to the measure $(\pi_z)_{z\in\mathbb{T}_N^d}$, we have

$$\begin{aligned}\sum_{z\in A\setminus\{x\}} \pi_x P_x[H_{\mathbb{T}_N^d\setminus A'} < \tilde{H}_A, X_{\tilde{H}_A}=z] &= \pi_x P_x[H_{\mathbb{T}_N^d\setminus A'} < \tilde{H}_A, X_{\tilde{H}_A}\neq x] \\ &= \sum_{z\in A\setminus\{x\}} \pi_z P_z[H_{\mathbb{T}_N^d\setminus A'} < \tilde{H}_A, X_{\tilde{H}_A}=x].\end{aligned} \tag{2.22}$$

By the strong Markov property applied at time $H_{\mathbb{T}_N^d\setminus A'}$, we have for any $z\in A$,

$$\pi_z P_z[H_{\mathbb{T}_N^d\setminus A'} < \tilde{H}_A, X_{\tilde{H}_A}=x] = \pi_z E_z\big[\mathbf{1}_{\{H_{\mathbb{T}_N^d\setminus A'}<\tilde{H}_A\}} P_{X_{H_{\mathbb{T}_N^d\setminus A'}}}[X_{H_A}=x]\big].$$

With (2.21) and (2.15), this yields

$$\left|\pi_z P_z[H_{\mathbb{T}_N^d\setminus A'} < \tilde{H}_A, X_{\tilde{H}_A}=x] - e_{\mathsf{A}}(\phi(z)) P_y[X_{H_A}=x]\right| \le N^{-\delta}, \tag{2.23}$$

for any $z \in A$. With this estimate applied to both sides of (2.22), we obtain

$$\begin{aligned}\pi_x e_{\mathsf{A}}(\phi(x))\big(1 - P_y[X_{H_A} = x]\big) =& P_y[X_{H_A} = x]\big(\mathrm{cap}(\mathsf{A}) - \pi_x e_{\mathsf{A}}(\phi(x))\big) \\ &+ O\big(|A|N^{-\delta}\big),\end{aligned}$$

implying (2.20). □

We observe that the entrance distribution $P_y[X_{H_B} = \cdot]$ was approximated in Lemma 2.3 by a quantity that is independent of N and solely relates to the infinite lattice random walk. This motivates the construction of the so-called 'local picture' that we develop next in order to construct the random interlacements measure.

2.3 Local measure

In this section we study the trace that a random walk X_n on $\mathbb{T}_N^d$ leaves inside a small box $A \subset \mathbb{T}_N^d$.

We already know from the previous section that the random walk typically enters the box A from a point x chosen with distribution $e_{\mathsf{A}}(\phi(x))/\,\mathrm{cap}(\mathsf{A})$. After entering the box A, the random walk behaves the same way as in the infinite lattice $\mathbb{Z}^d$ until it gets far away from A again. This motivates the following procedure of splitting the random walk trajectory into what we call 'excursions' . For this, recall the definition of A' and the shift operators θ_k from Section 2.2 and let

$$R_0 = H_A, \qquad D_0 = H_{\mathbb{T}_N^d \setminus A'} \circ \theta_{R_0} + R_0, \tag{2.24}$$

$$R_l = H_A \circ \theta_{D_{l-1}} + D_{l-1}, \qquad D_l = H_{\mathbb{T}_N^d \setminus A'} \circ \theta_{R_l} + R_l, \qquad \text{for } l \geq 1. \tag{2.25}$$

(2.26)

These will be respectively called return and departure times of the random walk between A and A'.

Observe that every time n for which the random walk is inside A has to satisfy $R_k \leq t < D_k$ for some $k \geq 0$. This implies that

$$\{X_0, X_1, \ldots, X_{D_k}\} \cap A = \bigcup_{j=0}^{k} \{X_{R_j}, X_1, \ldots, X_{D_j}\} \cap A. \tag{2.27}$$

Or in other words, the trace left by the random walk trajectory in A up to time D_k is given by the trace of the k separate excursions.

We now include a heuristic discussion that motivates the definition of what we call the 'local measure' Q_{A}, see (2.28) below. From Lemma 2.2 and the Strong Markov Property applied to $H_{\mathbb{T}_N^d \setminus A'}$, we can conclude that the set of points visited by the random walk between times R_0 and D_0 is roughly independent of R_1. Therefore, the excursions $\{X_{R_j}, X_1, \ldots, X_{D_j}\}$ of the random walk between A and A' are roughly independent from each other, for $j = 1, \ldots, k$. If we now use Lemma 2.3, we conclude that the entrance points X_{R_j} of these trajectories in A are roughly distributed as $e_{\mathsf{A}}(\phi(\cdot))/\operatorname{cap}(\mathsf{A})$. While the rest of the excursion $\{X_{R_j+1}, \ldots, X_{D_j}\}$ is a simple random walk that, as N grows, behaves more and more like a simple random walk on $\mathbb{Z}^d$ (note that this heuristic claim is only true because the random walk on $\mathbb{Z}^d$, for $d \geq 3$, is transient).

This motivates the definition of the following measure on the space W_+ of nearest neighbor trajectories in $\mathbb{Z}^d$.

$$Q^+_{\mathsf{A}}[X_0 = x, (X_n)_{n\geq 0} \in B] = e_{\mathsf{A}}(x) P^{\mathbb{Z}^d}_x[B], \text{ for } x \in \mathbb{Z}^d, \tag{2.28}$$

where B is any event in the σ-algebra of the space of random walk trajectories to be defined in the next chapter. Note that Q^+_{A} is a finite (but not necessarily a probability) measure, selecting a starting point x according to e_{A} and following a simple random walk from x.

We now have to understand how many excursions are typically performed by the random walk between A and A' until some fixed time n.

We will conduct the following discussion on a heuristic level, but we reefer to [17] for a rigorous description.

Fix a given time $n \geq 0$ and a site $x \in A$. Let us estimate the probability that $X_n = x$ and n is a return time R_j for some $j \geq 0$. This probability can be written as

$$\begin{aligned} &P[X_n = x \text{ and } n = R_j \text{ for some } j \geq 0] = \\ &\quad = P\Big[X_n = x, \bigcup_{m \leq n} \Big\{ \begin{array}{c} X_m \notin A' \text{ and } X \text{ stays in } A' \setminus A \\ \text{between times } m+1 \text{ and } n-1 \end{array} \Big\}\Big] \\ &\quad = \sum_{m=0}^{n} \frac{2^{-(n-m)}}{N^d} \#\Big\{ \begin{array}{c} \text{paths of length } n-m \text{ from } \mathbb{T}_N^d \setminus A' \text{ to } x \\ \text{and otherwise contained in } A' \setminus A \end{array} \Big\} \end{aligned}$$

$$= \sum_{m=0}^{n} \frac{2^{-(n-m)}}{N^d} \# \left\{ \begin{array}{c} \text{paths of length } n-m, \text{ joining } x \text{ to } \mathbb{T}_N^d \setminus A' \\ \text{and otherwise contained in } A' \setminus A \end{array} \right\}$$

$$= \sum_{m=0}^{n} \frac{1}{N^d} P_x[m = H_{\mathbb{T}_N^d \setminus A'} < \tilde{H}_A] = \frac{1}{N^d} P_x[H_{\mathbb{T}_N^d \setminus A'} < \min\{n, \tilde{H}_A\}].$$

We now use (2.21) to obtain that

$$\lim_{N\to\infty} \lim_{n\to\infty} \Big| N^d P\big[X_n = x \text{ and } n = R_j \text{ for some } j \geq 0\big] - e_{\mathsf{A}}(x) \Big| = 0.$$

Fix now some $u > 0$. Since the number of excursions starting at x between time 0 and uN^d is given by

$$\sum_{n=0}^{N^d} \mathbf{1}\{X_n = x \text{ and } n = R_j \text{ for some } j \geq 0\}, \tag{2.29}$$

we expect that

$$\lim_{N\to\infty} E\left[\begin{array}{c} \text{number of excursions starting at } x \\ \text{between times } 0 \text{ and } uN^d \end{array} \right] = u e_{\mathsf{A}}(x). \tag{2.30}$$

Note that (2.29) is a sum of (weakly dependent) Bernoulli random variables with parameter summing up to approximately $ue_{\mathsf{A}}(x)$. Therefore, we could guess that the number of such excursions should approach a Poisson random variable with parameter $ue_{\mathsf{A}}(x)$. We end this section with a list of conclusions that we obtained from the above informal discussion:

- the random walk on $\mathbb{T}_N^d$ up to time uN^d, intersected with A can be split into roughly independent excursions,
- for each point x in A, the distribution of the number of excursions starting at x is roughly a Poisson random variable with mean $ue_{\mathsf{A}}(x)$.

In order to make the above description into a formal construction, we choose an elegant description in terms of Poisson point processes. This is done in the next chapter.

Chapter 3

Random interlacements

In this chapter we extend the definition of the local picture, appearing in Chapter 2. This extension will allow us to define an invariant percolation on $\mathbb{Z}^d$, which we call random interlacements. Later we discuss some of its main properties, comparing with Bernoulli site percolation.

3.1 Definition of the model

In the first lecture, we studied the trace left by a random walk on the torus, when it runs up to time uN^d, where u is a fixed positive constant. For a fixed box $A \subset \mathbb{T}_N^d$, we obtained a somewhat informal description of how the random walk visits A:

- the random walk trajectory is split into roughly independent excursions,
- for each $x \in A$, the number of excursions starting at x is approximately an independent Poisson random variable with mean $ue_{\mathsf{A}}(x)$,
- the trace left by the random walk on A is given by the union of all these excursions intersected with A.

The above informal construction will be made precise below, using the formalism of Poisson point processes. For this, let us first introduce some notation. Let W_+ be the space of infinite nearest-neighbor trajectories that spend only a finite time in finite sets of $\mathbb{Z}^d$.

$$\begin{aligned} W_+ = \big\{ w : \mathbb{N} \to \mathbb{Z}^d : \|w(n) - w(n+1)\|_1 = 1 \text{ for each } n \geq 0 \text{ and} \\ \{n : w(n) = y\} \text{ is finite for all } y \in \mathbb{Z}^d \big\}. \end{aligned} \tag{3.1}$$

Let X_n, for $n \geq 0$ denote its canonical coordinates . We endow the space W_+ with the sigma algebra $\mathcal{W}_+$ generated by the coordinate maps X_i, $i \geq 0$.

We recall the definition of the measure Q^+_{A} on W_+:

$$Q^+_{\mathsf{A}}[X_0 = x, (X_n)_{n \geq 0} \in B] = e_{\mathsf{A}}(x) P_x[B], \quad B \in \mathcal{W}_+, x \in \mathbb{Z}^d. \tag{3.2}$$

From the transience of the simple random walk on $\mathbb{Z}^d$ (see Exercise 5.14) it follows that W_+ has a full measure under Q^+_{A}. We also need to construct the space of point measures

$$\Omega_+ = \Big\{ \omega_+ = \sum_{i=1}^{n} \delta_{w_i}; n \in \mathbb{Z}_+, w_1, \ldots, w_n \in W_+ \Big\} \tag{3.3}$$

Endowed with the sigma algebra generated by the evaluation maps $\omega_+ \mapsto \omega_+(D)$, where $D \in \mathcal{W}_+$. Above, δ_w stands for the Dirac's measure on w.

Now let $\mathbb{P}^u_{\mathsf{A}}$ be the law of a Poisson point on process with intensity measure uQ^+_{A}. It is interesting to note that this more abstract construction elegantly implements what was done in the informal procedure described in the first paragraph of this chapter.

In the first lecture we have seen that the asymptotic local picture left by the random walk on the torus should be related to $\mathbb{P}^u_{\mathsf{A}}$. This leaves the question whether there exists an infinite volume model (i.e. a model on the whole lattice $\mathbb{Z}^d$) whose restriction to a finite set A is described by $\mathbb{P}^u_{\mathsf{A}}$. In this lecture we are going to answer this question affirmatively: we will construct such model, called *random interlacement* . We will also study the existence of a phase transition for this model, and prove some of its basic properties. The results of this chapter appeared for the first time in [14].

We wish to construct the infinite volume analog to $\mathbb{P}^u_{\mathsf{A}}$, or intuitively speaking, the limit as A covers the whole lattice $\mathbb{Z}^d$. The first step is to introduce the measure space where this Poisson process will be defined. To this end we need few definitions.

Similarly to (3.1), let W be the space of doubly-infinite nearest-neighbor trajectories that spend only a finite time in finite subsets of $\mathbb{Z}^d$, i.e.

$$\begin{aligned} W = \{ w : &\mathbb{Z} \to \mathbb{Z}^d : \|w(n) - w(n+1)\|_1 = 1 \text{ for each } n \geq 0 \text{ and} \\ &\{n : w(n) = y\} \text{ is finite for all } y \in \mathbb{Z}^d \}. \end{aligned} \tag{3.4}$$

We again denote with X_n, $n \in \mathbb{Z}$, the canonical coordinates W, and write θ_k, $k \in \mathbb{Z}$, for the canonical shifts,

$$\theta_k(w)(\cdot) = w(\cdot + k), \text{ for } k \in \mathbb{Z} \quad (\text{resp. } k \geq 0 \text{ when } w \in W_+). \tag{3.5}$$

We endow and W with the σ-algebra $\mathcal{W}$, generated by the canonical coordinates.

Given $\mathsf{A} \subset \mathbb{Z}^d$, $w \in W$ (resp. $w \in W_+$), we define the entrance time in A and the exit time from A for the trajectory w:

$$\begin{aligned} H_{\mathsf{A}}(w) &= \inf\{n \in \mathbb{Z}\,(\text{resp. } \mathbb{N}) : X_n(w) \in \mathsf{A}\}, \\ T_{\mathsf{A}}(w) &= \inf\{n \in \mathbb{Z}\,(\text{resp. } \mathbb{N}) : X_n(w) \notin \mathsf{A}\}. \end{aligned} \tag{3.6}$$

When $\mathsf{A} \subset\subset \mathbb{Z}^d$ (meaning that $\mathsf{A} \subset \mathbb{Z}^d$ and is finite), we consider the subset of W of trajectories entering A:

$$W_{\mathsf{A}} = \{w \in W : X_n(w) \in \mathsf{A} \text{ for some } n \in \mathbb{Z}\}. \tag{3.7}$$

We can write W_{A} as a countable partition into measurable sets

$$W_{\mathsf{A}} = \bigcup_{n \in \mathbb{Z}} W_{\mathsf{A}}^n, \qquad \text{where } W_{\mathsf{A}}^n = \{w \in W : H_{\mathsf{A}}(w) = n\}. \tag{3.8}$$

The measure Q_{A}^+ is, up to a multiplicative factor u, the intensity of the Poisson point process $\mathbb{P}_{\mathsf{A}}^u$. However, it is not appropriate to take part in the infinite volume limit on $\mathbb{Z}^d$. Intuitively speaking, this is due to the fact that its trajectories have a starting point which depend on the choice of A.

The first step to obtain the infinite volume random interlacements is to extend the measure Q_{A}^+ to the space W, by requiring that $(X_{-n})_{n\geq 0}$ is a simple random walk started at X_0 conditioned not to return to A. That is, abusing slightly the notation, we define on $(W, \mathcal{W})$ the measure Q_{A} by

$$Q_{\mathsf{A}}[(X_{-n})_{n\geq 0} \in A, X_0 = x, (X_n)_{n\geq 0} \in B] = P_x[A|\widetilde{H}_{\mathsf{A}} = \infty] e_{\mathsf{A}}(x) P_x[B], \tag{3.9}$$

for $A, B \in \mathcal{W}_+$ and $x \in \mathbb{Z}^d$.

Observe that Q_{A} gives full measure to W_{A}^0. Which means that the set A is still registered somehow in the trajectories. Therefore it will be more convenient to consider the space W^* of trajectories in W modulo time shift

$$W^* = W/\sim, \text{ where } w \sim w' \text{ iff } w(\cdot) = w'(\cdot + k) \text{ for some } k \in \mathbb{Z}, \tag{3.10}$$

which allows us to 'ignore' the rather arbitrary (and A-dependent) time parametrisation of the random walks. We denote with π^* the canonical projection from W to W^*. The map π^* induces a σ-algebra in W^* given by $\mathcal{W}^* = \{A \subset W^*; (\pi^*)^{-1}(A) \in \mathcal{W}\}$, which is the largest σ-algebra on W^* for which $(W, \mathcal{W}) \stackrel{\pi^*}{\to} (W^*, \mathcal{W}^*)$ is measurable. We use W^*_{A} to denote the set of trajectories modulo time shift entering $\mathsf{A} \subset \mathbb{Z}^d$,

$$W^*_{\mathsf{A}} = \pi^*(W_{\mathsf{A}}). \tag{3.11}$$

It is easy to see that $W^*_{\mathsf{A}} \in \mathcal{W}^*$.

The random interlacement process that we are defining will be governed by a Poisson point process on the space $(W^* \times \mathbb{R}_+, \mathcal{W}^* \otimes \mathcal{B}(R_+))$. To this end we define Ω in analogy to (3.3):

$$\Omega = \Big\{ \omega = \sum_{i \geq 1} \delta_{(w^*_i, u_i)} : w^*_i \in W^*, u_i \in \mathbb{R}_+ \\ \text{such that } \omega(W^*_{\mathsf{A}} \times [0,u]) < \infty, \text{ for every } \mathsf{A} \subset\subset \mathbb{Z}^d \text{ and } u \geq 0 \Big\}. \tag{3.12}$$

This space is endowed with the σ-algebra $\mathcal{A}$ generated by the evaluation maps $\omega \mapsto \omega(D)$ for $D \in \mathcal{W}^* \otimes \mathcal{B}(\mathcal{R}_+)$.

The intensity measure of the Poisson point process governing the random interlacement will be given by $\nu \otimes \mathrm{d}u$. Here, $\mathrm{d}u$ is the Lebesgue measure on $\mathbb{R}_+$ and the measure ν on W^* is constructed as an appropriate extension of Q_{A} to W^* in the following theorem.

Theorem 3.1 ([14], Theorem 1.1)**.** *There exists a unique σ-finite measure ν on the space $(W^*, \mathcal{W}^*)$ satisfying, for each finite set $\mathsf{A} \subset \mathbb{Z}^d$,*

$$1_{W^*_{\mathsf{A}}} \cdot \nu = \pi^* \circ Q_{\mathsf{A}} \tag{3.13}$$

where the finite measure Q_{A} on W_{A} is given by (3.9)

Proof. The uniqueness of ν satisfying (3.13) is clear since, given a sequence of sets $\mathsf{A}_k \uparrow \mathbb{Z}^d$, $W^* = \cup_k W^*_{\mathsf{A}_n}$.

For the existence, what we need to prove is that, for fixed $\mathsf{A} \subset \mathsf{A}' \subset \mathbb{Z}^d$,

$$\pi^* \circ (1_{W_{\mathsf{A}}} \cdot Q_{\mathsf{A}'}) = \pi^* \circ Q_{\mathsf{A}}. \tag{3.14}$$

We can then set, for arbitrary $\mathsf{A}_k \uparrow \mathbb{Z}^d$,

$$\nu = \sum_k \mathbf{1}\{W^*_{\mathsf{A}_k} \setminus W^*_{\mathsf{A}_{k-1}}\}\, \pi^* \circ Q_{\mathsf{A}_k}. \tag{3.15}$$

We introduce the space

$$W_{\mathsf{A},\mathsf{A}'} = \{w \in W_{\mathsf{A}} : H_{\mathsf{A}'}(w) = 0\} \tag{3.16}$$

and the bijection $s_{\mathsf{A},\mathsf{A}'} : W_{\mathsf{A},\mathsf{A}'} \to W_{\mathsf{A},\mathsf{A}}$ given by

$$[s_{\mathsf{A},\mathsf{A}'}(w)](\cdot) = w(H_{\mathsf{A}}(w) + \cdot), \tag{3.17}$$

which moves the origin of time from the entrance time to A' to the entrance time of A.

To prove (3.14), it is enough to show that

$$s_{\mathsf{A},\mathsf{A}'} \circ (1_{W_{\mathsf{A},\mathsf{A}'}} \cdot Q_{\mathsf{A}'}) = Q_{\mathsf{A}}, \tag{3.18}$$

Indeed, from (3.9) it follows that $1_{W_{\mathsf{A},\mathsf{A}'}} \cdot Q_{\mathsf{A}'} = 1_{W_{\mathsf{A}}} \cdot Q_{\mathsf{A}'}$ and thus (3.14) follows just by applying π^* on both sides (3.18).

We now consider the set Σ of finite paths $\sigma : \{0, \cdots, N_\sigma\} \to \mathbb{Z}^d$ such that $\sigma(0) \in \mathsf{A}'$, $\sigma(n) \notin \mathsf{A}$ for $n < N_\sigma$ and $\sigma(N_\sigma) \in \mathsf{A}$. We split the left hand-side of (3.18) by partitioning $W_{\mathsf{A},\mathsf{A}'}$ into the sets

$$W^\sigma_{\mathsf{A},\mathsf{A}'} = \{w \in W_{\mathsf{A},\mathsf{A}'} : w \text{ restricted to } \{0, \cdots, N_\sigma\} \text{ equals } \sigma\}, \qquad \sigma \in \Sigma. \tag{3.19}$$

For $w \in W^\sigma_{\mathsf{A},\mathsf{A}'}$, we have $H_{\mathsf{A}}(w) = N_\sigma$, so that we can write

$$s_{\mathsf{A},\mathsf{A}'} \circ (1_{W_{\mathsf{A},\mathsf{A}'}} \cdot Q_{\mathsf{A}'}) = \sum_{\sigma \in \Sigma} \theta_{N_\sigma} \circ (1_{W^\sigma_{\mathsf{A},\mathsf{A}'}} \cdot Q_{\mathsf{A}'}). \tag{3.20}$$

To prove (3.18), consider an arbitrary collection of sets $A_i \subset \mathbb{Z}^d$, for $i \in \mathbb{Z}$, such that $A_i \neq \mathbb{Z}^d$ for at most finitely many $i \in \mathbb{Z}$. Then,

$$\begin{aligned} s_{\mathsf{A},\mathsf{A}'} &\circ (1_{W_{\mathsf{A},\mathsf{A}'}} \cdot Q_{\mathsf{A}'})[X_i \in A_i, i \in \mathbb{Z}] \\ &= \sum_{\sigma \in \Sigma} Q_{\mathsf{A}'}[X_{i+N_\sigma}(w) \in A_i, i \in \mathbb{Z}, w \in W^\sigma_{\mathsf{A},\mathsf{A}'}] \\ &= \sum_{\sigma \in \Sigma} Q_{\mathsf{A}'}[X_i(w) \in A_{i-N_\sigma}, i \in \mathbb{Z}, w \in W^\sigma_{\mathsf{A},\mathsf{A}'}]. \end{aligned} \tag{3.21}$$

Using the formula (3.9), the identity $e_{\mathsf{A}'}(x)P_x[\,\cdot\,|\widetilde{H}_{\mathsf{A}}=\infty]=P_x[\,\cdot\,,\widetilde{H}_{\mathsf{A}}=\infty]$, for $x\in\operatorname{supp}e_{\mathsf{A}'}$, and the Markov property, the above expression equals

$$\begin{aligned}&\sum_{x\in\operatorname{supp}e_{\mathsf{A}'}}\sum_{\sigma\in\Sigma}P_x\big[X_j\in A_{-j-N_\sigma},j\geq 0,\widetilde{H}_{\mathsf{A}'}=\infty\big]\\&\qquad\times P_x\big[X_n=\sigma(n)\in A_{n-N_\sigma},0\leq n\leq N_\sigma\big]P_{\sigma(N_\sigma)}\big[X_n\in A_n,n\geq 0\big]\\&\quad=\sum_{x\in\operatorname{supp}e_{\mathsf{A}'}}\sum_{y\in\mathsf{A}}\sum_{\sigma:\sigma(N_\sigma)=y}P_x\big[X_j\in A_{-j-N_\sigma},j\geq 0,\widetilde{H}_{\mathsf{A}'}=\infty\big]\\&\qquad\times P_x\big[X_n=\sigma(n)\in A_{n-N_\sigma},0\leq n\leq N_\sigma\big]P_y\big[X_n\in A_n,n\geq 0\big].\end{aligned}\tag{3.22}$$

For fixed $x\in\operatorname{supp}e_{\mathsf{A}'}$ and $y\in\mathsf{A}$, we have, using the reversibility in the first step and the Markov property in the second,

$$\begin{aligned}&\sum_{\sigma:\sigma(N_\sigma)=y}P_x\big[X_j\in A_{-j-N_\sigma},j\geq 0,\widetilde{H}_{\mathsf{A}'}=\infty\big]\\&\qquad\qquad\times P_x\big[X_n=\sigma(n)\in A_{n-N_\sigma},0\leq n\leq N_\sigma\big]\\&=\sum_{\substack{\sigma:\sigma(N_\sigma)=y\\ \sigma(0)=x}}P_x\big[X_j\in A_{-j-N_\sigma},j\geq 0,\widetilde{H}_{\mathsf{A}'}=\infty\big]\\&\qquad\qquad\times P_y\big[X_m=\sigma(N_\sigma-m)\in A_{-m},0\leq m\leq N_\sigma\big]\\&=\sum_{\substack{\sigma:\sigma(N_\sigma)=y\\ \sigma(0)=x}}P_y\left[\begin{matrix}X_m=\sigma(N_\sigma-m)\in A_{-m},0\leq m\leq N_\sigma,\\ X_m\in A_{-m},m\geq N_\sigma,\widetilde{H}_{\mathsf{A}'}\circ\theta_{N_\sigma}=\infty\end{matrix}\right]\\&=P_y\left[\begin{matrix}\widetilde{H}_{\mathsf{A}}=\infty,\text{ the last visit to }\mathsf{A}'\\ \text{occurs at }x,\ X_m\in A_{-m},m\geq 0\end{matrix}\right].\end{aligned}\tag{3.23}$$

Using (3.23) in (3.22) and summing over $x\in\operatorname{supp}e_{\mathsf{A}'}$, we obtain

$$\begin{aligned}&s_{\mathsf{A},\mathsf{A}'}\circ(1_{W_{\mathsf{A},\mathsf{A}'}}\cdot Q_{\mathsf{A}'})[X_i\in A_i,i\in\mathbb{Z}]\\&\quad=\sum_{y\in\mathsf{A}}P_y[\widetilde{H}_{\mathsf{A}}=\infty,X_m=A_{-m},m\geq 0]P_y[X_m\in A_m,m\geq 0]\\&\quad\overset{(3.9)}{=}Q_{\mathsf{A}}[X_m\in A_m,m\in\mathbb{Z}].\end{aligned}\tag{3.24}$$

This shows (3.18) and concludes the proof of the existence of the measure ν satisfying (3.13). Moreover, ν is clearly σ-finite, it is sufficient to observe that $\nu(W^*_{\mathsf{A}},[0,u])<\infty$ for any $\mathsf{A}\subset\subset\mathbb{Z}^d$ and $u\geq 0$. □

We can now complete the construction of the random interlacement model. On the space $(\Omega, \mathcal{A})$ we consider the law $\mathbb{P}$ of a Poisson point process with intensity $\nu(\mathrm{d}w^* \otimes \mathrm{d}u)$, recall that ν is σ-finite. With the usual identification of point measures and subsets, under $\mathbb{P}$, the configuration ω can be viewed as an infinite random cloud of doubly-infinite random walk trajectories (modulo time-shift) with attached non-negative labels u_i.

Finally, for $\omega = \sum_{i \geq 0} \delta_{(w_i^*, u_i)} \in \Omega$ we define two subsets of $\mathbb{Z}^d$, the *interlacement set at level* u, that is the set of sites visited by the trajectories with label smaller than u,

$$\mathcal{I}^u(\omega) = \bigcup_{i: u_i \leq u} \mathrm{Range}(w_i^*), \tag{3.25}$$

and its complement, the *vacant set at level* u,

$$\mathcal{V}^u(\omega) = \mathbb{Z}^d \setminus \mathcal{I}^u(\omega). \tag{3.26}$$

Let Π^u be the mapping from Ω to $\{0,1\}^{\mathbb{Z}^d}$ given by

$$\Pi^u(\omega) = (\mathbf{1}\{x \in \mathcal{V}^u(\omega)\} : x \in \mathbb{Z}^d). \tag{3.27}$$

We endow the space $\{0,1\}^{\mathbb{Z}^d}$ with the σ-field $\mathcal{Y}$ generated by the canonical coordinates $(Y_x : x \in \mathbb{Z}^d)$. As for $\mathsf{A} \subset\subset \mathbb{Z}^d$, we have

$$\mathcal{V}^u \supset \mathsf{A} \qquad \text{if and only if} \qquad \omega(W_{\mathsf{A}}^* \times [0,u]) = 0, \tag{3.28}$$

the mapping $\Pi^u : (\Omega, \mathcal{A}) \to (\{0,1\}^{\mathbb{Z}^d}, \mathcal{Y})$ is measurable. We can thus define on $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{Y})$ the law Q^u of the vacant set at level u by

$$Q^u = \Pi^u \circ \mathbb{P}. \tag{3.29}$$

3.2 Basic properties

We now prove several important properties of the random interlacement model. But first, let $s_{\mathsf{A}} : W_{\mathsf{A}}^* \to W$ be defined as

$$s_{\mathsf{A}}(w^*) = w^0, \quad \text{where } w^0 \text{ is the unique element of } W_{\mathsf{A}}^0 \text{ with } \pi^*(w^0) = w^*. \tag{3.30}$$

We also define measurable map μ_{A} from Ω to the space of point measures on $(W_+ \times \mathbb{R}_+, \mathcal{W}_+ \otimes \mathcal{B}(\mathbb{R}_+))$ via

$$\mu_{\mathsf{A}}(\omega)(f) = \int_{W^*_{\mathsf{A}} \times \mathbb{R}_+} f(s_{\mathsf{A}}(w^*)_+, u)\omega(\mathrm{d}w^*, \mathrm{d}u), \qquad \text{for } \omega \in \Omega, \tag{3.31}$$

where f is a non-negative measurable function on $W_+ \times \mathbb{R}_+$ and for $w \in W$, $w_+ \in W_+$ is its restriction to $\mathbb{N}$. In words, μ_{A} selects from ω those trajectories that touch A and erases their parts prior to the first visit to A. We further define measurable function $\mu_{\mathsf{A},u}$ from Ω to the space of point measures on $(W_+, \mathcal{W}_+)$ by

$$\mu_{\mathsf{A},u}(\omega)(\mathrm{d}w) = \mu_{\mathsf{A}}(\omega)(\mathrm{d}w \times [0,u]), \tag{3.32}$$

which 'selects' from $\mu_{\mathsf{A}}(\omega)$ only those trajectories whose labels are smaller than u. Observe that

$$\mathcal{I}^u(\omega) \cap \mathsf{A} = \bigcup_{w \in \operatorname{supp} \mu_{\mathsf{A},u}(\omega)} \operatorname{Range} w \cup \mathsf{A}. \tag{3.33}$$

It follows from the construction of the measure $\mathbb{P}$ and from the defining property (3.13) of ν that

$$\mu_{\mathsf{A},u} \circ \mathbb{P} = \mathbb{P}^u_{\mathsf{A}}. \tag{3.34}$$

This has some important implications. But let us first define the Green function

$$g(x,y) = \sum_{n \geqslant 0} P_x[X_n = y], \text{ for } x,y \in \mathbb{Z}^d. \tag{3.35}$$

We write $g(x)$ for $g(x,0)$. We refer to [8], Theorem 1.5.4 p.31 for the following estimate

$$c' \frac{1}{1+|x-y|^{d-2}} \leqslant g(x,y) \leqslant c \frac{1}{|x-y|^{d-2}}, \text{ for } x,y \in \mathbb{Z}^d. \tag{3.36}$$

We can now state the following

Lemma 3.2. *For every $u \geq 0$, $x, y \in \mathbb{Z}^d$, $\mathsf{A} \subset\subset \mathbb{Z}^d$,*

$$\mathbb{P}[\mathsf{A} \subset \mathcal{V}^u] = \exp\{-u \operatorname{cap}(\mathsf{A})\}, \tag{3.37}$$

$$\mathbb{P}[x \in \mathcal{V}^u] = \exp\{-u/g(0)\}, \tag{3.38}$$

$$\mathbb{P}[\{x,y\} \in \mathcal{V}^u] = \exp\Big\{ - \frac{u}{g(0)+g(y-x)} \Big\}. \tag{3.39}$$

Proof. Observe that $\mathsf{A} \subset \mathcal{V}^u(\omega)$ if and only if $\mu_{\mathsf{A},u}(\omega) = 0$. Claim (3.37) then follows from

$$\mathbb{P}[\mu_{\mathsf{A},u}(\omega) = 0] \overset{(3.34)}{=} \exp\{-uQ_{\mathsf{A}}(W_+)\} \overset{(3.2)}{=} \exp\{-ue_{\mathsf{A}}(\mathbb{Z}^d)\} = \exp\{-u\,\mathrm{cap}(\mathsf{A})\}.$$

Recalling that

$$\mathrm{cap}(\{x\}) = g(0)^{-1}, \qquad \text{and} \qquad \mathrm{cap}(\{x,y\}) = \frac{2}{g(0)+g(x-y)}. \tag{3.40}$$

(3.38) and (3.39) follows directly from (3.37) □

The last lemma and (3.36) imply that

$$\mathrm{Cov}_{\mathbb{P}}(\mathbf{1}_{x\in\mathcal{V}^u}, \mathbf{1}_{y\in\mathcal{V}_u}) \sim \frac{2u}{g(0)^2} e^{-2u/g(0)} g(x-y) \geq c_u |x-y|^{2-d}, \quad \text{as } |x-y| \to \infty.$$

Long range correlation are thus present in the random set $\mathcal{V}^u$.

As another consequence of (3.37) and the sub additivity of the capacity, $\mathrm{cap}(\mathsf{A}\cup\mathsf{A}') \leq \mathrm{cap}\,\mathsf{A} + \mathrm{cap}\,\mathsf{A}'$, we see that

$$\mathbb{P}[\mathsf{A}\cup\mathsf{A}' \subset \mathcal{V}^u] \geq \mathbb{P}[\mathsf{A}\subset\mathcal{V}^u]\,\mathbb{P}[\mathsf{A}'\subset\mathcal{V}^u], \qquad \text{for } \mathsf{A},\mathsf{A}' \subset\subset \mathbb{Z}^d,\ u\geq 0, \tag{3.41}$$

that is the events $\mathsf{A}\subset\mathcal{V}^u$ and $\mathsf{A}'\subset\mathcal{V}^u$ are positively correlated.

The inequality (3.41) is the special case for the FKG inequality for the measure Q^u (see (3.29)) which was proved in [16]. We present it here for the sake of completeness without proof.

Theorem 3.3 (FKG inequality for random interlacement)**.** *Let $A, B \in \mathcal{Y}$ be two increasing events. Then*

$$Q^u[A\cap B] = Q^u[A]Q^u[B]. \tag{3.42}$$

The measure Q^u thus satisfies the one of the principal inequalities that hold for the Bernoulli percolation. Many of the difficulties appearing when studying random interlacements originate in the fact that the second important inequality, the van den Berg-Kesten one, does not hold for Q^u.

3.3 Translation invariance and ergodicity

We next consider the translation invariance and ergodicity of random interlacement. For $x \in \mathbb{Z}^d$ and $w \in W$ we define $w + x \in W$ by $(w+x)(n) = w(n) + x, n \in \mathbb{Z}$. For $w \in W^*$, we then set $w^\star + x = \pi^\star(w+x)$ for $\pi^*(w) = w^*$. Finally, for $\omega = \sum_{i\geq 0} \delta_{(w_i^*, u_i)} \in \Omega$ we define

$$\tau_x \omega = \sum_{i\geq 0} \delta_{(w_i^* - x, u_i)}. \tag{3.43}$$

We let t_x, $x \in \mathbb{Z}^d$, stand for the canonical shifts of $\{0,1\}^{\mathbb{Z}^d}$.

Proposition 3.4.

(i) ν *is invariant under translations* τ_x *of* W^* *for any* $x \in \mathbb{Z}^d$.

(ii) $\mathbb{P}$ *is invariant under translation* τ_x *of* Ω *for any* $x \in \mathbb{Z}^d$.

(iii) For any $u \geq 0$*, the translation maps* $(t_x)_{x\in\mathbb{Z}^d}$ *define a measure preserving ergodic flow on* $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{Y}, Q^u)$.

Proof. The proofs of parts (i), (ii) and of the fact that $(t_x)_{x\in\mathbb{Z}^d}$ is measure preserving flow are left as an exercise. They can be found in [14, (1.28) and Theorem 2.1]. We will only show the ergodicity, as its proof is instructive.

As we know that (t_x) is a measure preserving flow, to prove the ergodicity we only need to show that it is mixing, that is for any $\mathsf{A} \subset\subset \mathbb{Z}^d$ and for any $[0,1]$-valued $\sigma(Y_x : x \in \mathsf{A})$-measurable function f on $\{0,1\}^{\mathbb{Z}^d}$, one has

$$\lim_{|x|\to\infty} E^{Q^u}[f\, f \circ t_x] = E^{Q^u}[f]^2 \tag{3.44}$$

In view of (3.33), (3.44) will follow once we show that for any $\mathsf{A} \subset\subset \mathbb{Z}^d$ and any $[0,1]$-valued measurable function F on the set of finite point measures on W_+ endowed with the canonical σ-field,

$$\lim_{|x|\to\infty} \mathbb{E}[F(\mu_{\mathsf{A},u}) F(\mu_{\mathsf{A},u}) \circ \tau_x] = \mathbb{E}[F(\mu_{\mathsf{A},u})]^2. \tag{3.45}$$

As, due to definition of τ_x and $\mu_{\mathsf{A},u}$, there exists a function G with similar properties as F, such that $F(\mu_{\mathsf{A},u}) \circ \tau_x = G(\mu_{\mathsf{A}+x,u})$, (3.45) follows from the next lemma.

Lemma 3.5. *Let $u \geq 0$ and A_1 and A_2 be finite disjoint subsets of $\mathbb{Z}^d$. Let F_1 and F_2 be $[0,1]$-valued measurable functions on the set of finite point-measures on W_+ endowed with its canonical σ-field. Then*

$$\begin{aligned} &\left|\mathbb{E}[F_1(\mu_{\mathsf{A}_1,u})\, F_2(\mu_{\mathsf{A}_2,u})] - \mathbb{E}[F_1(\mu_{\mathsf{A}_1,u})]\, \mathbb{E}[F_2(\mu_{\mathsf{A}_2,u})]\right| \\ &\qquad \leq 4u\,\mathrm{cap}(\mathsf{A}_1)\,\mathrm{cap}(\mathsf{A}_2) \sup_{x\in\mathsf{A}_1, y\in\mathsf{A}_2} g(x-y). \end{aligned} \tag{3.46}$$

Proof. We write $\mathsf{A} = \mathsf{A}_1 \cup \mathsf{A}_2$ and decompose the Poisson point process $\mu_{\mathsf{A},u}$ into four point processes on $(W_+, \mathcal{W}_+)$ as follows:

$$\mu_{\mathsf{A},u} = \mu_{1,1} + \mu_{1,2} + \mu_{2,1} + \mu_{2,2}, \tag{3.47}$$

where

$$\begin{aligned} \mu_{1,1}(\mathrm{d}w) &= 1\{X_0 \in \mathsf{A}_1, H_{\mathsf{A}_2} = \infty\}\mu_{\mathsf{A},u}(\mathrm{d}w), \\ \mu_{1,2}(\mathrm{d}w) &= 1\{X_0 \in \mathsf{A}_1, H_{\mathsf{A}_2} < \infty\}\mu_{\mathsf{A},u}(\mathrm{d}w), \\ \mu_{2,1}(\mathrm{d}w) &= 1\{X_0 \in \mathsf{A}_2, H_{\mathsf{A}_1} < \infty\}\mu_{\mathsf{A},u}(\mathrm{d}w), \\ \mu_{2,2}(\mathrm{d}w) &= 1\{X_0 \in \mathsf{A}_2, H_{\mathsf{A}_1} = \infty\}\mu_{\mathsf{A},u}(\mathrm{d}w),. \end{aligned} \tag{3.48}$$

In words, the support of $\mu_{1,1}$ are trajectories in the support of $\mu_{\mathsf{A},u}$ which enter A_1 but not A_2, the support $\mu_{1,2}$ are trajectories that enter first A_1 and then A_2, and similarly $\mu_{2,1}$, $\mu_{2,2}$.

The $\mu_{i,j}$'s are independent Poisson point processes, since they are supported on disjoint sets (recall that A_1 and A_2 are disjoint). Their corresponding intensity measures are given by

$$\begin{aligned} &u\,1\{X_0 \in \mathsf{A}_1, H_{\mathsf{A}_2} = \infty\}P_{e_\mathsf{A}}, \\ &u\,1\{X_0 \in \mathsf{A}_1, H_{\mathsf{A}_2} < \infty\}P_{e_\mathsf{A}}, \\ &u\,1\{X_0 \in \mathsf{A}_2, H_{\mathsf{A}_1} < \infty\}P_{e_\mathsf{A}}, \\ &u\,1\{X_0 \in \mathsf{A}_2, H_{\mathsf{A}_1} = \infty\}P_{e_\mathsf{A}}. \end{aligned} \tag{3.49}$$

We observe that $\mu_{\mathsf{A}_1,u} - \mu_{1,1} - \mu_{1,2}$ is determined by $\mu_{2,1}$ and therefore independent of $\mu_{1,1}, \mu_{2,2}$ and $\mu_{1,2}$. In the same way, $\mu_{\mathsf{A}_2,u} - \mu_{2,2} - \mu_{2,1}$ is independent of $\mu_{2,2}, \mu_{2,1}$ and $\mu_{1,1}$. We can therefore introduce the auxiliary Poisson processes $\mu'_{2,1}$ and $\mu'_{1,2}$ such that they have the same law as $\mu_{\mathsf{A}_1,u}-\mu_{1,1}-\mu_{1,2}$ and

$\mu_{\mathsf{A}_2,u}-\mu_{2,2}-\mu_{2,1}$ respectively, and $\mu'_{2,1}$, $\mu'_{1,2}$, $\mu_{i,j}$, $1\le i,j\le 2$ are independent. Then

$$\begin{aligned}\mathbb{E}[F_1(\mu_{\mathsf{A}_1,u})] &= \mathbb{E}[F_1((\mu_{\mathsf{A}_1,u}-\mu_{1,1}-\mu_{1,2})+\mu_{1,1}+\mu_{1,2})]\\ &= \mathbb{E}[F_1(\mu'_{2,1}+\mu_{1,1}+\mu_{1,2})],\end{aligned} \tag{3.50}$$

and in the same way

$$\mathbb{E}[F_2(\mu_{\mathsf{A}_2})] = \mathbb{E}[F_2(\mu'_{1,2}+\mu_{2,2}+\mu_{2,1})]. \tag{3.51}$$

Using (3.50), (3.51) and the independence of the Poisson processes $\mu'_{2,1}+\mu_{1,1}+\mu_{1,2}$ and $\mu'_{1,2}+\mu_{2,2}+\mu_{2,1}$ we get

$$\mathbb{E}[F_1(\mu_{\mathsf{A}_1})]\,\mathbb{E}[F_2(\mu_{\mathsf{A}_2})] = \mathbb{E}[F_1(\mu'_{2,1}+\mu_{1,1}+\mu_{1,2})F_2(\mu'_{1,2}+\mu_{2,2}+\mu_{2,1})]. \tag{3.52}$$

From (3.52) we see that

$$\begin{aligned}&|\mathbb{E}[F_1(\mu_{\mathsf{A}_1})\,F_2(\mu_{\mathsf{A}_2})]-\mathbb{E}[F_1(\mu_{\mathsf{A}_1})]\,\mathbb{E}[F_2(\mu_{\mathsf{A}_2})]|\\ &\quad\le P[\mu'_{2,1}\neq 0 \text{ or } \mu'_{1,2}\neq 0 \text{ or } \mu_{2,1}\neq 0 \text{ or } \mu_{1,2}\neq 0]\\ &\quad\le 2(\mathbb{P}[\mu_{2,1}\neq 0]+\mathbb{P}[\mu_{1,2}\neq 0])\\ &\quad\le 2u(P_{e_{\mathsf{A}}}[X_0\in\mathsf{A}_1, H_{\mathsf{A}_2}<\infty]+P_{e_{\mathsf{A}}}[X_0\in\mathsf{A}_2, H_{\mathsf{A}_1}<\infty]).\end{aligned} \tag{3.53}$$

We now bound the two last terms in the above equation

$$\begin{aligned}P_{e_{\mathsf{A}_1\cup\mathsf{A}_2}}[X_0\in\mathsf{A}_1, H_{\mathsf{A}_2}<\infty] &\le \sum_{x\in\mathsf{A}_1} e_{\mathsf{A}_1}(x)P_x[H_{\mathsf{A}_2}<\infty]\\ &= \sum_{x\in\mathsf{A}_1, y\in\mathsf{A}_2} e_{\mathsf{A}_1}(x)g(x,y)e_{\mathsf{A}_2}(y)\\ &\le \mathrm{cap}(\mathsf{A}_1)\ \mathrm{cap}(\mathsf{A}_2)\sup_{x\in\mathsf{A}_1,\, y\in\mathsf{A}_2} g(x,y).\end{aligned} \tag{3.54}$$

A similar estimate holds for $P_{e_{\mathsf{A}_1\cup\mathsf{A}_2}}[X_0\in\mathsf{A}_2, H_{\mathsf{A}_1}<\infty]$ and the lemma follows. □

As (3.45) follows easily from Lemma 3.5, the proof of Proposition 3.4 is completed. □

Proposition 3.4(iii) has the following standard corollary.

Corollary 3.6 (zero-one law). *Let $A \in \mathcal{Y}$ be invariant under the flow $(t_x : x \in \mathbb{Z}^d)$. Then, for any $u \geq 0$,*

$$Q^u[A] = 0 \text{ or } 1. \tag{3.55}$$

In particular, the event

$$\mathrm{Perc}(u) := \{\omega \in \Omega : \mathcal{V}^u(\omega) \text{ contains an infinite connected component}\}, \tag{3.56}$$

satisfies for any $u \geq 0$

$$\mathbb{P}[\mathrm{Perc}(u)] = 0 \text{ or } 1. \tag{3.57}$$

Proof. The first statement follows from the ergodicity by usual techniques. The second statement follows from

$$\mathbb{P}[\mathrm{Perc}(u)] = Q^u\Big[\Big\{y \in \{0,1\}^{\mathbb{Z}^d} : \begin{array}{l} y \text{ contains an infinite} \\ \text{connected component of 1's} \end{array}\Big\}\Big] \tag{3.58}$$

and the fact that the event on the right-hand side is in $\mathcal{Y}$ and t_x invariant. □

We now let

$$\eta(u) = \mathbb{P}[0 \text{ belongs to an infinite connected component of } \mathcal{V}^u], \tag{3.59}$$

it follows by standard arguments that

$$\eta(u) > 0 \iff \mathbb{P}[\mathrm{Perc}(u)] = 1. \tag{3.60}$$

In particular defining

$$u_\star = \sup\{u \geq 0 : \eta(u) > 0\}, \tag{3.61}$$

we see than the random interlacement model exhibits a phase transition at $u = u_\star$. The non-trivial issue is of course to deduce that $0 < u_\star < \infty$ which we will (partially) do in the next lecture.

Let us now collect some important properties of random interlacements. In doing this, we will try to draw a parallel between this process and Bernoilli percolation defined as follows. Fixed $p \in [0,1]$, we define in some probability space R^p a collection of i.i.d random variables $(Y_x)_{x \in \mathbb{Z}^d}$. We will say that a given site x is open if $Y_x = 1$, otherwise we say that it is closed. Let us now try to understand how this random configuration in $\{0,1\}^{\mathbb{Z}^d}$ compares with the one obtained by the measure Q^u defined in (3.29).

The first important observation is that under the measure R^p every configuration inside a finite set A has positive probability. This is not the case with Q^u, as we note in the following

Remark 3.7. Using (3.29) and the definition of Ω, we conclude that

$$\begin{array}{c}\text{for every } u \geq 0, \text{ almost surely under the measure } Q^u, \text{ the set} \\ \{x \in \mathbb{Z}^d; Y_x = 1\} \text{ has no finite connected components.}\end{array} \tag{3.62}$$

One particular consequence of this fact is that the random interlacements measure Q^u will not satisfy the so-called finite energy property. We say that a measure Q on $\{0,1\}^{\mathbb{Z}^d}$ satisfies the finite energy property if

$$0 < Q(Y_y = 1 | Y_z, z \neq y) < 1, \; Q\text{-a.s., for all } y \in \mathbb{Z}^d; \tag{3.63}$$

for more details, see [5] (Section 12). Intuitively speaking, this says that not all configurations on a finite set have positive probability under the measure Q^u. Due to the absence of this property, some percolation techniques, such as Burton and Keane's uniqueness argument, will not be directly applicable to Q^u.

Remark 3.8. Another important technique in Bernoulli independent percolation is the so-called Peierls-type argument. This argument makes use of the so-called $*$-paths defined as follows. We say that a sequence $x_0, x_1, \ldots, x_n$ is a $*$-path if the supremum norm $|x_i - x_{i+1}|_\infty$ equals one for every $i = 0, \ldots, n-1$. The Peierl's argument strongly relies on the fact that, for p sufficiently close to one,

$$\begin{array}{c}\text{the probability that there is some } *\text{-path of 0's (closed sites)} \\ \text{from the origin to } B(0,2N) \text{ decays exponentially with } N.\end{array} \tag{3.64}$$

This can be used for instance to show that for such values of p there is a positive probability that the origin belongs to an infinite connected component of 1's (open sites).

This type of argument fails in the case of random interlacements. Actually, using (3.38) together with (3.63) we obtain that

$$\text{for every } u > 0\text{, with positive probability there is an infinite } *\text{-path of 0's starting from the origin.} \tag{3.65}$$

It is actually possible to show that the probability to find a long planar $*$-path decays, see Chapter 4. However, this is done using a different technique than in Peierl's argument.

In the next lemma we show that Bernoulli percolation does not dominate (or is dominated) by random interlacements. For measures Q and Q' in $\{0,1\}^{\mathbb{Z}^d}$, we say that Q dominates Q' if

$$\int f \, \mathrm{d}Q \geq \int f \, \mathrm{d}Q', \text{ for every increasing function } f : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}_+. \tag{3.66}$$

Lemma 3.9. *For any values of $p \in (0,1)$ and $u > 0$, the measure Q^u neither dominates nor is dominated by R^p.*

Proof. We start by showing that Q^u is not dominated R^p. For this, consider the function $f = \mathbf{1}\{Y_x = 1 \text{ for every } x \in [0,L)^d\}$. This fuction is clearly monotone increasing and for every choice of p,

$$\int f \, \mathrm{d}R^p = p^{L^d}. \tag{3.67}$$

While for every $u > 0$,

$$\int f \, \mathrm{d}Q^u = \exp\{-u \, \mathrm{cap}([0,L)^d)\}, \text{ see (3.37).} \tag{3.68}$$

Which by Exercise 5.17 is at most $\exp\{-cuL^{d-2})\}$. From these considerations, it is clear that for any $u > 0$ and any $p \in (0,1)$ we have $\int f \mathrm{d}R^p < \int f \mathrm{d}Q^u$ for some L large enough. This finishes the proof that R^p does not dominate Q^u.

Let us now turn to the proof that R^p is not dominated by Q^u. For this, we consider the function $g = \mathbf{1}\{Y_x = 1 \text{ for some } x \in [0,L)^d\}$, which is clearly increasing and satisfies

$$\int g \, \mathrm{d}R^p = 1 - (1-p)^{L^d}. \tag{3.69}$$

In order to estimate the integral of g with respect to Q^u, we observe that if the whole cube $[0,L)^d$ is covered by the random interlacements, then $g = 0$. Therefore, writing A for $[0,L)^d$,

$$\int g \,\mathrm{d}Q^u \leq 1 - \mathbb{P}[\mathsf{A} \subset \mathcal{I}^u] \overset{(3.34)}{=} 1 - \mathbb{P}^u_{\mathsf{A}}\Big[\mathsf{A} \subset \bigcup_{w_{+,i} \in \mathrm{supp}(\omega_+)} \mathrm{Range}(w_{+,i})\Big]. \tag{3.70}$$

In order to evaluate the above probability, let us first condition on the number of points in the support of ω_+.

$$\begin{aligned} \mathbb{P}^u_{\mathsf{A}}\Big[\mathsf{A} \subset \bigcup_{w_{+,i} \in \mathrm{supp}(\omega_+)} \mathrm{Range}(w_{+,i})\Big] &\geq \mathbb{P}^u_{\mathsf{A}}\big[\omega_+(W_+) = \lfloor \log^2(L) L^{d-2} \rfloor\big] \\ &\quad \times P^{\otimes \lfloor \log^2(L) L^{d-2} \rfloor}_{e_{\mathsf{A}}/\operatorname{cap}(A)}\Big[\mathsf{A} \subset \bigcup_{i=1}^{\lfloor \log^2(L) L^{d-2} \rfloor} \mathrm{Range}(X_i)\Big] \end{aligned} \tag{3.71}$$

where the above probability is the independent product of $\lfloor \log^2(L) L^{d-2} \rfloor$ simple random walks X_i's, starting with distribution $e_{\mathsf{A}}/\operatorname{cap}(A)$.

Let us first evaluate the first term, corresponding to the Poisson distribution of $\omega_+(W_+)$. For this, we write $\alpha = u \operatorname{cap}(\mathsf{A})$ and $\beta = \lfloor \log^2(L) L^{d-2} \rfloor$. Then, using de Moivre-Stirling's approximation, we obtain that the left term in the above equation is

$$\frac{e^{-\alpha} \alpha^\beta}{\beta!} \geq c \frac{e^{-\alpha+\beta}}{\sqrt{\beta}} \Big(\frac{\alpha}{\beta}\Big)^\beta$$

and using Exercises 5.17 and 5.18, for L lager then some c_u,

$$\begin{aligned} &\geq \exp\{-c_u L^{d-2} + \lfloor (\log^2 L) L^{d-2} \rfloor\} \Big(\frac{c_u}{\log^2 L}\Big)^\beta \\ &\geq \Big(\frac{c_u}{\log^2 L}\Big)^\beta \geq \exp\big\{ - c_u \log(\log^2 L) \cdot (\log^2 L) L^{d-2}\big\} \\ &\geq \exp\big\{ - c_u (\log^3 L) L^{d-2}\big\}. \end{aligned} \tag{3.72}$$

Let us now bound the second term in (3.71). Fix first some $z \in \mathsf{A}$ and estimate

$$\begin{aligned} P^{\otimes \beta}_{e_{\mathsf{A}}/\operatorname{cap}(A)}\big[z \in \cup_{i=1}^{\beta} \mathrm{Range}(X_i)\big] &= 1 - \big(P_{e_{\mathsf{A}}/\operatorname{cap}(A)}\big[z \notin \mathrm{Range}(X_1)\big]\big)^\beta \\ &\overset{(3.36)}{\geq} 1 - \big(1 - cL^{2-d}\big)^{c(\log^2 L) L^{d-2}} \geq 1 - e^{-c \log^2 L}. \end{aligned}$$

Therefore, by a simple union bound, we obtain that therm in the right hand side of (3.71) is bounded from below by $1/2$ as soon as L is large enough depending on u. Putting this fact together with (3.71) and (3.72), we obtain that

$$\int g \, \mathrm{d}Q^u \leq 1 - c \exp\big\{ - c_u (\log^3 L) L^{d-2} \big\}, \tag{3.73}$$

which is smaller than the right hand side of (3.69) for L large enough depending on p and u. This proves that Q^u does not dominate R^p for any values of $p \in (0,1)$ or $u > 0$, finishing the proof of the lemma. $\square$

Chapter 4

Renormalization

In this section we are going to prove that $u^* > 0$ for d sufficiently large ($d \geq 7$ is enough). This only establishes one side of the non-triviality of u_*, but ilustrates the multi-scale renormalization, which is employed in several other problems of dependent percolation and particle systems. The biggest advantage of the renormalization scheme is that it does not enter too much on the kind of dependence involved in the problem. Roughly speaking, only having a control on the decay of dependence (such as in Lemma 3.5) we may have enough to obtain global statements about the measure under consideration.

To make more sense of the control of dependences established in Lemma 3.5, we need to control the decay of the Green's function for the simple random walk on $\mathbb{Z}^d$. We quote from Theorem 1.5.4 of [8] that

$$g(x) \leq c|x|^{2-d}. \tag{4.1}$$

The main result of this section is

Theorem 4.1. *For $d \geq 7$, we have that $u_* > 0$.*

Proof. The proof we present here follows the arguments of Proposition 4.1 in [14] with some minor modifications.

We will use this bound in the renormalization argument we mentioned above. This renormalization will take place on $\mathbb{Z}^2 \subset \mathbb{Z}^d$, which is identifyed by the isometry $(x_1, x_2) \mapsto (x_1, x_2, 0, \ldots, 0)$. Throughout the text we make no distinction between $\mathbb{Z}^2$ and its isometric copy inside $\mathbb{Z}^d$.

We say that $\tau : \{0, \cdots, n\} \to \mathbb{Z}^2$ is a $*$-path if

$$|\tau(k+1) - \tau(k)|_\infty = 1, \text{ for all } k \in \{0, \cdots, n-1\},$$

where $|p|_\infty$ is the maximum of the absolute value of the two coordinates of $p \in \mathbb{Z}^2$. Roughly speaking, the strategy of the proof is to prove that with positive probability there is no $*$-path in $\mathcal{I}^u \cap \mathbb{Z}^2$ surrounding the origin. This will imply by a duality argument that there exists an infinite connected componenet in $\mathcal{V}^u$.

We now define a sequence of non-negative integers which will represent the scales involved in the renormalization proceedure. For any $L_0 \geq 2$, let

$$\begin{gathered} L_{n+1} = l_n L_n, \text{ for every } n \geq 0, \\ \text{where } l_n = 100 \lfloor L_n^a \rfloor \text{ and } a = \tfrac{1}{1000}. \end{gathered} \tag{4.2}$$

Here $\lfloor a \rfloor$ represent the largest integer smaller or equal to a.

In what follows, we will consider a sequence of boxes in $\mathbb{Z}^2$ of size L_n, but before, let us consider the set of indices

$$J_n = \{n\} \times \mathbb{Z}^2, \text{ for } n \geq 0. \tag{4.3}$$

For $m = (n, q) \in J_n$, we consider the box

$$D_m = (L_n q + [0, L_n)^2) \cap \mathbb{Z}^2, \tag{4.4}$$

And also

$$\widetilde{D}_m = \bigcup_{i,j \in \{-1,0,1\}} D_{(n, q+(i,j))}. \tag{4.5}$$

As we mentioned, our strategy is to prove that the probability of finding a $*$-path in the set $\mathcal{I}^u \cap \mathbb{Z}^2$ that separates the origin from infinite in $\mathbb{Z}^2$ is smaller than one. We do this by bounding the probabilities of the following crossing events

$$B_m^u = \left\{ \begin{array}{c} \omega \in \Omega; \text{there exists a } *\text{-path in } \mathcal{I}^u \cap \mathbb{Z}^2 \\ \text{connecting } D_m \text{ to the complement of } \widetilde{D}_m \end{array} \right\}, \tag{4.6}$$

where $m \in J_n$. For $u > 0$, we write

$$q_n^u = \mathbb{P}[B_{(n,0)}^u] \overset{\text{Proposition 3.4}}{=} \sup_{m \in J_n} \mathbb{P}[B_{(n,m)}^u]. \tag{4.7}$$

In order to show that for u small enough q_n^u decays with n, we are going to obtain an induction relation between q_n^u and q_{n+1}^u (that were defined in terms

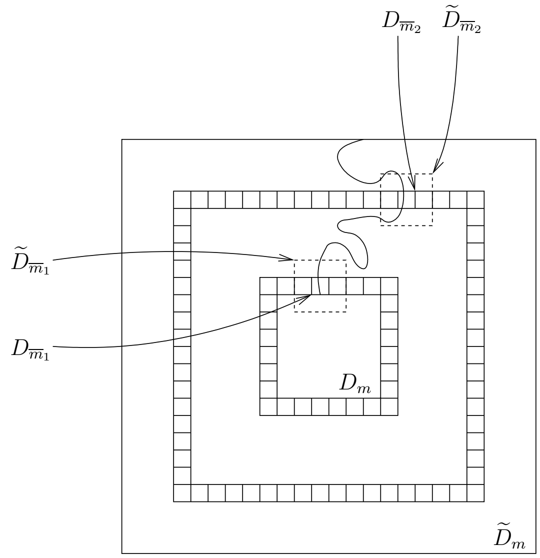

Figure 4.1: The figure shows all the boxes with indexes in $\mathcal{K}_1$ and $\mathcal{K}_2$. Note that the event B^u_m implies $B^u_{\overline{m}_1}$ and $B^u_{\overline{m}_2}$ for some $\overline{m}_1 \in \mathcal{K}_1$ and $\overline{m}_2 \in \mathcal{K}_2$.

of two different scales). For this we consider, for a fixed $m \in J_{n+1}$, the indexes of boxes in the scale n that are in the "boundary of D_m". More precisely

$$\mathcal{K}^m_1 = \{\overline{m}_1 \in J_n; D_{\overline{m}_1} \subset D_m \text{ and } D_{\overline{m}_1} \text{ is neighbor of } \mathbb{Z}^2 \setminus D_m\}. \tag{4.8}$$

And the indexes of boxes at the scale n and that have some point at distance $L_{n+1}/2$ of D_m

$$\mathcal{K}^m_2 = \{\overline{m}_2 \in J_n; D_{\overline{m}_2} \cap \ \{x \in \mathbb{Z}^2; d_{\mathbb{Z}^2}(z, D_m) = L_{n+1}/2\} \neq \emptyset\}. \tag{4.9}$$

The boxes associated with the two sets of indexes above are shown in Figure 4.1. In this figure we also illustrate that the event B^u_m implies the occurrence of both $B^u_{\overline{m}_1}$ and $B^u_{\overline{m}_2}$ for some choice of $\overline{m}_1 \in \mathcal{K}^m_1$ and $\overline{m}_2 \in \mathcal{K}^m_2$.

This, with a rough counting argument, allows us to conclude that

$$q^u_m \leqslant c l^2_n \sup_{\substack{\overline{m}_1 \in \mathcal{K}^m_1 \\ \overline{m}_2 \in \mathcal{K}^m_2}} \mathbb{P}[B^u_{\overline{m}_1} \cap B^u_{\overline{m}_2}], \text{ for all } u \geqslant 0. \tag{4.10}$$

We now want to control the dependence of the process in the two boxes $\widetilde{D}_{\overline{m}_1}$

and $\widetilde{D}_{\overline{m}_2}$. For this we will use Lemma 3.5, which provides that

$$\begin{aligned}\mathbb{P}[B^u_{\overline{m}_1} \cap B^u_{\overline{m}_1}] \leq \mathbb{P}[B^u_{\overline{m}_1}]\mathbb{P}[B^u_{\overline{m}_1}] + 4u\,\mathrm{cap}(\widetilde{D}_{\overline{m}_1})\,\mathrm{cap}(\widetilde{D}_{\overline{m}_2}) \sup_{x\in\widetilde{D}_{\overline{m}_1}, y\in\widetilde{D}_{\overline{m}_2}} g(x-y) \\ \overset{(4.1)}{\leqslant} (q^u_n)^2 + cL_n^2\frac{L_n^2}{L_{n+1}^5}\end{aligned}$$

where we assumed in the last step that $u \leqslant 1$. Using (4.10) and taking the supremum over $m \in J_{n+1}$, we conclude that

$$q^u_{n+1} \leqslant cl_n^2\left((q^u_n)^2 + L_n^4L_{n+1}^{-5}\right). \tag{4.11}$$

With help of this recurrence relation, we prove the next Lemma, which shows that for some choice of L_0 and for u taken small enough, q^u_n goes to zero sufficiently fast with n.

Lemma 4.2. *There exist L_0 and $\bar{u} = \bar{u}(L_0) > 0$, such that*

$$q^u_n \leqslant \frac{c_0}{l_n^2L_n^{1/2}} \tag{4.12}$$

for every $u < \bar{u}$.

Proof of Lemma 4.2. We define the sequence

$$b_n = c_0l_n^2q^u_n, \text{ for } n \geqslant 0. \tag{4.13}$$

The equation (4.11) can now be rewritten as

$$b_{n+1} \leqslant c\left(\left(\frac{l_{n+1}}{l_n}\right)^2 b_n^2 + (l_{n+1}l_n)^2L_n^4L_{n+1}^{-5}\right), \text{ for } n \geqslant 0. \tag{4.14}$$

With (4.2) one concludes that $(l_{n+1}l_n)^2 \leqslant cL_n^{2a}L_{n+1}^{2a} \leqslant cL_n^{4a+2a^2}$. Inserting this in (4.14) and using again (4.2), we obtain

$$b_{n+1} \leqslant c_1(L_n^{2a^2}b_n^2 + L_n^{2a^2-a-1}) \leqslant c_1L_n^{2a^2}(b_n^2 + L_n^{-1}). \tag{4.15}$$

We use this to show that, if for some $L_0 > (2c_1)^4$ and $u \leqslant 1$ we have $b_n \leqslant L_n^{-1/2}$, then the same inequality also holds for $n+1$. Indeed, supposing $b_n \leqslant L_n^{-1/2}$, we have

$$b_{n+1} \leqslant 2c_1L_n^{2a^2-1} \overset{(4.2)}{\leqslant} 2c_1L_{n+1}^{-1/2}L_n^{1/2(1+a)+2a^2-1} \overset{(4.2)}{\leqslant} 2c_1L_{n+1}^{-1/2}L_0^{-1/4} \leqslant L_{n+1}^{-1/2}. \tag{4.16}$$

Which is the statement of the lemma. So all we still have to prove is that $b_0 \leqslant L_0^{-1/2}$ for $L_0 > (2c_1)^4$ and small enough u. Indeed,

$$\begin{aligned} b_0 &\overset{(4.13)}{=} c_0 l_0^2 q_0^u \leqslant c_0 l_0^2 \sup_{m \in J_0} \mathbb{P}[\mathcal{I}^u \cap \widetilde{D}_m \neq \emptyset] \\ &\leqslant c_1 L_0^{2a+2} \sup_{x \in V} \mathbb{P}[x \in \mathcal{I}^u] \overset{(3.37)}{\leqslant} c_1 L_0^{2a+2}(1 - e^{-\operatorname{cap}(\{x\})u}). \end{aligned} \tag{4.17}$$

For some $L_0 > (2c_1)^4$, we take $u(L_0)$ small enough such that $b_0 \leqslant L_0^{-1/2}$ for any $u \leqslant u(L_0)$. This concludes the proof of Lemma 4.2 □

We now use this lemma to show that with positive probability, one can find an infinite connection from $(0,0)$ to infinite in the set $\mathcal{V}^u \cap \mathbb{Z}^d$. For this we choose L_0 and $u < u(L_0)$ as in the lemma. Writing B_M for the set $[-M, M] \times [-M, M] \subset \mathbb{Z}^2$, we have

$$\begin{aligned} 1 - \eta(u, (0,0)) &\leqslant \mathbb{P}[(0,0) \text{ is not in an infinite component of } \mathcal{V}^u \cap \mathbb{Z}^2] \\ &\leqslant \mathbb{P}[\mathcal{I}^u \cap B_M \neq \emptyset] + \mathbb{P}\Big[\begin{array}{c} \text{there is a } *\text{-path in } \mathbb{Z}^2 \setminus B_M \\ \text{surrounding the point } (0,0) \text{ in } \mathbb{Z}^2 \end{array} \Big] \\ &\leqslant \big(1 - \exp(-u \cdot \operatorname{cap}(B_M))\big) \\ &\quad + \sum_{n \geqslant n_0} \mathbb{P}\Big[\begin{array}{c} \mathcal{I}^u \cap \mathbb{Z}^2 \setminus B_M \text{ contains a } *\text{-path surrounding } (0,0) \text{ and} \\ \text{passing through some point in } [L_n, L_{n+1} - 1] \times \{0\} \in \mathbb{Z}^2 \end{array} \Big] \end{aligned} \tag{4.18}$$

The last sum can be bounded by $\sum_{n \geqslant n_0} \sum_m \mathbb{P}[B_m^u]$ where the index m runs over all labels of boxes D_m at level n that intersect $[L_n, L_{n+1} - 1] \times \{0\} \subset \mathbb{Z}^2$. Since the number of such m's is at most $l_n \leqslant c L_n^a$,

$$1 - \eta(u, (0,0)) \leqslant c L_{n_0}^2 u + \sum_{n \geqslant n_0} c L_n^a L_n^{-1/2} \overset{(4.2)}{\leqslant} c(L_{n_0}^2 u + \sum_{n \geqslant n_0} L_n^{-1/4}). \tag{4.19}$$

Choosing n_0 large and $u \leqslant u(L_0, n_0)$, we obtain that the percolation probability is positive. So that $u_* > 0$ finishing the proof of Theorem 4.1. □

Chapter 5

Locally tree-like graphs

In the previous lectures we have studied the random walk on the torus and the corresponding random interlacement on $\mathbb{Z}^d$. We have seen that in that case many interesting questions are still open, including the existence of the phase transition in the behavior of the vacant set of the random walk, and its correspondence to the phase transition of random interlacement. Answering these questions requires a better control of the random interlacement in both subcritical and supercritical phase which is not available at present.

In this chapter we are going to explore random interlacement on graphs where such control is available, namely on trees. We will then explain how such control can be used to show the phase transition for the vacant set of random walk on finite 'locally tree-like' graphs, and to give the equivalence of critical points in both models.

5.1 Random interlacement on trees

We start by considering random interlacement on trees. We will show that vacant clusters of this model behave like Galton-Watson trees, which allows for many exact computations. As in this lecture notes we only deal with random walks and random interlacement on regular graphs, we restrict our attention to regular trees only.

Let $\mathbb{T}_d$ be infinite d-regular tree, $d \geq 3$, for which the simple random walk is transient, see Exercise 5.19. We may therefore define random interlacement on $\mathbb{T}_d$ similarly as we deed for $\mathbb{Z}^d$, as we discuss below.

We write P_x for the law of the canonical simple random walk (X_n) on $\mathbb{T}_d$ started at $x \in \mathbb{T}_d$, and denote by e_K, $K \subset\subset \mathbb{T}_d$ the equilibrium measure,

$$e_K(x) = P_x[\tilde{H}_K = \infty]\mathbf{1}\{x \in K\}. \tag{5.1}$$

Observe that if K is connected, e_K can be easily computed. Indeed, on $\mathbb{T}_d$, under P_x, the process $d(X_n, x)$ has the same law as a drifted random walk on $\mathbb{N}$ started at 0. If not at 0, this walk jumps to the right with probability $(d-1)/d$ and to the left with probability $1/d$; at 0 it goes always to the right. Using standard computation for the random walk with drift, see e.g. [18], Lemma 1.24 (see also Exercise 5.19), it is then easy to show that

$$P_x[\tilde{H}_x = \infty] = P_y[H_x = \infty] = \frac{d-2}{d-1}, \tag{5.2}$$

for every neighbor y of x. For K connected, we then get

$$e_K(x) = \frac{1}{d}\#\{y : y \sim x, y \notin K\}\frac{d-2}{d-1}, \tag{5.3}$$

where the first two terms give the probability that the first step of the random walk exists K.

We consider spaces W_+, W, W^*, Ω and measures Q_K defined similarly as in Section 3.1, replacing $\mathbb{Z}^d$ by $\mathbb{T}_d$ in these definitions when appropriate. As in Theorem 3.1, it can be proved that there exists a unique σ-finite measure ν on $(W^*, \mathcal{W}^*)$ satisfying the restriction property (3.13). Using this measure, we can then construct a Poisson point process ω on $W^* \times \mathbb{R}_+$ with intensity measure $\nu(\mathrm{d}w^*) \otimes \mathrm{d}u$ and define the interlacement at level u and its vacant set as in (3.25), (3.26).

The main result of this section is the following theorem.

Theorem 5.1 ([16], Theorem 5.1). *Let $x \in \mathbb{T}_d$ and define $f_x : \mathbb{T}_d \to [0,1]$ by*

$$\begin{aligned} f_x(z) = P_z[d(X_n, x) > d(x,z) \text{ for all } n > 0] \\ \times P_z[d(X_n, x) \geq d(x,z) \text{ for all } n \geq 0]. \end{aligned} \tag{5.4}$$

Then the vacant cluster of $\mathcal{V}^u$ containing x in the random interlacement has the same law as the open cluster containing x in the independent Bernoulli site percolation on $\mathbb{T}_d$ characterized by

$$\mathrm{Prob}[z \text{ is open}] = \exp\{-u f_x(z)\}. \tag{5.5}$$

Remark 5.2. 1. Observe that on $\mathbb{T}_d$, $f_z(x)$ is the same for all $z \neq x$. Hence, the cluster of $\mathcal{V}^u$ containing x can be viewed as a Galton-Watson tree with a particular branching law in the first generation.

2. Beware that the joint law of, e.g., vacant clusters containing two points $x \neq y \in \mathbb{T}_d$ is *not* the same as in the Bernoulli percolation.

Proof. We partition the space W^* into disjoint subsets $W^{*,z}$ according to the position where $w^* \in W^*$ get closest to the given point x,

$$W^\star = \bigsqcup_{z \in \mathbb{T}_d} W^{*,z}, \tag{5.6}$$

where

$$W^{*,z} = \{w^* \in W^* : z \in \mathrm{Ran}(w^*), d(x, \mathrm{Ran}(w^*) = d(x,z))\}. \tag{5.7}$$

(The fact that $W^{*,z}$ are disjoint follows easily from the fact that $\mathbb{T}_d$ is a tree.)

As a consequence of disjointness we see that the random variables $\omega(W^{*,z} \times [0,u])$ are independent. We may thus define independent site Bernoulli percolation on $\mathbb{T}_d$ by setting

$$Y_z^u(\omega) = \mathbf{1}\{\omega(W^{*,z} \times [0,u]) \geq 1\} \qquad \text{for } z \in \mathbb{T}_d. \tag{5.8}$$

By (3.9), (3.13) and (5.7), we see that

$$\mathbb{P}[Y_z^u = 0] = \exp\{-u f_x(z)\}. \tag{5.9}$$

To finish the proof of the theorem, it remains to observe that the null cluster of $(Y_\cdot^u)$ containing x coincides with the component of $\mathcal{V}^u$ containing x. The easy proof of this claim is left as exercise. □

As a corollary of Theorem 5.1 and (5.3) we obtain the value of critical point of random interlacement on $\mathbb{T}_d$ which, similarly as on $\mathbb{Z}^d$, is defined by

$$u_\star(\mathbb{T}_d) = \inf\big\{u \geq 0 : \mathbb{P}[\text{the cluster of } x \text{ in } \mathcal{V}^u \text{ is infinite}] = 0\big\}. \tag{5.10}$$

Corollary 5.3. *The critical point of the random interlacement on* $\mathbb{T}_d$ *is given by*

$$u_*(\mathbb{T}_d) = \frac{d(d-1)\log(d-1)}{(d-2)^2}. \tag{5.11}$$

Proof. For $z \neq x$, by considering drifted random walk as above (5.1), it is easy to see that

$$f_x(z) = \frac{d-2}{d-1} \times \frac{d-1}{d}\frac{d-2}{d-1} = \frac{(d-2)^2}{d(d-1)}. \tag{5.12}$$

Hence, the Galton-Watson process mentioned in Remark 5.2 has (except in the first generation) binomial offspring distribution with parameters $(d-1, \exp\{-u\frac{(d-2)^2}{d(d-1)}\})$. This Galton-Watson process is critical if the mean of its offspring distribution is equal one, implying that $u_*(\mathbb{T}_d)$ is the solution of

$$(d-1)\exp\Big\{-u\frac{(d-2)^2}{d(d-1)}\Big\} = 1, \tag{5.13}$$

yielding (5.11). □

Remark 5.4. For the previous result, the offspring distribution in the first generation is irrelevant. Using (5.1) and Theorem 5.1, it is however easy to see that (for $k = 0, \dots, d$)

$$\mathbb{P}[x \in \mathcal{V}^u] = e^{-u\,\mathrm{cap}(x)} = e^{-uf_x(x)} = e^{-u(d-2)/(d-1)}, \tag{5.14}$$

$$\mathbb{P}\big[|\mathcal{V}^u \cap \{y : y \sim x\}| = k \big| x \in \mathcal{V}^u\big] = \binom{d}{k} e^{-uk\frac{(d-2)^2}{d(d-1)}}\big(1 - e^{-u\frac{(d-2)^2}{d(d-1)}}\big)^{d-k}. \tag{5.15}$$

We will need this formulas later.

Remark 5.5. 1. Many results of this section do hold for general (weighted) trees, not only for $\mathbb{T}_d$. However, as the invariant measure of the random walk is then in general not uniform, a slight care should be taken in defining the random interlacement.

2. Apart $\mathbb{T}_d$, there is to our knowledge only one other case where the critical value of random interlacement can be computed explicitly (and is non-trivial), namely for the base graph being a Galton-Watson tree. In this case, it was shown by M. Tassy [15] that $u_\star$ is a.s. constant (i.e. 'does not depend' on the realization of the Galton-Watson tree) and can be computed as a solution to a particular equation.

5.2 Random walk on tree-like graphs

We now return to the problem of the vacant set of the random walk on finite graphs. However, instead of considering the torus as in Chapter 2 we are going to study graphs that locally look like a tree, in hope to use the results of the previous section.

Actually, the most of this section will deal with so-called *random regular graphs*. Random d-regular graph with n vertices is a graph that is chosen uniformly from the set $\mathcal{G}_{n,d}$ of all simple (i.e. without loops and multiple edges) graphs with the vertex set $V_n = [n] := \{1, \dots, n\}$ and all vertices of degree d. We let $\mathbb{P}_{n,d}$ to denote the distribution of such graph, that is the uniform distribution on $\mathcal{G}_{n,d}$.

It is well know that with probability tending to 1 as n increases, the majority of vertices in random regular graph has a neighborhood with radius $c \log n$ which is graph-isomorph to a ball in $\mathbb{T}_d$.

For a fixed graph $G = (V, \mathcal{E})$ let P^G be the law of random walk on G started from the uniform distribution and $(X_t)_{t \geq 0}$ the canonical process. As before we will be interested in the vacant set

$$\mathcal{V}^u = V \setminus \{X_t : 0 \leq t \leq u|V|\}, \tag{5.16}$$

and denote by $\mathcal{C}_{\max}$ its maximal connected component.

We will study the properties of the vacant set under the *annealed measure* $\mathbf{P}_{n,d}$ given by

$$\mathbf{P}_{n,d}(\cdot) = \int P^G(\cdot)\mathbb{P}_{n,d}(\mathrm{d}G). \tag{5.17}$$

The following theorem states that a phase transition in the behavior of the vacant set on random regular graph.

Theorem 5.6 ($d \geq 3$, $u_\star := u_*(\mathbb{T}_d)$)**.**

(a) For every $u < u_\star$ there exist constant $c(u) \in (0,1)$ such that

$$n^{-1}|\mathcal{C}_{\max}| \xrightarrow{n\to\infty} c(u) \qquad \textit{in } \mathbf{P}_{n,d}\textit{-probability.} \tag{5.18}$$

(b) When $u > u_\star$, then for every ε there is $K(u,\varepsilon) < \infty$ such that for all n large

$$\mathbf{P}_{n,d}[|\mathcal{C}_{\max}| \geq K(u,\varepsilon)\log n] \leq \varepsilon. \tag{5.19}$$

Observe that this theorem not only proves the phase transition, but also confirms that the critical point coincides with the critical point of random interlacement on $\mathbb{T}_d$. Theorem 5.6 was for the first time proved (in a weaker form but for a larger class of graphs) by [3]. We are going to use a simple proof given by Cooper and Frieze [4] which uses in a clever way the randomness of the graph. Besides being simple, this proof has an additional advantage that it can be used also in the vicinity of the critical point: By very similar techniques that we are going to present here, [2] proves that the vacant set of the random walk exhibits a double-jump behavior analogical to the maximal connected cluster in Bernoulli percolation.

Theorem 5.7.

(a) Critical window. *Let $(u_n)_{n\geq 1}$ be a sequence satisfying*

$$|n^{1/3}(u_n - u_\star)| \leq \lambda < \infty \quad \textit{for all } n \textit{ large enough.} \tag{5.20}$$

Then for every $\varepsilon > 0$ there exists $A = A(\varepsilon, d, \lambda)$ such that for all n large enough

$$\mathbf{P}_{n,d}[A^{-1}n^{2/3} \leq |\mathcal{C}^{u_n}_{\max}| \leq An^{2/3}] \geq 1 - \varepsilon. \tag{5.21}$$

(b) Above the window. *When $(u_n)_{n\geq 1}$ satisfies*

$$u_\star - u_n \xrightarrow{n\to\infty} 0, \qquad \textit{and} \qquad n^{1/3}(u_\star - u_n) \xrightarrow{n\to\infty} \infty, \tag{5.22}$$

then

$$|\mathcal{C}^{u_n}_{\max}|/n^{2/3} \xrightarrow{n\to\infty} \infty, \qquad \textit{in } \mathbf{P}_{n,d}\textit{-probability.} \tag{5.23}$$

(c) Below the window. *When $(u_n)_{n\geq 1}$ satisfies*

$$u_\star - u_n \xrightarrow{n\to\infty} 0, \qquad \textit{and} \qquad n^{1/3}(u_\star - u_n) \xrightarrow{n\to\infty} -\infty, \tag{5.24}$$

then

$$|\mathcal{C}^{u_n}_{\max}|/n^{2/3} \xrightarrow{n\to\infty} 0. \qquad \textit{in } \mathbf{P}_{n,d}\textit{-probability.} \tag{5.25}$$

We will now sketch the main steps of the proof of Theorem 5.6. Detailed proofs can be found in [4, 2].

5.2.1 Very short introduction to random graphs

We start by reviewing some properties of random regular graphs (For more about these graphs see e.g. [1, 19].) that is the graphs distributed according to $\mathbb{P}_{n,d}$. It turns out that it is easier to work with multigraphs instead of simple graphs. Therefore we introduce $\mathcal{M}_{n,d}$ for the set of all d-regular multigraphs with vertex set $[n]$.

For reasons that will be explained later, we also define random graphs with a given degree sequence $\boldsymbol{d} : [n] \to \mathbb{N}$. We will use $\mathcal{G}_{\boldsymbol{d}}$ to denote the set of graphs for which every vertex $x \in [n]$ has the degree $\boldsymbol{d}_x = \boldsymbol{d}(x)$. Similarly, $\mathcal{M}_{\boldsymbol{d}}$ stands for the set of such multigraphs; here loops are counted twice when considering the degree. $\mathbb{P}_{n,d}$ and $\mathbb{P}_{\boldsymbol{d}}$ denote the uniform distributions on $\mathcal{G}_{n,d}$ and $\mathcal{G}_{\boldsymbol{d}}$ respectively.

We first introduce the *pairing construction*, which allows to generate $\mathbb{P}_{n,d}$-distributed graphs starting from a random pairing of a set with dn elements. The same construction can be used to generate a random graph chosen uniformly at random from $\mathcal{G}_{\boldsymbol{d}}$.

We consider a sequence $\boldsymbol{d} : V_n \to \mathbb{N}$ such that $\sum_{x \in V_n} \boldsymbol{d}_x$ is even. Given such a sequence, we associate to every vertex $x \in V_n$, $\boldsymbol{d}_x$ half-edges. The set of half-edges is denoted by $H_{\boldsymbol{d}} = \{(x,i) : x \in V_n, i \in [\boldsymbol{d}_x]\}$. We write $H_{n,d}$ for the case $\boldsymbol{d}_x = d$ for all $x \in V_n$. Every perfect matching M of $H_{\boldsymbol{d}}$ (i.e. partitioning of $H_{\boldsymbol{d}}$ into $|H_{\boldsymbol{d}}|/2$ disjoint pairs) corresponds to a multigraph $G_M = (V_n, \mathcal{E}_M) \in \mathcal{M}_{\boldsymbol{d}}$ with

$$\mathcal{E}_M = \big\{\{x,y\} : \big\{(x,i),(y,j)\big\} \in M \text{ for some } i \in [\boldsymbol{d}_x],\, j \in [\boldsymbol{d}_y]\big\}. \tag{5.26}$$

We say that the matching M is simple, if the corresponding multigraph G_M is simple, that is G_M is a graph. With a slight abuse of notation, we write $\bar{\mathbb{P}}_{\boldsymbol{d}}$ for the uniform distribution on the set of all perfect matchings of $H_{\boldsymbol{d}}$, and also for the induced distribution on the set of multigraphs $\mathcal{M}_{\boldsymbol{d}}$. It is well known (see e.g. [1] or [11]) that a $\bar{\mathbb{P}}_{\boldsymbol{d}}$ distributed multigraph G conditioned on being simple has distribution $\mathbb{P}_{\boldsymbol{d}}$, that is

$$\bar{\mathbb{P}}_{\boldsymbol{d}}[G \in \cdot \,|G \in \mathcal{G}_{\boldsymbol{d}}] = \mathbb{P}_{\boldsymbol{d}}[G \in \cdot\,], \tag{5.27}$$

and that, for d constant, there is $c > 0$ such that for all n large enough

$$c < \bar{\mathbb{P}}_{n,d}[G \in \mathcal{G}_{n,d}] < 1 - c. \tag{5.28}$$

These two claims allow to deduce $\mathbb{P}_{n,d}$-a.a.s. statements directly from $\bar{\mathbb{P}}_{n,d}$-a.a.s. statements.

The main advantage of dealing with matchings is that they can be constructed sequentially: To construct a uniformly distributed perfect matching of $H_{\boldsymbol{d}}$ one samples *without replacements* a sequence $h_1, \dots, h_{|H_{\boldsymbol{d}}|}$ of elements of $H_{\boldsymbol{d}}$ in the following way. For i odd, h_i can be chosen by an arbitrary rule (which might also depend on the previous $(h_j)_{j<i}$), while if i is even, h_i must be chosen uniformly among the remaining half-edges. Then, for every $1 \le i \le |H_{\boldsymbol{d}}|/2$ one matches h_{2i} with h_{2i-1}.

It is clear from the above construction that, conditionally on $M' \subseteq M$ for a (partial) matching M' of $H_{\boldsymbol{d}}$, $M \setminus M'$ is distributed as a uniform perfect matching of $H_{\boldsymbol{d}} \setminus \{(x,i) : (x,i) \text{ is matched in } M'\}$. Since the law of the graph G_M does not depend on the labels 'i' of the half-edges, we obtain for all partial matchings M' of $H_{\boldsymbol{d}}$ the following *restriction property*,

$$\bar{\mathbb{P}}_{\boldsymbol{d}}[G_{M\setminus M'} \in \cdot \,|M \supset M'] = \bar{\mathbb{P}}_{\boldsymbol{d}'}[G_M \in \cdot], \tag{5.29}$$

where $\boldsymbol{d}'_x$ is the number of half-edges incident to x in $H_{\boldsymbol{d}}$ that are not yet matched in M', that is $\boldsymbol{d}'_x = \boldsymbol{d}_x - \big|\{\{(y_1,i),(y_2,j)\} \in M' : y_1 = x, i \in [\boldsymbol{d}_x]\}\big|$, and $G_{M\setminus M'}$ is the graph corresponding to a non-perfect matching $M \setminus M'$, defined in the obvious way.

5.2.2 Distribution of the vacant set

Instead of the vacant set, it is more suitable to consider the following object that we call *vacant graph* $\mathbf{V}^u$. It is defined by $\mathbf{V}^u = (V, \mathcal{E}^u)$ with

$$\mathcal{E}^u = \{\{x,y\} \in \mathcal{E} : x, y \in \mathcal{V}^u_G\}. \tag{5.30}$$

It is important to notice that the vertex set of $\mathbf{V}^u$ is a deterministic set V and not the random set $\mathcal{V}^u$, in particular $\mathbf{V}^u$ is not the graph induced by $\mathcal{V}^u$ in G. Observe however that the maximal connected component of the vacant set

$\mathcal{C}_{\max}$ (defined before in terms of the graph induced by $\mathcal{V}^u$ in G) coincides with the maximal connected component of the vacant graph $\mathbf{V}^u$ (except when $\mathcal{V}^u$ is empty, but this difference can be ignored in our investigations).

We use $\mathcal{D}^u : V \to \mathbb{N}$ to denote the (random) degree sequence of $\mathbf{V}^u$, and write $Q^u_{n,d}$ for the distribution of this sequence under the annealed measure $\bar{\mathbf{P}}_{n,d}$, defined by $\bar{\mathbf{P}}_{n,d}(\cdot) := \int P^G(\cdot)\bar{\mathbb{P}}_{n,d}(\mathrm{d}G)$.

The following important but simple observation due to [4] allows to reduce questions on the properties of the vacant set $\mathcal{V}^u$ of the random walk on random regular graphs to questions on random graphs with given degree sequences.

Proposition 5.8 (Lemma 6 of [4])**.** *For every $u \geq 0$, the distribution of the vacant graph $\mathbf{V}^u$ under $\bar{\mathbf{P}}_{n,d}$ is given by $\bar{\mathbb{P}}_{\boldsymbol{d}}$ where $\boldsymbol{d}$ is sampled according to $Q^u_{n,d}$, that is*

$$\bar{\mathbf{P}}_{n,d}[\mathbf{V}^u \in \cdot\,] = \int \bar{\mathbb{P}}_{\boldsymbol{d}}[G \in \cdot\,]Q^u_{n,d}(\mathrm{d}\boldsymbol{d}). \tag{5.31}$$

Proof. The full proof is given in [2], here we give less rigorous but more transparent proof. The main observation behind this proof is the following joint construction of a $\bar{\mathbf{P}}_{n,d}$ distributed multigraph and a (discrete-time) random walk on it.

1. Pick X_0 in V uniformly.

2. Pair all half-edges incident to X_0 according to the pairing construction given above.

3. Pick uniformly a number Z_0 in $[d]$ and set X_1 to be the vertex paired with (X_0, Z_0).

4. Pair all not-yet paired half-edges incident to X_1 according to the pairing construction.

5. Pick uniformly a number Z_1 in $[d]$ and set X_2 to be the vertex paired with (X_1, Z_1).

6. ...

7. Stop when $X_{|V|u}$ and its neighbors are known.

At this moment we constructed first $|V|u$ steps of the random walk trajectory and determined all edges in the graph that are incident to vertices visited by this trajectory. To finish the construction of the graph we should

(8) Pair all remaining half-edges according to the pairing construction.

It is not hard to observe that the edges created in step (8) are exactly the edges of the vacant graph $\mathbf{V}^u$ and that the degree of x in $\mathbf{V}^u$ is known already at step (7). Using the restriction property of partial matchings (5.29), it is then not difficult to prove the proposition. $\square$

Due to the last proposition, in order to show Theorem 5.6 we need information about two objects: the maximal connected component of $\mathbb{P}_{\boldsymbol{d}}$-distributed random graph, and the distribution $Q^u_{n,d}$. We deal with them in the next two subsections.

5.2.3 Behavior of random graphs with a given degree sequence.

The random graphs with a given degree sequence are well studied. A rather surprising fact, due to Molloy and Reed [12] is that the phase transition in its behavior is characterized by a single real parameter computed from a degree sequence. We give a very weak version of [12] result:

Theorem 5.9. *For a degree sequence $\boldsymbol{d} : [n] \to \mathbb{N}$, let*

$$Q(\boldsymbol{d}) = \frac{\sum_{x=1}^n \boldsymbol{d}_x^2}{\sum_{x=1}^n \boldsymbol{d}_x} - 2. \tag{5.32}$$

Consider now a sequence of degree sequences $(\boldsymbol{d}^n)_{n\geq 1}$, $\boldsymbol{d}^n : [n] \to \mathbb{N}$, and assume that the degrees d^n_x are uniformly bounded by some Δ and that and that $|\{x \in [n] : d^n_x = 1\}| \geq \zeta n$ for a $\zeta > 0$. Then

- *If $\liminf Q(\boldsymbol{d}^n) > 0$, then there is $c > 0$ such that with $\bar{\mathbf{P}}_{\boldsymbol{d}}$ probability tending to one the maximal connected component of the graph is larger than cn.*
- *When $\limsup Q(\boldsymbol{d}^n) < 0$, then the size of maximal connected component of $\bar{\mathbf{P}}_{\boldsymbol{d}}$-distributed graph is with high probability $o(n)$.*

Later works, see e.g. [7, 6], give a more detailed description of random graphs with given degree sequences, including the description of the critical window which allows to deduce Theorem 5.7.

5.2.4 Distribution of the degree sequence of the vacant graph

We will show that the distribution of the degree sequence of the vacant graph is the same as the distribution of the number of vacant neighbors of any given vertex x in a random interlacement on $\mathbb{T}_d$. More precisely, it follows from Remark 5.4 that the probability that $x \in \mathcal{V}^u_{\mathbb{T}_d}$ and its degree in $\mathcal{V}^u_{\mathbb{T}^d}$ is i, $i = 0, \dots, d$, is given by

$$d^u_i := e^{-u\frac{d-2}{d-1}} \binom{d}{i} p^i_u (1-p_u)^{d-i}, \tag{5.33}$$

with $p^i_u = \exp\{-u\frac{(d-2)^2}{d(d-1)}\}$.

Recall $\mathcal{D}^u$ denotes the degree sequence of the vacant graph $\mathbf{V}^u$. For any degree sequence $\boldsymbol{d}$, $n_i(\boldsymbol{d})$ denotes the number of vertices with degree i in $\boldsymbol{d}$. The following theorem states that quenched expectation of $n_i(\mathcal{D}^u)$ concentrates around nd^i_u.

Theorem 5.10. *For every $u > 0$ and every $i \in \{0, \dots, d\}$,*

$$\left|E^G[n_i(\mathcal{D}^u)] - nd^u_i\right| \le c(\log^5 n) n^{1/2}, \quad \bar{\mathbb{P}}_{n,d}\text{-}a.a.s. \tag{5.34}$$

Although we do not present the proof of the above theorem, let us mention that it is similar to the derivation of (2.29) in Chapter 2. The main difference lies on the fact that here we have to use the quasi-stationary distribution and the matching construction of G in order to obtain good error bounds as above.

In order to control $Q^u_{n,d}$ we need to show that $n_i(\mathcal{D}^u)$ concentrates around its mean. This is the result of the following theorem that holds for *deterministic* graphs.

Theorem 5.11. *Let G be a d-regular (multi)graph on n vertices whose spectral gap λ_G is larger than some $\alpha > 0$. Then, for every $\varepsilon \in (0, \frac{1}{4})$, and for every $i \in \{0, \dots, d\}$,*

$$P^G\left[|n_i(\mathcal{D}^u) - E^G[n_i(\mathcal{D}^u)]| \ge n^{1/2+\varepsilon}\right] \le c_{\alpha,\varepsilon} e^{-c_{\alpha,\varepsilon} n^{\varepsilon}}. \tag{5.35}$$

The proof of this theorem uses concentration inequalities for Lipschitz functions of sequences of not-independent random variables. We omit it in these notes, it can be found in [2].

From Theorems 5.10 and 5.11, it is easy to compute the typical value of $Q(\mathcal{D}^u)$. It turns out that it is positive when $u < u_\star$ and negative when $u > u_\star$. This proves via Theorem 5.9 and Proposition 5.8 the existence of phase transition of the vacant set.

In fact, the above results allow to compute $Q(\mathcal{D}^u)$ up to an additive error which is $o(n^{-1/2+\varepsilon})$. This precision is more than enough to apply the stronger results on the behavior of random graphs with given degree sequences [6] and to show Theorem 5.7.

Exercises

Exercise 5.12. Consider the matrix $C(\cdot,\cdot)$ defined in (2.1). Show that for each vector $k = (k_1, \dots, k_d)$ (for $k_j = 0, \dots, N-1$) the functions

$$\psi_k(x_1, \dots, x_d) = \prod_{j=1}^{d} \exp\Big\{\frac{2\pi i x_j k_j}{N}\Big\} \tag{5.36}$$

are eigenvectors of C, with eigenvalues given respectively by

$$\lambda_k = \prod_{j=1}^{d} \big(1 + \cos(2\pi k_j/N)\big)/2. \tag{5.37}$$

Define the spectral gap γ_N to be the subtraction of the largest and the second largest eigenvalues from λ_k above. Conclude that γ_N satisfies

$$\gamma_N \geq c/N^2, \tag{5.38}$$

for some constant $c > 0$.

Exercise 5.13. Let $C(x,y)$ (for $x, y \in \mathbb{T}_N^d$) be the adjacency matrix defined in (2.1) as in Exercise 5.12 above. Use the spectral decomposition to show that

$$\mathbf{1}_x = \sum_{k=(k_1,\dots,k_d)} \prod_{j=1}^{d} \exp\Big\{\frac{2\pi i x_j k_j}{N}\Big\}\psi_k. \tag{5.39}$$

where $\mathbf{1}_x$ stands for the indicator function of the point x. Then show that for any x and y in $\mathbb{T}_N^d$,

$$P_y[X_n = x] = C^{(n)}\mathbf{1}_x = 1/N^d + O(\exp\{-cn/N^2\}). \tag{5.40}$$

Exercise 5.14. Show that the probability that the simple (non-lazy) random walk on $\mathbb{Z}^d$ satisfies the following

$$P_0^{\mathbb{Z}^d}[X_n = 0] = \mathbf{1}\{n \text{ even}\} \sum_{k_1+\dots+k_d=n/2} \frac{n!}{k_1! \dots k_d!}(2d)^{-n}. \tag{5.41}$$

Recalling the Stirling's approximation formula, show that if $d \geq 3$, the random walk never returns to the origin with positive probability. Note that the same can be concluded to the lazy random walk.

Exercise 5.15. Consider a one dimensional (non-lazy) random walk $0 = X_0, X_1, X_2, \ldots$. Now fix a sequence $\ell_n > 0$ and show that for some $c > 0$

$$P_0[|X_n| > \ell_n n] \leq \exp\{-c\ell_n^2 n\}. \tag{5.42}$$

Hint: Observe that $|X_n| > \ell_n n$ if and only if $\exp\{\theta|X_n|\} > \exp\{\theta \ell_n n\}$. Now use Markov's inequality and optimize in θ.

Note that $\exp\{X_n\}$ is a submartingale. Now, using Doob's inequality and the same argument as above, show that

$$P_0[\max_{k \leq n} |X_k| > \ell_n n] \leq 2\exp\{-c\ell_n^2 n\}. \tag{5.43}$$

Exercise 5.16. Given a set $\mathsf{A} \subset \mathbb{Z}^d$, show using reversibility that

$$\operatorname{cap}(A) = \lim_{n \to \infty} \sum_{z \in B(0,n)^c} P_z[H_{\mathsf{A}} < \tilde{H}_{B(0,n)^c}]. \tag{5.44}$$

In particular, conclude that if $\mathsf{A} \subset \mathsf{A}'$, then $\operatorname{cap}(A) \leq \operatorname{cap}(A)'$.

Exercise 5.17. Let A_N be the box $[0, L)^d \subset \mathbb{Z}^d$ and show that for some positive constant $c = c(d)$,

$$c^{-1} L^{d-2} \operatorname{cap}(\mathsf{A}_N) \leq cL^{d-2}, \text{ for all } N \geq 1. \tag{5.45}$$

Hint: Use Exercise 5.16 to write the capacity of A and $x = (L/2, L/2, \ldots, L/2)$ (which you know how to bound). Now use the Strong Markov Property to relate the two, together with (3.36).

Exercise 5.18. Again, let A_N be the box $[0, L)^d \subset \mathbb{Z}^d$ and use this other hint to show that for some positive constant $c = c(d)$,

$$\operatorname{cap}(\mathsf{A}_N) \geq cL^{d-2}, \text{ for all } N \geq 1. \tag{5.46}$$

Note that this procedure only gives a lower bound for the capacity of A.

Hint: Use a Gambler's Ruin argument to show that the probability that the random walk (starting from the boundary of A_N) leaves $[-L, 2L)^d$ before returning to A_N is at least c/N. Then use the invariance principle to conclude the proof.

Exercise 5.19. Consider the distance between a random walker on the infinite d-regular tree $\mathbb{T}^d$ for $d \geq 3$. Show using this comparison that this simple random walk is transient.

Show also (5.2) using a recursion relation.

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Terms

Symbols