Partially hyperbolic diffeomorphisms with 2-dimensional center

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Invariance Principle

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Used by: Wilkinson (Livsič theory of partially hyperbolic maps), Yang, V (SRB measures), Hertz, Hertz, Tahzibi, Ures (measures of maximal entropy), Kocsard, Potrie (Livsič theory of smooth cocycles)

Stable ergodicity

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Assume that no eigenvalue is a root of unity. Then A is ergodic relative to the volume (Haar) measure.

Federico Rodriguez Hertz proved that A is stably ergodic: every volume preserving diffeomorphism in a neighborhood is ergodic.

Stable Bernoulli property

Fix any symplectic form ω on $\mathbb{T}^4$ invariant under A. Then

Theorem (Artur Avila, MV)

Every ω-symplectic diffeomorphism $f : \mathbb{T}^4 \to \mathbb{T}^4$ in a neighborhood of A is ergodically equivalent to a Bernoulli shift. In fact,

- either f is non-uniformly hyperbolic (all Lyapunov exponents are different from zero)
- or else f is conjugate to A by some volume preserving diffeomorphism.

Some extensions

We consider C^∞ diffeomorphisms. The theorem extends to finite differentiability (C^k with $k \geq 22$, say).

The theorem also remains true for any symplectic pseudo-Anosov $A : \mathbb{T}^d \to \mathbb{T}^d$ in any (even) dimension $d \geq 4$, with $\dim E^c = 2$. But the conjugacy is only a volume preserving homeomorphism.

Lyapunov exponents

Every nearby diffeomorphism $f : \mathbb{T}^4 \to \mathbb{T}^4$ is partially hyperbolic, with invariant splitting $E^u \oplus E^c \oplus E^s$ having $\dim E^c = 2$.

All the iterates of f are ergodic, by F. Rodriguez Hertz.

Let $\lambda^u > \lambda_1^c \geq \lambda_2^c > \lambda^s$ be the Lyapunov exponents. Symplecticity implies that $\lambda^u + \lambda^s = \lambda_1^c + \lambda_2^c = 0$.

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Case 1: $\lambda_1^c > 0 > \lambda_2^c$

Then f is non-uniformly hyperbolic and so, by Ornstein, Weiss, it is equivalent to a Bernoulli shift.

Vanishing Lyapunov exponents

Case 2: $\lambda_1^c = \lambda_2^c = 0$

The hard case. To prove conjugacy to the linear automorphism we must recover an Abelian group structure on the torus compatible with the dynamics of f.

In the hardest (accessible) case, this is produced from an invariant translation structure on the center leaves, which is itself an upgrade of an invariant conformal structure on the center leaves.

Stable and unstable holonomies

Every f close to A is partially hyperbolic, dynamically coherent, and center bunched: for some choice of the norm,

$$\|D_x^c f\| \, \|(D_x^c f)^{-1}\| < \min\{\frac{1}{\|D_x^s f\|}, \frac{1}{\|(D_x^u f)^{-1}\|}\}.$$

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Given x, y in the same strong stable leaf, the strong stable leaf of any $z \in W^c_x$ intersects W^c_y in exactly one point $H^s_{x,y}(z)$.

The map $H^s_{x,y} : W^c_x \to W^c_y$ is a C^1 diffeomorphism. Consider the stable holonomies

$$h^s_{x,y} = \mathbb{P}(DH^s_{x,y}) : \mathbb{P}(E^c_x) \to \mathbb{P}(E^c_y)$$

Unstable holonomies are defined analogously.

Invariance Principle

Remember that we are dealing with the case $\lambda_c^1 = \lambda_c^2 = 0$.
The main step is to prove that f can not be accessible.

Theorem

If f is accessible then there exists a family $\{m_x : x \in M\}$ satisfying

1. each m_x is a probability measure on projective space $\mathbb{P}(E_x^c)$.
2. $\mathbb{P}(D_x^c f)_* m_x = m_{f(x)}$ for every x.
3. $(h_{x,y}^s)_* m_x = m_y$ for all x, y in the same strong stable leaf.
4. $(h_{x,y}^u)_* m_x = m_y$ for all x, y in the same strong unstable leaf.
5. $x \mapsto m_x$ is continuous, with respect to weak* topology.

From probability measures to conformal structures

Let 0 be a fixed point of f. The derivative $D_0^c f$ is close to $A \mid E_A^c$, which is an irrational rotation (no eigenvalue is a root of unity).

Then, m_0 has no atom of mass $\geq 1/2$ on $\mathbb{P}(E_0^c)$. The same is true for every m_x, by accessibility and holonomy invariance.

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Then, by the barycenter construction of Douady, Earle, each m_x determines a conformal structure on E_x^c. This provides each W_x^c with the conformal structure of the complex plane $\mathbb{C}$.

This structure is continuous and is invariant under the dynamics, the stable holonomies and the unstable holonomies.

From conformal structure to translation structure

Fix any uniformization $\mathbb{C} \to W_0^c$. This also chooses a translation structure on W_0^c. Push this structure to all the other center leaves by stable/unstable holonomy, using accessibility.

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We prove that there is $C(\gamma) > 0$ such that $d(H_\gamma(z), z) \leq C(\gamma)$ for every $z \in W_0^c$. This uses that center leaves W_x^c are at uniformly bounded distance from the center spaces E_x^c (F. Rodriguez Hertz).

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Fix any uniformization $\mathbb{C} \to W^c_0$. This also chooses a translation structure on W^c_0. Push this structure to all the other center leaves by stable/unstable holonomy, using accessibility.

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Then we deduce that $a = 1$.

From translation structure to algebraic model

The translation structure on central leaves defines an $\mathbb{R}^2$ action

$$\mathbb{R}^2 \times \mathbb{T}^4 \to \mathbb{T}^4, \quad (v, x) \mapsto \tau_v(x)$$

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$G = \overline{\{\tau_v : v \in \mathbb{R}^2\}}$ is a compact group of homeomorphisms of $\mathbb{T}^4$. Its action on $\mathbb{T}^4$ is Abelian, transitive and free.

So, $\phi : G \to \mathbb{T}^4$, $g \mapsto g(0)$ is a homeomorphism from G to $\mathbb{T}^4$. $\tilde{f} = \phi^{-1} \circ f \circ \phi$ is a group automorphism, and it is conjugate to A.

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This proves that f is conjugate to A. This conjugacy preserves the strong stable, strong unstable and center foliations.

Since A is not accessible, it follows that f is not accessible.

The non-accessible case

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The *su*-holonomy (respectively, center holonomy) preserves the area measure defined by the symplectic form ω on the center leaves (respectively, *su*-leaves).

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When $d = 4$ (hence $\dim E^u = \dim E^s = 1$), we can use methods of Avila, V, Wilkinson to show that the conjugacy is C^∞.

Area preserving cocycles

Consider $F : M \times N \to M \times N$, $(x, y) \mapsto (f(x), g(x, y))$, where N is a surface and f is Anosov.

Assume: F is volume preserving, partially hyperbolic with $E^c =$ vertical bundle, center bunched and accessible (hence, ergodic).

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Consider the Lyapunov exponents

$$\lambda_+(F) = \lim_n \frac{1}{n} \log \|\partial_y g^n(x, y)\|$$
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($M \times N$ may be replaced by any fiber bundle over M whose fiber is a surface)

Area preserving cocycles

Theorem

If genus$(N) \geq 2$ then $\lambda_+ > 0 > \lambda_-$ and F is a continuity point for the Lyapunov exponents.

Area preserving cocycles

Theorem

If genus(N) ≥ 2 then $\lambda_+ > 0 > \lambda_-$ and F is a continuity point for the Lyapunov exponents.

Rough idea: By an application of the Invariance Principle, for the Lyapunov exponents to vanish there must exist either an invariant continuous line field, or an invariant pair of transverse continuous line fields, on N.

Either alternative is incompatible with genus(N) ≥ 2.