Theory of metric Lie groups and $H$-surfaces in homogeneous 3-manifolds.

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Based on joint work with Mira, Pérez, Ros and Tinaglia.

Definition
A 2-dimensional submanifold with constant mean curvature $H \geq 0$ in a Riemannian 3-manifold is called an $H$-surface.

Definition
If the isometry group of a Riemannian manifold $Y$ acts transitively, then $Y$ is called homogeneous.

Definition
A Lie group with left invariant metric is called a metric Lie group.
Notation and Language

- \( Y \) = simply connected homogeneous 3-manifold.
- \( X \) = simply connected 3-dimensional Lie group with left invariant metric (\( X \) is a metric Lie group).
- \( H(Y) = \inf \{ \max |H_M| : M = \text{immersed closed surface in } Y \} \), where \( \max |H_M| \) denotes max of absolute mean curvature function \( H_M \).
- The number \( H(Y) \) is called the critical mean curvature of \( Y \).
- \( \text{Ch}(Y) = \inf_{K \subset Y \text{ compact}} \frac{\text{Area}(\partial K)}{\text{Volume}(K)} \) = Cheeger constant of \( Y \).

Goals of Lecture 2

- Classification and quasi-isometric classification of the possible \( Y \).
- Right cosets of 2-dimensional subgroups \( H \subset X \) and the existence of algebraic open book decompositions.
- Uniqueness and embeddedness of minimal spheres in \( X \approx SU(2) \).
- Isoperimetric domains and the isoperimetric profile of \( Y \).
- Explain the formula: \( \text{Ch}(Y) = 2H(Y) \) for non-compact \( Y \).
- Discuss CMC foliations, Isoperimetric Inequality Conjectures, Stability Conjecture, Product CMC Foliation Conjectures.
Theorem (Simply connected homogeneous 3-dimensional \( Y \))

If \( Y \) is a simply connected homogenous 3-manifold, then:
- \( Y \) is isometric to a **metric Lie group** (Lie group with left invariant metric - these examples form a 3-parameter family of non-isometric homogeneous 3-manifolds), or
- \( Y \) is isometric to \( S^2(\kappa) \times \mathbb{R} \) for some \( \kappa > 0 \).

Brief Sketch of Proof.

\( D = \) dimension of identity component \( \text{Iso}_e(Y) \) of isometry group of \( Y \).
- If \( D = 3 \), then one identifies \( \text{Iso}_e(Y) \) with \( Y \) by its action on \( Y \).
- If \( D = 4 \), then \( Y \) has the structure of an \( E(\kappa, \tau) \)-space, and so it is either isometric to \( S^2(\kappa) \times \mathbb{R} \) for some \( \kappa \) or to one of the Lie groups \( \text{SU}(2), \text{Nil}_3, \widetilde{\text{SL}}(2, \mathbb{R}), \mathbb{H} \times \mathbb{R} \) with some left invariant metric, where \( \mathbb{H} \) is the group of affine transformations \( \{ f(x) = ax + b : \mathbb{R} \to \mathbb{R} \mid a > 0, b \in \mathbb{R} \} \).
- If \( D = 6 \), then \( Y \) has constant curvature, and so \( Y \) is isometric to a metric Lie group.
Theorem (Simply connected 3-dimensional Lie groups $X$)

Every $X$ is isomorphic to $SU(2) = \{\text{group of unit length quaterions}\}$ or to the universal covering group of a 3-dimensional subgroup of the 6-dim affine group $F = \{f(x) = Ax + b : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \mid b \in \mathbb{R}^2, A \in GL(2, \mathbb{R})\}$, which is the natural semidirect product of $\mathbb{R}^2$ with $GL(2, \mathbb{R}) = Aut(\mathbb{R}^2)$.

The 3-dimensional subgroups of $F$ are one of the following types:

- $SL(2, \mathbb{R}) = \{A \in GL(2, \mathbb{R}) \mid \det(A) = 1\}$.
- The semidirect product of the subgroup $\mathbb{R}^2 \subset F$ of translations with any particular 1-parameter subgroup $\Gamma$ of $GL(2, \mathbb{R})$.

Example

- The group $E(2)$ of rigid motions of $\mathbb{R}^2$ is the semidirect product of $\mathbb{R}^2 \subset F$ with the 1-parameter subgroup $S^1$ of rotations in $GL(2, \mathbb{R})$.
- The group of conformal affine transformations of $\mathbb{R}^2$, $H^3 = \{f(x) = ax + b : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \mid a > 0, b \in \mathbb{R}^2\}$, is the semidirect product of $\mathbb{R}^2 \subset F$ with the 1-parameter subgroup of $GL(2, \mathbb{R})$ of positive multiples of the identity matrix. This group only admits left invariant metrics of constant negative curvature.
**Definition**

An \( H \)-foliation of \( Y \) is a foliation by surfaces (leaves) of constant mean curvature \( H \geq 0 \).

**Definition**

A \textbf{CMC} foliation of \( Y \) is a foliation by surfaces (leaves) of constant mean curvature, with the mean curvature possibly varying from leaf to leaf.

**Example**

- Since a 2-dimensional subgroup \( H \) of \( X \) has constant mean curvature \( H \geq 0 \) and left translations are isometries of \( X \), then the set of left cosets \( \mathcal{K} = \{ aH \mid a \in X \} \) of \( H \) is an example of an \( H \)-foliation of \( X \).

- Since every right coset of \( H \) is the left coset of a conjugate subgroup, then the set of right cosets \( \mathcal{F} = \{ Ha \mid a \in X \} \) of \( H \) is an example of a \textbf{CMC} foliation of \( X \).
Theorem

- Let $H \subset X$ be a 2-dimensional connected subgroup.
- Then the set of right cosets $\mathcal{F} = \{Ha \mid a \in X\}$ of $H$ coincides with the set of surfaces in $X$ at constant distances from $H$.
- In particular, the set $\mathcal{F}$ of equidistant surfaces from $H$ forms a CMC foliation of $X$.
- If $H$ is normal, then $\mathcal{F}$ is an $H$-foliation.

Proof.

- It suffices to check that for $d > 0$ small, a surface $\Sigma_d$ of constant distance from $H$ (there are 2 such surfaces) is the right coset $pH$ for any $p \in \Sigma_d$.
- Let $h \in H$. Since $l_h(H) = hH = H$ and $l_h$ is an isometry that leaves $H$ invariant, then for any $p \in \Sigma_d$, $Hp$ is a connected surface of distance $d$ from $H$.
- As $Hp$ and $\Sigma_d$ are connected and $Hp \cap \Sigma_d \neq \emptyset$, then $Hp = \Sigma_d$.
- Since the right coset $Hp$ is the left coset $pH'$ of the subgroup $H' = p^{-1}Hp$, $Hp$ has the same constant mean curvature as $H'$.
Theorem

- Let $G$ be an $n$-dimensional metric Lie group with isometry group $\text{Iso}(G)$ of dimension $n$.
- If $C$ is a component of the fixed point set of an isometry $I \in \text{Iso}(G)$, then $C$ is a left coset of some totally geodesic subgroup of $G$.

Sketch of the proof.

- Assume $I \in \text{Iso}(G)$ and $I(e) = e$.
- $I$ induces a Lie isomorphism of the $n$-dimensional space of Killing fields = the Lie algebra $\mathfrak{R}(G)$ of right invariant vector fields.
- Let $(\hat{G}, \star)$ be the related Lie group to $\mathfrak{R}(G)$, which is isomorphic to $G$ with the opposite multiplication: $x \star y = yx$.
- By integration, $I$ induces an isomorphism of $\hat{G}$, and hence of $G$.
- Since the fixed point set of a group isomorphism is a subgroup, then $C$ is a subgroup of $G$.
- $C$ is totally geodesic since the fixed point set of an isometry is totally geodesic.
Remark (Existence of algebraic open book decompositions)

- Consider a semidirect product $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, where $A$ is diagonal, i.e., $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}$.

- Reflection in the $(x, z)$-plane $H_{xz}$ or the $(y, z)$-plane $H_{yz}$ is an isometry of the canonical metric and each plane is a subgroup.

- For each $t \in \mathbb{R}$, the plane $P(t)$ parallel to $H_{xz}$ of signed distance $t$, $\{p = (x, y, z) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mid \text{dist}(p, H_{xz}) = |t|, ty > 0\}$ is a right coset of $H_{xz}$ and a left coset of $H_{xz}(t) = (0, 0 - t) \cdot H_{xz} \cdot (0, 0, t)$.

- Each $H_{xz}(t)$ contains the 1-parameter subgroup $\Gamma = \text{the x-axis}$.

- $[0, \frac{1}{2} \text{Trace}(A)]$ parameterizes the mean curvatures of these subgroups, where $\frac{1}{2} \text{Trace}(A)$ is the mean curvature of $\mathbb{R}^2 \rtimes_A \{0\}$.

Algebraic open book decomposition of $\text{Sol}_3$, where all subgroups are minimal and the only planar leaves are the $(x, y)$ and $(x, z)$-planes. Here the binding $\Gamma$ is the $x$-axis.
Theorem (Milnor)

Let $X$ be a 3-dimensional metric Lie group that is unimodular with unimodular basis $\{E_1, E_2, E_3\}$. For $i = 1, 2, 3$:

- At each point $p \in X$, $E_i(p)$ is a principal Ricci curvature direction.
- The integral curves of $E_i$ are geodesics of rotational symmetry by angle $\pi$.

Theorem (Meeks-Mira-Pérez-Ros)

Suppose $X$ is a metric Lie group isomorphic to $SU(2)$ with unimodular basis $\{E_1, E_2, E_3\}$ and let $\Gamma$ be an integral curve of one of these vector fields. Then:

- If $\Sigma$ is a least-area orientable surface with $\partial \Sigma = \Gamma$ and $R_\Gamma : X \to X$ is rotation by $\pi$ around $\Gamma$, then $S = \Sigma \cup R_\Gamma(\Sigma)$ is an embedded minimal 2-sphere in $X$.
- Up to left translation in $X$, $S$ is the unique immersed minimal 2-sphere in $X$.
- $S$ separates $X$ into isometric regions that are interchanged under $R_\Gamma$. 
Proof.

- Let $\Gamma = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ be the 1-parameter subgroup which is the integral curve of $E_1$ passing through $e$.

- $\Gamma$ is the fixed point set of $R_\Gamma$ and it is an unknotted geodesic in $X$.

- By **Hardt-Simon**, $\exists$ a smooth, compact, embedded, least-area orientable surface $\Sigma$ with $\partial \Sigma = \Gamma$, and any two such least-area surfaces intersect only along their common boundary $\partial \Sigma = \Gamma$.

- $\Sigma$ is not invariant under the left action of $\Gamma$, since the linking number of distinct $\Gamma$-orbits is 1 and $\Sigma$ is orientable.

- Thus, the set of left $\Gamma$ translates $\mathcal{F} = \{\theta \text{Int}(\Sigma) \mid \theta \in \Gamma\}$ of the interior of $\Sigma$ forms a minimal foliation of $X - \Gamma$ and every least-area orientable surface with boundary $\Gamma$ is a leaf of $\mathcal{F}$.

- Since the fundamental group $\Pi_1(X - \Gamma) = \mathbb{Z}$ contains as a subgroup $\Pi_1(\Sigma)$, then $\Sigma$ is a disk.

- Hence, $S = \Sigma \cup R_\Gamma(\Sigma)$ is an embedded minimal sphere.

- The uniqueness of the minimal sphere $S$ follows from the uniqueness of $H$-spheres in $X$ (discussed in Lecture 4) and the theorem follows.
Notation and Language

- \( H(Y) = \inf \{ \max |H_M| : M = \text{immersed closed surface in } Y \} \), where \( \max |H_M| \) denotes max of absolute mean curvature function \( H_M \).
- The number \( H(Y) \) is called the **critical mean curvature** of \( Y \).
- \( \text{Ch}(Y) = \inf_{K \subset Y \text{ compact}} \frac{\text{Area}(\partial K)}{\text{Volume}(K)} = \text{Cheeger constant of } Y \).

Remark

- If \( Y \) is diffeomorphic to \( S^3 \) or \( S^2 \times \mathbb{R} \), then \( H(Y) = 0 \) since there exist minimal spheres in such an \( Y \).

Theorem (Meeks-Mira-Pérez-Ros)

- If \( Y \) is noncompact, then:
  \[
  2H(Y) = \inf_{K \subset Y \text{ compact}} \frac{\text{Area}(\partial K)}{\text{Volume}(K)} = \text{Cheeger constant of } Y.
  \]
- If \( Y = \mathbb{R}^2 \rtimes_A \mathbb{R} \), then \( \text{Ch}(Y) = \text{Trace}(A) \).
- In particular, \( H(Y) = 1 \) if \( Y = \mathbb{H}^3 \) and \( H(Y) = 1/2 \) if \( Y = \mathbb{H}^2 \times \mathbb{R} \).
The shaded surface in \( \widetilde{SL}(2, \mathbb{R}) \) is the horocylinder \( \mathcal{C} = \) the inverse image by the projection \( \Pi \) of the horocycle \( \alpha_0 \subset \mathbb{H}^2 \).

The 1-parameter parabolic subgroup \( \Gamma^P \) is contained in \( \mathcal{C} \), as is the center \( \mathbb{Z} \) of \( \widetilde{SL}(2, \mathbb{R}) \).
Theorem ($\mathbb{E}(\kappa, \tau)$ spaces diffeomorphic to $\mathbb{R}^3$)

- Suppose $X$ is $\widetilde{SL}(2, \mathbb{R})$ with a left invariant metric and $\text{Ch}(X) = 2$.
- If $\text{dim}(\text{Iso}(X)) = 4$, then:
  - There exists a unique $b > 0$ such that $X$ is isometric to
    $$X_A = \mathbb{R}^2 \rtimes A \mathbb{R}, \quad A = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$  
  - Consider $X, X_A$ to be subgroups of $\text{Iso}(X)$ and let $I: \mathbb{R}^2 \rtimes A \mathbb{R} \to X$ be an isometry preserving identity elements.
  - Under $I$, horizontal planes correspond to parallel horocylinders.
  - $G = X \cap X_A = X \cap [\mathbb{R}^2 \rtimes A \mathbb{R}] \approx \mathbb{Z} \times [\mathbb{R} \rtimes (1) \mathbb{R}]$ is subgroup of $\text{Iso}(X)$. 
Subgroups of $G = X \cap X_A = X \cap [\mathbb{R}^2 \rtimes_A \mathbb{R}] \approx \mathbb{Z} \times [\mathbb{R} \rtimes (1) \mathbb{R}]$ in $X$, $\mathbb{R}^2 \rtimes_A \mathbb{R}$:

- $\{0\} \times [\{0\} \rtimes (1) \mathbb{R}] \subset \mathbb{Z} \times [\mathbb{R} \rtimes (1) \mathbb{R}]$ is the 1-parameter subgroup $[(0,0) \rtimes_A \mathbb{R}] \subset \mathbb{R}^2 \rtimes_A \mathbb{R} = X_A$.

- $\{0\} \times [\{0\} \rtimes (1) \mathbb{R}] \subset \mathbb{Z} \times [\mathbb{R} \rtimes (1) \mathbb{R}]$ is the parabolic subgroup $\Gamma^P$ of $X$ contained in the horocylinder $C = I(\mathbb{R}^2 \rtimes \{0\})$.

- $\{0\} \times [(0,0) \rtimes_A \mathbb{R}] \subset G = \text{subgroup} [(0,0) \rtimes_A \mathbb{R}] \subset \mathbb{R}^2 \rtimes_A \mathbb{R} = X_A$.

- $\{0\} \times [(0,0) \rtimes (1) \mathbb{R}] \subset G = 1$-parameter hyperbolic subgroup of $X$ orthogonal to $C$ at $e$.

- $\mathbb{Z} \times (0,0) \subset G$ corresponds to the center of $X$ and to $\mathbb{Z}$ subgroup of $\mathbb{R}^2 \rtimes_A 0 \subset \mathbb{R}^2 \rtimes_A \mathbb{R}$.
Definition

A diffeomorphism $f : M_1 \to M_2$ between two Riemannian manifolds is a **quasi-isometry** if there is a $c \geq 1$ such that for any vector $v_p \in T M_1$, $c^{-1} |v_p| \leq |f_*(v_p)| \leq c |v_p|$, where $|w|$ denotes the length of a tangent vector $w$.

Remark

- The definition of the Cheeger constant of a Riemannian manifold $Y$ implies $\text{Ch}(Y) \neq 0$ is a quasi-isometric property of the manifold.
- The definition also implies that if the manifold has polynomial volume growth, then the degree of that volume growth is a quasi-isometric property of the manifold.
**Theorem (Partial Quasi-isometric Classification)**

1. **Any two left invariant metrics on metric Lie group yield quasi-isometric manifolds (the identity map is a quasi-isometry).**

2. **Every 3-dimensional simply connected metric Lie group is quasi-isometric to one of the following Lie groups with any of its left invariant metrics:**
   
   \[\text{SU}(2), \ R^3, \ \text{Nil}_3, \ \text{Sol}_3, \ \mathbb{H}^3, \ X_{D \leq 1}\].

Furthermore:

- \(\tilde{E}(2)\) admits a flat left invariant metric.
- Metrics on \(\text{Nil}_3\) have polynomial volume growth of degree 4 and those of \(\text{Sol}_3\) have exponential volume growth.
- Left invariant metrics on \(\mathbb{H}^2 \times \mathbb{R}\) different from product metric correspond to left invariant metrics on \(\tilde{\text{SL}}(2, \mathbb{R})\) with 4-dimensional isometry group.
- Simply connected 3-dimensional non-unimodular groups with \(D > 1\) admit left invariant metrics of constant negative curvature.
- The Cheeger constant of a non-compact \(X\) vanishes **iff** it is isomorphic to \(R^3, \tilde{E}(2), \text{Nil}_3\) or \(\text{Sol}_3\).
Theorem (Solutions to the Isoperimetric Problem)

- Let $\mathbf{Y}$ be a homogeneous $3$-manifold.
- Then for each $\mathbf{V} > 0$, there exist a smooth solution to the isoperimetric problem with volume $\mathbf{V}$.
- In other words, there exists a smooth compact domain $\overline{\Omega}$ with volume $\mathbf{V}$ and with $\partial \Omega$ having smallest possible area.

Definition

- The isoperimetric profile of $\mathbf{Y}$ is defined as the function $I: (0, \infty) \rightarrow (0, \infty)$ given by

\[ I(t) = \inf \{ \text{Area}(\partial \Omega) \}, \]

where $\overline{\Omega} \subset \mathbf{Y}$ is a smooth compact domain with $\text{Volume}(\Omega) = t$.
- Note that $\text{Ch}(\mathbf{Y}) = \inf \{ \frac{I(t)}{t} \mid t \in (0, \infty) \}$. 
Definition

The **radius** of a compact Riemannian manifold with boundary is the maximum distance from points in the manifold to its boundary.

Definition

The **diameter** of a compact Riemannian manifold with boundary is the maximum distance between points in the manifold.
Theorem (Meeks-Mira-Pérez-Ros)

Suppose $Y$ is a non-compact, simply connected homogeneous 3-manifold with $Ch(X)$. Then:

1. $Ch(Y) = 2H(Y) = \lim_{t \to \infty} \frac{I(t)}{t}$.

2. If $Y$ is not isometric to $S^2(\kappa) \times \mathbb{R}$ for some $\kappa > 0$ and $\Omega \subset Y$ is an isoperimetric domain in $Y$ with volume $t$, then:
   a. $\partial \Omega$ is connected and has $H > 0$ as the boundary of $\Omega$.
   b. $Ch(Y) < \min \left\{ 2H_{\partial \Omega}, \frac{I(t)}{t} \right\}$, where $H_{\partial \Omega}$ is the constant mean curvature of the boundary of $\Omega$.

3. Let $\Omega_n \subset Y$ be any sequence of isoperimetric domains with volumes tending to infinity and let $R_n$ be the radius of $\Omega_n$. Then:
   a. $\lim_{n \to \infty} R_n = \infty$.
   b. $\lim_{n \to \infty} H_{\partial \Omega_n} = H(Y)$. 
Corollary (Meeks-Mira-Pérez-Ros)

- Let $X$ be a metric Lie group diffeomorphic to $\mathbb{R}^3$.
- Given $L, R > 0$, there exists a $C > 0$ such that for all compact immersed minimal surfaces $\Sigma$ with boundary of total length at most $L$ and contained in an extrinsic ball of radius $R$, then

$$\text{Area}(\Sigma) \leq C \cdot \text{Length}(\partial \Sigma).$$

Proof.

- Fix the length $L > 0$ and the radius $R$.
- Let $\Sigma_n \subset B_X(e, R)$ be a sequence of compact surfaces with $|H_{\Sigma}| \leq H(X)$, length of boundaries at most $L_n \leq L$ and areas $A_n \to \infty$.
- Since radii of isoperimetric domains with volume $\to \infty$ are arbitrarily large, $\exists$ an isoperimetric domain $\Omega$ such that $\Sigma_n \subset B_X(e, R) \subset \Omega$.
- By White, a subsequence of the $\Sigma_n$ converges to a varifold $\Sigma(\infty)$ with $H_{\Sigma(\infty)} \leq H(X) < H_{\partial \Omega}$ by previous theorem.
- Translate $\Sigma(\infty)$ until its support touches $\partial \Omega$ a first time.
- This is impossible by a maximum principle (White) for 2-varifolds $V$ with $H_V \leq H_{\partial \Omega}$. \qed
Isoperimetric Inequality and Radius Estimate in $\mathbb{H}^3$ (Meeks-Mira-Pérez-Ros)

- Let $X = \mathbb{H}^3$.

- By the mean curvature comparison principle, every compact immersed surface in $X$ with absolute mean curvature function $|H_\Sigma|$ less than or equal to $1 = H(X)$ and $1$ boundary curve of length at most $L$ lies in an extrinsic ball of radius less than $R = L/2$.

- More generally, given $L > 0$, $\exists D(L) > 0$ such that every compact immersed surface $\Sigma$ in $X$ with $|H_\Sigma| \leq 1 = H(X)$ and boundary of length at most $L$ has diameter less than $D(L)$; the argument here is nontrivial.

- Thus, the previous corollary implies that an isoperimetric inequality holds for such surfaces in $X$.

Isoperimetric Inequality for general $X$ for connected boundary surfaces

- **Meeks-Mira-Pérez-Ros** prove for $X$ diffeomorphic to $\mathbb{R}^3$, compact immersed surfaces with absolute mean curvature function $|H_\Sigma|$ less than or equal to $H(X)$ and $1$ boundary curve of length at most $L$ have a uniform bound on their intrinsic radii.

- This fact is deep and uses results concerning $H$-spheres from Lecture 4.

- **Meeks-Mira-Pérez-Ros** also prove a similar result for minimal surfaces in $X$ with at most two boundary curves.
Theorem (Isoperimetric Inequality, Meeks-Mira-Pérez-Ros)

Let $X$ be a metric Lie group diffeomorphic to $\mathbb{R}^3$.

Given $L > 0$, $\exists$ $C > 0$ such that $\forall$ compact immersed surfaces $\Sigma$ with one boundary curve of total length at most $L$ and absolute mean curvature function $|H_\Sigma|$ less than or equal to $H(X)$, then

$$\text{Area}(\Sigma) \leq C \cdot \text{Length}(\partial \Sigma).$$

Theorem (Minimal Isoperimetric Inequality, Meeks-Mira-Pérez-Ros)

Let $X$ be a metric Lie group diffeomorphic to $\mathbb{R}^3$.

Given $L > 0$, $\exists$ $C > 0$ such that $\forall$ compact immersed minimal surfaces $\Sigma$ with one or two boundary curves of total length at most $L$, then

$$\text{Area}(\Sigma) \leq C \cdot \text{Length}(\partial \Sigma).$$
Theorem (DeChang Chen)

- Given $H_0 \geq 0$, $\exists R_0 > 0$ such that the following hold.
- Let $Y$ be a simply connected Riemannian $3$-manifold with absolute sectional curvature at most $1$.
- $\forall$ compact immersed surfaces $\Sigma \subset Y$ with $|H_\Sigma| \leq H_0$, then
  \[\text{Radius}(\Sigma) \leq R_0 \cdot \text{Area}(\Sigma).\]

Corollary (Meeks-Mira-Pérez-Ros)

- Let $X$ be a metric Lie group diffeomorphic to $\mathbb{R}^3$.
- Given $L > 0$, $\exists D_0$ such that $\forall$ compact immersed surfaces $\Sigma$ with one boundary curve of total length at most $L$ and absolute mean curvature function $|H_\Sigma|$ less than or equal to $H(X)$, then
  \[\text{Radius}(\Sigma) < \text{Diameter}(\Sigma) \leq D_0.\]
- Furthermore, this same result holds for minimal surfaces in $X$ with at most $2$ boundary components.
Theorem (Isoperimetric Inequality in $\mathbb{R}^3$)

Given $L > 0$, then $\forall$ compact immersed minimal surfaces $\Sigma \subset \mathbb{R}^3$ with one boundary curves of total length at most $L$, then

$$\text{Area}(\Sigma) \leq \frac{1}{4\pi} \cdot [\text{Length}(\partial \Sigma)]^2.$$ 

Furthermore, if one has equality in the above formula, then $\Sigma$ is a round disk in a flat plane in $\mathbb{R}^3$.

Conjecture (Isoperimetric Inequality Conjecture in $\mathbb{R}^3$)

Given $L > 0$, then $\forall$ compact immersed minimal surfaces $\Sigma$ with boundary $\partial \Sigma$ of length at most $L > 0$, then:

$$\text{Area}(\Sigma) \leq \frac{1}{4\pi} \cdot [\text{Length}(\partial \Sigma)]^2.$$ 

Furthermore, if one has equality in the above formula, then $\Sigma$ is a round disk in a plane in $\mathbb{R}^3$. 
Conjecture (Isoperimetric Inequality Conjecture, Meeks-Mira-Pérez-Ros)

Let $X$ be a metric Lie group diffeomorphic to $\mathbb{R}^3$.

1. $\exists C > 0$ such that $\forall$ compact immersed surfaces $\Sigma$ with boundary $\partial \Sigma$ of length at most $L > 0$ and absolute mean curvature function $|H_\Sigma|$ less than or equal to $H(X)$, then

$$\text{Area}(\Sigma) \leq C \cdot [\text{Length}(\partial \Sigma)]^2.$$

2. Furthermore, if $X$ is $\mathbb{H}^3$ with it usual metric, then the constant $C = \frac{1}{4\pi}$ works in the above formula and if one has equality in the above formula, then $\Sigma$ is a round disk in a horosphere in $\mathbb{H}^3$.

Remark

- Item 1 holds for minimal surfaces with at most 2 boundary components (see Lecture 4 for the proof).

- One important consequence of this conjecture is that complete embedded $H$-surfaces of finite topology in any $X$ would have bounded second fundamental form when $H \in (0, H(X))$.

- This bounded curvature result will be proved in Lecture 4.
Definition

- Let $Y$ be a 3-dimensional homogeneous manifold and $\Gamma$ be a 1-parameter subgroup of the isometry group of $Y$.
- We say that a properly embedded surface $\Sigma \subset Y$ is an entire $\Gamma$-Killing graph if each orbit of the left action of $\Gamma$ on $Y$ intersects $\Sigma$ in exactly 1 point.

Example

If $H$ is a 2-dimensional subgroup of $X$, then $H$ is a Killing graph with respect to some 1-parameter subgroup of $X$.

A relationship of Killing graphs with the critical mean curvature

- If $\Sigma \subset Y$ is an entire $\Gamma$-Killing graph with respect to a 1-parameter subgroup $\Gamma$ and $\Sigma$ is a noncompact $H$-surface, then $\mathcal{F} = \{ a\Sigma \mid a \in \Gamma \}$ is an $H$-foliation of $Y$.
- If $Y$ is noncompact, then every compact immersed surface $\Delta$ in $Y$ intersects and lies on the mean convex side of one of the leaves $b\Sigma$ of $\mathcal{F}$.
- Let $p \in \Delta \cap b\Sigma$. By the mean curvature comparison principle, $\max(|H_\Delta|) \geq |H_\Delta|(p) > H$.
- By definition of the critical mean curvature, $H(Y) \geq H$. 
Crucial in the proof of the equality $\text{Ch}(Y) = 2\text{H}(Y)$ is the next theorem.

**Theorem (F(X)-Foliation Theorem, Meeks-Mira-Pérez-Ros)**

Let $X$ be diffeomorphic to $\mathbb{R}^3$ with $\text{Ch}(X) > 0$. Then:

- $X$ contains a properly embedded $\text{H}(X)$-surface $\Sigma$ that is a $\Gamma_1$-Killing graph for some 1-parameter subgroup $\Gamma_1$.

- $\Sigma$ is invariant under elements of a 1-parameter subgroup $\Gamma_2$ and an infinite normal cyclic subgroup $\mathbb{Z} \not\subset \Gamma_2$ whose elements commute with $\Gamma_2$.

- $\Gamma_2$ contains a cyclic subgroup $\mathbb{Z}'$ such that

  $$[\Sigma/(\mathbb{Z} \times \mathbb{Z}')] \subset [Y = X/(\mathbb{Z} \times \mathbb{Z}')]$$

  is a torus.

- $\Sigma/(\mathbb{Z} \times \mathbb{Z}')$ bounds a region of finite volume in $Y$ and it is the unique solution to the isoperimetric problem in $Y$ with this volume.

- Given any sequence of isoperimetric domains $\Omega_n \subset X$ with volumes tending to infinity, after left translations, the $\Omega_n$ converge to the mean convex component of $X - \Sigma$ and $\Sigma = \lim_{n \to \infty} \partial \Omega_n$. 
Remark (Related CMC foliations, Meeks-Pérez-Ros)

- Suppose \( X = \mathbb{H}^3 = \mathbb{R}^2 \times_A \mathbb{R} \), where \( A \) is the identity matrix and suppose \( \mathcal{F} \) is a CMC foliation of \( X \).
- If every leaf of \( \mathcal{F} \) has constant mean curvature at least \( H(X) = 1 \), then \( \mathcal{F} \) is a foliation by horospheres.
- In this case the surface \( \Sigma \) in the previous theorem must be a horosphere, and so, it is unique up to ambient isometry.

Remark (Related CMC foliations, Meeks-Mira-Pérez-Ros)

- In the case of \( X = \mathbb{H} \times \mathbb{R} \) with the product metric, there are many vertical \( H(X) \)-graphs over \( \mathbb{H} \) (and so they are Killing graphs).
- But any complete, embedded doubly-periodic \( H(X) \)-surface \( \Sigma' \) in \( X \) must be a leaf of the \( H(X) \)-foliation arising from \( \Sigma \) and \( \Gamma_1 \) in the previous theorem.
Theorem (Curvature Estimates for CMC Foliations, Meeks-Pérez-Ros)

- Suppose that $\mathcal{F}$ is a CMC foliation of a homogeneous 3-dim $Y$.
- Then the leaves of $\mathcal{F}$ have bounded second fundamental form and any leaf $L$ of $\mathcal{F}$ with maximal mean curvature is strongly stable, i.e., it admits a positive Jacobi function.
- $Y$ always admits a limit ”weak” CMC foliation $\mathcal{F}'$ of some divergent sequence of translations of $\mathcal{F}$ such that $\mathcal{F}'$ has a leaf having constant mean curvature equal to the supremum of the absolute mean curvatures of the leaves of $\mathcal{F}$, and any such leaf is strongly stable.

Conjecture (Strong Stability Conjecture, Meeks-Mira-Pérez-Ros)

- A complete strongly stable $H$-surface in $X$ with $H \geq H(X)$ is a Killing graph and so $H = H(X)$.
- In particular, if $H(X) = 0$, then any complete, strongly stable minimal surface $\Sigma$ in $X$ is a leaf of a minimal foliation of $X$ and so, $\Sigma$ is actually homologically area-minimizing in $X$.
- Hence, by the above theorem, any CMC foliation of an $X$ isomorphic to $\mathbb{R}^3$, $\text{Nil}_3$, $\tilde{E}(2)$ or $\text{Sol}_3$ would be a minimal foliation.
Conjecture (Product CMC-foliation Conjecture, Meeks-Mira-Pérez-Ros)

Let $\mathcal{F}$ be a CMC foliation of a homogeneous 3-dimensional $Y$.

- The constant mean curvatures of the leaves of $\mathcal{F}$ are at most $H(Y)$.
- Topologically, $\mathcal{F}$ is a product foliation by planes or by spheres.
- If $Y \approx \mathbb{R}^3$ and $p \in Y$, $\exists$ a product foliation of $Y - \{p\}$ by $H$-spheres.
- If $Y \approx \mathbb{R}^3$ and $\mathcal{F}$ is an $H(X)$-foliation, then:
  - Every leaf of $\mathcal{F}$ is some $\Gamma$-Killing graph and $\mathcal{F}$ is the related ”Killing”-foliation.
  - If $\text{Ch}(X) > 0$ and $\mathcal{F}$ has a leaf of quadratic area growth, then, up to ambient isometry, $\mathcal{F}$ is the foliation given in the $H(X)$-Foliation Theorem of Meeks-Mira-Pérez-Ros.
  - If $\text{Ch}(X) = 0$ and $\mathcal{F}$ has a leaf of quadratic area growth, then, up to ambient isometry, $\mathcal{F}$ is the foliation of horizontal planes in a semidirect product structure $\mathbb{R}^2 \rtimes_{A} \mathbb{R}$ for $X$. 