Theory of metric Lie groups and H-surfaces in homogeneous 3-manifolds.

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Based on joint work with Mira, Pérez, Ros and Tinaglia.

Definition

A **2**-dimensional submanifold with constant mean curvature $\mathbf{H} \geq 0$ in a Riemannian **3**-manifold is called an **H-surface**.

Definition

If the isometry group of a Riemannian manifold **Y** acts transitively, then **Y** is called **homogeneous**.

Definition

A Lie group with left invariant metric is called a **metric Lie group**.

Notation and Language

- $\mathbf{Y}$ = simply connected homogeneous $\mathbf{3}$-manifold.
- $\mathbf{X}$ = simply connected $\mathbf{3}$-dimensional Lie group with left invariant metric ($\mathbf{X}$ is a **metric Lie group**).
- $\mathbf{H}(\mathbf{Y}) = \mathbf{Inf}\{\max|\mathbf{H}_\mathbf{M}| : \mathbf{M} = \text{immersed closed surface in } \mathbf{Y}\}$, where $\max|\mathbf{H}_\mathbf{M}|$ denotes max of absolute mean curvature function $\mathbf{H}_\mathbf{M}$.
- The number $\mathbf{H}(\mathbf{Y})$ is called the **critical mean curvature** of $\mathbf{Y}$.
- $\mathrm{Ch}(\mathbf{Y}) = \mathbf{Inf}_{\mathbf{K}\subset\mathbf{Y} \text{ compact}} \dfrac{\mathbf{Area}(\partial\mathbf{K})}{\mathbf{Volume}(\mathbf{K})}$ = Cheeger constant of $\mathbf{Y}$.

Goals of Lecture 2

- Classification and quasi-isometric classification of the possible $\mathbf{Y}$.
- Right cosets of $\mathbf{2}$-dimensional subgroups $\mathbf{H} \subset \mathbf{X}$ and the existence of **algebraic open book decompositions**.
- Uniqueness and embeddedness of minimal spheres in $\mathbf{X} \approx \mathbf{SU}(\mathbf{2})$.
- Isoperimetric domains and the isoperimetric profile of $\mathbf{Y}$.
- Explain the formula: $\mathrm{Ch}(\mathbf{Y}) = 2\mathbf{H}(\mathbf{Y})$ for non-compact $\mathbf{Y}$.
- Discuss **CMC** foliations, Isoperimetric Inequality Conjectures, Stability Conjecture, Product **CMC** Foliation Conjectures.

Theorem (Simply connected homogeneous **3-dimensional Y**)

If $\mathbf{Y}$ is a simply connected homogenous $\mathbf{3}$-manifold, then:

- $\mathbf{Y}$ is isometric to a **metric Lie group** (Lie group with left invariant metric - these examples form a $\mathbf{3}$-parameter family of non-isometric homogeneous $\mathbf{3}$-manifolds), **or**
- $\mathbf{Y}$ is isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$ for some $\kappa > 0$.

Brief Sketch of Proof.

$\mathbf{D} =$ dimension of identity component $\mathrm{Iso}_e(\mathbf{Y})$ of isometry group of $\mathbf{Y}$.

- If $\mathbf{D} = \mathbf{3}$, then one identifies $\mathrm{Iso}_e(\mathbf{Y})$ with $\mathbf{Y}$ by its action on $\mathbf{Y}$.
- If $\mathbf{D} = \mathbf{4}$, then $\mathbf{Y}$ has the structure of an $\mathbb{E}(\kappa, \tau)$-space, and so it is either isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$ for some κ or to one of the Lie groups

  $$\mathbf{SU}(2), \ \mathbf{Nil_3}, \ \widetilde{\mathbf{SL}}(2, \mathbb{R}), \ \mathbb{H} \times \mathbb{R}$$

  with some left invariant metric, where $\mathbb{H}$ is the group of affine transformations $\{\mathbf{f}(\mathbf{x}) = \mathbf{a}\mathbf{x} + \mathbf{b} \colon \mathbb{R} \to \mathbb{R} \mid \mathbf{a} > 0, \, \mathbf{b} \in \mathbb{R}\}$.
- If $\mathbf{D} = \mathbf{6}$, then $\mathbf{Y}$ has constant curvature, and so $\mathbf{Y}$ is isometric to a metric Lie group. $\square$

Theorem (Simply connected **3-dimensional** Lie groups **X**)

Every **X** is isomorphic to $\mathbf{SU}(\mathbf{2}) = \{\text{group of unit length quaterions}\}$ or to the universal covering group of a **3**-dimensional subgroup of the **6-dim** affine group $\mathbf{F} = \{\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b} \colon \mathbb{R}^2 \to \mathbb{R}^2 \mid \mathbf{b} \in \mathbb{R}^2,\ \mathbf{A} \in \mathbf{GL}(\mathbf{2}, \mathbb{R})\}$, which is the natural semidirect product of $\mathbb{R}^2$ with $\mathbf{GL}(\mathbf{2}, \mathbb{R}) = \mathbf{Aut}(\mathbb{R}^2)$.

The **3**-dimensional subgroups of **F** are one of the following types:

- $\mathbf{SL}(\mathbf{2}, \mathbb{R}) = \{\mathbf{A} \ \in \mathbf{GL}(\mathbf{2}, \mathbb{R}) \mid \det(\mathbf{A}) = 1\}$.
- The semidirect product of the subgroup $\mathbb{R}^2 \subset \mathbf{F}$ of translations with any particular 1-parameter subgroup $\mathbf{\Gamma}$ of $\mathbf{GL}(\mathbf{2}, \mathbb{R})$.

Example

- The group $\mathbf{E}(\mathbf{2})$ of rigid motions of $\mathbb{R}^2$ is the semidirect product of $\mathbb{R}^2 \subset \mathbf{F}$ with the 1-parameter subgroup $\mathbb{S}^1$ of rotations in $\mathbf{GL}(\mathbf{2}, \mathbb{R})$.
- The group of conformal affine transformations of $\mathbb{R}^2$,

$$\mathbb{H}^3 = \{\mathbf{f}(\mathbf{x}) = \mathbf{a}\mathbf{x} + \mathbf{b} \colon \mathbb{R}^2 \to \mathbb{R}^2 \mid \mathbf{a} > 0,\, \mathbf{b} \in \mathbb{R}^2\},$$

  is the semidirect product of $\mathbb{R}^2 \subset \mathbf{F}$ with the 1-parameter subgroup of $\mathbf{GL}(\mathbf{2}, \mathbb{R})$ of positive multiples of the identity matrix. This group only admits left invariant metrics of constant negative curvature.

Definition

An $\mathbf{H}$-foliation of $\mathbf{Y}$ is a foliation by surfaces (leaves) of constant mean curvature $\mathbf{H} \geq 0$

Definition

A **CMC** foliation of $\mathbf{Y}$ is a foliation by surfaces (leaves) of constant mean curvature, with the mean curvature possibly varying from leaf to leaf.

Example

- Since a $\mathbf{2}$-dimensional subgroup $\mathbf{H}$ of $\mathbf{X}$ has constant mean curvature $\mathbf{H} \geq 0$ and left translations are isometries of $\mathbf{X}$, then the set of left cosets $\mathcal{K} = \{a\mathbf{H} \mid a \in \mathbf{X}\}$ of $\mathbf{H}$ is an example of an $\mathbf{H}$-foliation of $\mathbf{X}$.
- Since every right coset of $\mathbf{H}$ is the left coset of a conjugate subgroup, then the set of right cosets $\mathcal{F} = \{\mathbf{H}a \mid a \in \mathbf{X}\}$ of $\mathbf{H}$ is an example of a **CMC** foliation of $\mathbf{X}$.

Theorem

- Let $\mathbf{H} \subset \mathbf{X}$ be a $\mathbf{2}$-dimensional connected subgroup.
- Then the set of right cosets $\mathcal{F} = \{\mathbf{H}a \mid a \in \mathbf{X}\}$ of $\mathbf{H}$ coincides with the set of surfaces in $\mathbf{X}$ at constant distances from $\mathbf{H}$.
- In particular, the set $\mathcal{F}$ of equidistant surfaces from $\mathbf{H}$ forms a **CMC** foliation of $\mathbf{X}$.
- If $\mathbf{H}$ is normal, then $\mathcal{F}$ is an $\mathbf{H}$-foliation.

Proof.

- It suffices to check that for $d > 0$ small, a surface $\mathbf{\Sigma}_d$ of constant distance from $\mathbf{H}$ (there are $\mathbf{2}$ such surfaces) is the right coset $\mathbf{pH}$ for any $\mathbf{p} \in \mathbf{\Sigma}_d$.
- Let $\mathbf{h} \in \mathbf{H}$. Since $l_{\mathbf{h}}(\mathbf{H}) = \mathbf{hH} = \mathbf{H}$ and $l_{\mathbf{h}}$ is an isometry that leaves $\mathbf{H}$ invariant, then for any $\mathbf{p} \in \mathbf{\Sigma}_d$, $\mathbf{Hp}$ is a connected surface of distance d from $\mathbf{H}$.
- As $\mathbf{Hp}$ and $\mathbf{\Sigma}_d$ are connected and $\mathbf{Hp} \cap \mathbf{\Sigma}_d \neq \emptyset$, then $\mathbf{Hp} = \mathbf{\Sigma}_d$.
- Since the right coset $\mathbf{Hp}$ is the left coset $\mathbf{pH}'$ of the subgroup $\mathbf{H}' = \mathbf{p}^{-1}\mathbf{Hp}$, $\mathbf{Hp}$ has the same constant mean curvature as $\mathbf{H}'$. $\square$

Theorem

- Let $\mathbf{G}$ be an $\mathbf{n}$-dimensional metric Lie group with isometry group $\mathrm{Iso}(\mathbf{G})$ of dimension $\mathbf{n}$.
- If $\mathbf{C}$ is a component of the fixed point set of an isometry $\mathbf{I} \in \mathrm{Iso}(\mathbf{G})$, then $\mathbf{C}$ is a left coset of some totally geodesic subgroup of $\mathbf{G}$.

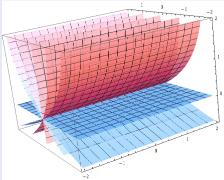
Sketch of the proof.

- Assume $\mathbf{I} \in \mathrm{Iso}(\mathbf{G})$ and $\mathbf{I}(e) = e$.
- $\mathbf{I}$ induces a Lie isomorphism of the $\mathbf{n}$-dimensional space of Killing fields = the Lie algebra $\mathbf{R}(\mathbf{G})$ of right invariant vector fields.
- Let $(\widehat{\mathbf{G}}, \star)$ be the related Lie group to $\mathbf{R}(\mathbf{G})$, which is isomorphic to $\mathbf{G}$ with the opposite multiplication: $\mathbf{x} \star \mathbf{y} = \mathbf{yx}$.
- By integration, $\mathbf{I}$ induces an isomorphism of $\widehat{\mathbf{G}}$, and hence of $\mathbf{G}$.
- Since the fixed point set of a group isomorphism is a subgroup, then $\mathbf{C}$ is a subgroup of $\mathbf{G}$.
- $\mathbf{C}$ is totally geodesic since the fixed point set of an isometry is totally geodesic. $\square$

Remark (Existence of algebraic open book decompositions)

- Consider a semidirect product $\mathbf{X} = \mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$, where $\mathbf{A}$ is diagonal, i.e., $\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}$.
- Reflection in the $(\mathbf{x}, \mathbf{z})$-plane $\mathbf{H_{xz}}$ or the $(\mathbf{y}, \mathbf{z})$-plane $\mathbf{H_{yz}}$ is an isometry of the canonical metric and each plane is a **subgroup**.
- For each $t \in \mathbb{R}$, the plane $P(t)$ parallel to $\mathbf{H_{xz}}$ of signed distance t, $\{\mathbf{p} = (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R} \mid \mathbf{dist}(\mathbf{p}, \mathbf{H_{xz}}) = |t|,\ t\mathbf{y} > 0\}$ is a right coset of $\mathbf{H_{xz}}$ and a left coset of $\mathbf{H_{xz}}(t) = (0, 0 - t) \cdot \mathbf{H_{xz}} \cdot (0, 0, t)$.
- Each $\mathbf{H_{xz}}(t)$ contains the 1-parameter subgroup $\mathbf{\Gamma} =$ the $\mathbf{x}$-axis.
- $[0, \frac{1}{2}\mathbf{Trace}(\mathbf{A})]$ parameterizes the mean curvatures of these subgroups, where $\frac{1}{2}\mathbf{Trace}(\mathbf{A})$ is the mean curvature of $\mathbb{R}^2 \rtimes_{\mathbf{A}} \{0\}$.

Algebraic open book decomposition of $\mathbf{Sol_3}$, where all subgroups are minimal and the only planar leaves are the $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{x}, \mathbf{z})$-planes. Here the binding $\mathbf{\Gamma}$ is the $\mathbf{x}$-axis.



Theorem (Milnor)

Let $\mathbf{X}$ be a $\mathbf{3}$-dimensional metric Lie group that is unimodular with unimodular basis $\{E_1, E_2, E_3\}$. For $i = 1, 2, 3$:

- At each point $p \in \mathbf{X}$, $E_i(p)$ is a principal Ricci curvature direction.
- The integral curves of E_i are geodesics of rotational symmetry by angle π.

Theorem (Meeks-Mira-Pérez-Ros)

Suppose $\mathbf{X}$ is a metric Lie group isomorphic to $\mathbf{SU}(\mathbf{2})$ with unimodular basis $\{E_1, E_2, E_3\}$ and let $\mathbf{\Gamma}$ be an integral curve of one of these vector fields. Then:

- If $\mathbf{\Sigma}$ is a least-area orientable surface with $\partial\mathbf{\Sigma} = \mathbf{\Gamma}$ and $R_{\mathbf{\Gamma}} : \mathbf{X} \to \mathbf{X}$ is rotation by π around $\mathbf{\Gamma}$, then $\mathbf{S} = \mathbf{\Sigma} \cup R_{\mathbf{\Gamma}}(\mathbf{\Sigma})$ is an embedded minimal $\mathbf{2}$-sphere in $\mathbf{X}$.
- Up to left translation in $\mathbf{X}$, $\mathbf{S}$ is the unique immersed minimal $\mathbf{2}$-sphere in $\mathbf{X}$.
- $\mathbf{S}$ separates $\mathbf{X}$ into isometric regions that are interchanged under $R_{\mathbf{\Gamma}}$.

Proof.

- Let $\boldsymbol{\Gamma} = \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ be the 1-parameter subgroup which is the integral curve of E_1 passing through e.
- $\boldsymbol{\Gamma}$ is the fixed point set of $R_{\boldsymbol{\Gamma}}$ and it is an unknotted geodesic in $\mathbf{X}$.
- By **Hardt-Simon**, $\exists$ a smooth, compact, embedded, least-area orientable surface $\boldsymbol{\Sigma}$ with $\partial\boldsymbol{\Sigma} = \boldsymbol{\Gamma}$, and any two such least-area surfaces intersect only along their common boundary $\partial\boldsymbol{\Sigma} = \boldsymbol{\Gamma}$.
- $\boldsymbol{\Sigma}$ is not invariant under the left action of $\boldsymbol{\Gamma}$, since the linking number of distinct $\boldsymbol{\Gamma}$-orbits is $\mathbf{1}$ and $\boldsymbol{\Sigma}$ is orientable.
- Thus, the set of left $\boldsymbol{\Gamma}$ translates $\mathcal{F} = \{\theta\,\mathrm{Int}(\boldsymbol{\Sigma}) \mid \theta \in \boldsymbol{\Gamma}\}$ of the interior of $\boldsymbol{\Sigma}$ forms a minimal foliation of $\mathbf{X} - \boldsymbol{\Gamma}$ and every least-area orientable surface with boundary $\boldsymbol{\Gamma}$ is a leaf of $\mathcal{F}$.
- Since the fundamental group $\Pi_1(\mathbf{X} - \boldsymbol{\Gamma}) = \mathbb{Z}$ contains as a subgroup $\Pi_1(\boldsymbol{\Sigma})$, then $\boldsymbol{\Sigma}$ is a disk.
- Hence, $\mathbf{S} = \boldsymbol{\Sigma} \cup R_{\boldsymbol{\Gamma}}(\boldsymbol{\Sigma})$ is an embedded minimal sphere.
- The uniqueness of the minimal sphere $\mathbf{S}$ follows from the uniqueness of $\mathbf{H}$-spheres in $\mathbf{X}$ (discussed in Lecture $\mathbf{4}$) and the theorem follows. $\square$

Notation and Language

- $\mathbf{H}(\mathbf{Y}) = \mathbf{Inf}\{\max|\mathbf{H_M}| : \mathbf{M} = \text{immersed closed surface in } \mathbf{Y}\}$, where $\max|\mathbf{H_M}|$ denotes max of absolute mean curvature function $\mathbf{H_M}$.
- The number $\mathbf{H}(\mathbf{Y})$ is called the **critical mean curvature** of $\mathbf{Y}$.
- $\mathrm{Ch}(\mathbf{Y}) = \mathbf{Inf}_{\mathbf{K}\subset\mathbf{Y}\ \text{compact}} \dfrac{\mathbf{Area}(\partial\mathbf{K})}{\mathbf{Volume}(\mathbf{K})}$ = Cheeger constant of $\mathbf{Y}$.

Remark

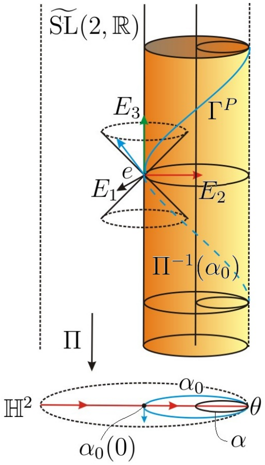
- If $\mathbf{Y}$ is diffeomorphic to $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$, then $\mathbf{H}(\mathbf{Y}) = 0$ since there exist minimal spheres in such an $\mathbf{Y}$.

Theorem (Meeks-Mira-Pérez-Ros)

- If $\mathbf{Y}$ is noncompact, then:

$$2\mathbf{H}(\mathbf{Y}) = \mathbf{Inf}_{\mathbf{K}\subset\mathbf{Y}\ \text{compact}} \frac{\mathbf{Area}(\partial\mathbf{K})}{\mathbf{Volume}(\mathbf{K})} = \text{Cheeger constant of } \mathbf{Y}.$$

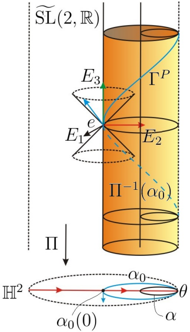
- If $\mathbf{Y} = \mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$, then $\mathrm{Ch}(\mathbf{Y}) = \mathbf{Trace}(\mathbf{A})$.
- In particular, $\mathbf{H}(\mathbf{Y}) = 1$ if $\mathbf{Y} = \mathbb{H}^3$ and $\mathbf{H}(\mathbf{Y}) = 1/2$ if $\mathbf{Y} = \mathbb{H}^2 \times \mathbb{R}$.



- The shaded surface in $\widetilde{\mathbf{SL}}(\mathbf{2},\mathbb{R})$ is the horocylinder $\mathbf{C}$ = the inverse image by the projection Π of the horocycle $\alpha_0 \subset \mathbb{H}^2$.
- The 1-parameter parabolic subgroup $\mathbf{\Gamma}^P$ is contained in $\mathbf{C}$, as is the center $\mathbb{Z}$ of $\widetilde{\mathbf{SL}}(\mathbf{2},\mathbb{R})$.

Theorem ($\mathbb{E}(\kappa,\tau)$ spaces diffeomorphic to $\mathbb{R}^3$)

- Suppose $\mathbf{X}$ is $\widetilde{\mathbf{SL}}(\mathbf{2},\mathbb{R})$ with a left invariant metric and $\mathrm{Ch}(\mathbf{X}) = 2$.
- If $\mathbf{dim}(\mathrm{Iso}(\mathbf{X})) = \mathbf{4}$, then:
  - There exists a unique $\mathbf{b} > 0$ such that $\mathbf{X}$ is isometric to
$$\mathbf{X_A} = \mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}, \quad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ \mathbf{b} & 1 \end{pmatrix}.$$
  - Consider $\mathbf{X}$, $\mathbf{X_A}$ to be subgroups of $\mathrm{Iso}(\mathbf{X})$ and let $\mathbf{I}\colon \mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R} \to \mathbf{X}$ be an isometry preserving identity elements.
  - Under $\mathbf{I}$, horizontal planes correspond to parallel horocylinders.
  - $\mathbf{G} = \mathbf{X} \cap \mathbf{X_A} = \mathbf{X} \cap [\mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}] \approx \mathbb{Z} \times [\mathbb{R} \rtimes_{(1)} \mathbb{R}]$ is subgroup of $\mathrm{Iso}(\mathbf{X})$.



Subgroups of $\mathbf{G} = \mathbf{X} \cap \mathbf{X_A} = \mathbf{X} \cap [\mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}] \approx \mathbb{Z} \times [\mathbb{R} \rtimes_{(1)} \mathbb{R}]$ in $\mathbf{X}$, $\mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$:

- $\{0\} \times [\{0\} \rtimes_{(1)} \mathbb{R}] \subset \mathbb{Z} \times [\mathbb{R} \rtimes_{(1)} \mathbb{R}]$ is the 1-parameter subgroup $[(0,0) \rtimes_{\mathbf{A}} \mathbb{R}] \subset \mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R} = \mathbf{X_A}$.
- $\{0\} \times [\{0\} \rtimes_{(1)} \mathbb{R}] \subset \mathbb{Z} \times [\mathbb{R} \rtimes_{(1)} \mathbb{R}]$ is the parabolic subgroup $\mathbf{\Gamma}^P$ of $\mathbf{X}$ contained in the horocylinder $\mathbf{C} = \mathbf{I}(\mathbb{R}^2 \rtimes \{0\})$.
- $\{0\} \times [\{(0,0)\} \rtimes_{(1)} \mathbb{R}] \subset \mathbf{G} =$ subgroup $[\{(0,0)\} \rtimes_{\mathbf{A}} \mathbb{R}] \subset \mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R} = \mathbf{X_A}$.
- $\{0\} \times [\{(0,0)\} \rtimes_{(1)} \mathbb{R}] \subset \mathbf{G} =$ 1-parameter hyperbolic subgroup of $\mathbf{X}$ orthogonal to $\mathbf{C}$ at e.
- $\mathbb{Z} \times (0,0) \subset \mathbf{G}$ corresponds to the center of $\mathbf{X}$ and to $\mathbb{Z}$ subgroup of $\mathbb{R}^2 \rtimes_{\mathbf{A}} 0 \subset \mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$.

Definition

A diffeomorphism $f\colon M_1 \to M_2$ between two Riemannian manifolds is a **quasi-isometry** if there is a $c \geq 1$ such that for any vector $v_p \in \mathbf{T}M_1$, $c^{-1}|v_p| \leq |f_*(v_p)| \leq c|v_p|$, where $|w|$ denotes the length of a tangent vector w.

Remark

- The definition of the Cheeger constant of a Riemannian manifold $\mathbf{Y}$ implies
$$\mathrm{Ch}(\mathbf{Y}) \neq 0$$
is a quasi-isometric property of the manifold.
- The definition also implies that if the manifold has polynomial volume growth, then the degree of that volume growth is a quasi-isometric property of the manifold.

Theorem (Partial Quasi-isometric Classification)

1. Any two left invariant metrics on metric Lie group yield quasi-isometric manifolds (the identity map is a quasi-isometry).
2. Every **3**-dimensional simply connected metric Lie group is quasi-isometric to one of the following Lie groups with any of its left invariant metrics:

$$\mathbf{SU}(2),\ \mathbf{R}^3,\ \mathbf{Nil_3},\ \mathbf{Sol_3},\ \mathbb{H}^3,\ \mathbf{X}_{\mathbf{D}\le 1}.$$

Furthermore:

- $\widetilde{\mathbf{E}}(2)$ admits a flat left invariant metric.
- Metrics on $\mathbf{Nil_3}$ have polynomial volume growth of degree **4** and those of $\mathbf{Sol_3}$ have exponential volume growth.
- Left invariant metrics on $\mathbb{H}^2 \times \mathbb{R}$ different from product metric correspond to left invariant metrics on $\widetilde{\mathbf{SL}}(2,\mathbb{R})$ with **4**-dimensional isometry group.
- Simply connected **3**-dimensional non-unimodular groups with $\mathbf{D} > 1$ admit left invariant metrics of constant negative curvature.
- The Cheeger constant of a non-compact $\mathbf{X}$ vanishes **iff** it is isomorphic to $\mathbf{R}^3$, $\widetilde{\mathbf{E}}(2)$, $\mathbf{Nil_3}$ or $\mathbf{Sol_3}$.

Theorem (Solutions to the Isoperimetric Problem)

- Let $\mathbf{Y}$ be a homogeneous $\mathbf{3}$-manifold.
- Then for each $\mathbf{V} > 0$, there exist a smooth solution to the isoperimetric problem with volume $\mathbf{V}$.
- In other words, there exists a smooth compact domain $\overline{\boldsymbol{\Omega}}$ with volume $\mathbf{V}$ and with $\partial\boldsymbol{\Omega}$ having smallest possible area.

Definition

- The **isoperimetric profile** of $\mathbf{Y}$ is defined as the function $\mathbf{I}\colon (0,\infty) \to (0,\infty)$ given by

$$\mathbf{I}(t) = \inf\{\mathbf{Area}(\partial\boldsymbol{\Omega})\},$$

where $\overline{\boldsymbol{\Omega}} \subset \mathbf{Y}$ is a smooth compact domain with $\mathbf{Volume}(\boldsymbol{\Omega}) = t$.

- Note that $\mathrm{Ch}(\mathbf{Y}) = \inf\{\frac{\mathbf{I}(t)}{t} \mid t \in (0,\infty)\}$.

Definition

The **radius** of a compact Riemannian manifold with boundary is the maximum distance from points in the manifold to its boundary.

Definition

The **diameter** of a compact Riemannian manifold with boundary is the maximum distance between points in the manifold.

Theorem (Meeks-Mira-Pérez-Ros)

Suppose $\mathbf{Y}$ is a non-compact, simply connected homogeneous $\mathbf{3}$-manifold with $\mathrm{Ch}(\mathbf{X})$. Then:

1. $\mathrm{Ch}(\mathbf{Y}) = 2\mathbf{H}(\mathbf{Y}) = \lim_{t\to\infty} \frac{\mathbf{I}(t)}{t}$.
2. If $\mathbf{Y}$ is not isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$ for some $\kappa > 0$ and $\mathbf{\Omega} \subset \mathbf{Y}$ is an isoperimetric domain in $\mathbf{Y}$ with volume t, then:
   a. $\partial\mathbf{\Omega}$ is connected and has $\mathbf{H} > 0$ as the boundary of $\mathbf{\Omega}$.
   b. $\mathrm{Ch}(\mathbf{Y}) < \min\left\{2\mathbf{H}_{\partial\mathbf{\Omega}}, \dfrac{\mathbf{I}(t)}{t}\right\}$, where $\mathbf{H}_{\partial\mathbf{\Omega}}$ is the constant mean curvature of the boundary of $\mathbf{\Omega}$.
3. Let $\mathbf{\Omega}_n \subset \mathbf{Y}$ be any sequence of isoperimetric domains with volumes tending to infinity and let $\mathbf{R}_n$ be the radius of $\mathbf{\Omega}_n$. Then:
   a. $\lim_{n\to\infty} \mathbf{R}_n = \infty$.
   b. $\lim_{n\to\infty} \mathbf{H}_{\partial\mathbf{\Omega}_n} = \mathbf{H}(\mathbf{Y})$.

Corollary (Meeks-Mira-Pérez-Ros)

- Let $\mathbf{X}$ be a metric Lie group diffeomorphic to $\mathbf{R}^3$.
- Given $\mathbf{L}$, $\mathbf{R} > 0$, there exists a $\mathbf{C} > 0$ such that for all compact immersed minimal surfaces $\boldsymbol{\Sigma}$ with boundary of total length at most $\mathbf{L}$ and contained in an extrinsic ball of radius $\mathbf{R}$, then

$$\mathbf{Area}(\boldsymbol{\Sigma}) \leq \mathbf{C} \cdot \mathbf{Length}(\partial\boldsymbol{\Sigma}).$$

Proof.

- Fix the length $\mathbf{L} > 0$ and the radius $\mathbf{R}$.
- Let $\boldsymbol{\Sigma}_n \subset B_{\mathbf{X}}(e, \mathbf{R})$ be a sequence of compact surfaces with $|\mathbf{H}_{\boldsymbol{\Sigma}}| \leq \mathbf{H}(\mathbf{X})$, length of boundaries at most $\mathbf{L}_n \leq \mathbf{L}$ and areas $\mathbf{A}_n \to \infty$.
- Since radii of isoperimetric domains with volume $\to \infty$ are arbitrarily large, $\exists$ an isoperimetric domain $\boldsymbol{\Omega}$ such that $\boldsymbol{\Sigma}_n \subset B_{\mathbf{X}}(e, \mathbf{R}) \subset \boldsymbol{\Omega}$.
- By White, a subsequence of the $\boldsymbol{\Sigma}_n$ converges to a varifold $\boldsymbol{\Sigma}(\infty)$ with $\mathbf{H}_{\boldsymbol{\Sigma}(\infty)} \leq \mathbf{H}(\mathbf{X}) < \mathbf{H}_{\partial\boldsymbol{\Omega}}$ by previous theorem.
- Translate $\boldsymbol{\Sigma}(\infty)$ until its support touches $\partial\boldsymbol{\Omega}$ a first time.
- This is impossible by a maximum principle (White) for $\mathbf{2}$-varifolds V with $\mathbf{H}_V \leq \mathbf{H}_{\partial\boldsymbol{\Omega}}$. $\square$

Isoperimetric Inequality and Radius Estimate in $\mathbb{H}^3$ (Meeks-Mira-Pérez-Ros)

- Let $\mathbf{X} = \mathbb{H}^3$.
- By the mean curvature comparison principle, every compact immersed surface in $\mathbf{X}$ with absolute mean curvature function $|\mathbf{H}_{\mathbf{\Sigma}}|$ less than or equal to $\mathbf{1} = \mathbf{H}(\mathbf{X})$ and $\mathbf{1}$ boundary curve of length at most $\mathbf{L}$ lies in an extrinsic ball of radius less than $\mathbf{R} = \mathbf{L}/\mathbf{2}$.
- More generally, given $\mathbf{L} > 0$, $\exists \mathbf{D}(\mathbf{L}) > 0$ such that every compact immersed surface $\mathbf{\Sigma}$ in $\mathbf{X}$ with $|\mathbf{H}_{\mathbf{\Sigma}}| \leq \mathbf{1} = \mathbf{H}(\mathbf{X})$ and boundary of length at most $\mathbf{L}$ has diameter less than $\mathbf{D}(\mathbf{L})$; the argument here is nontrivial.
- Thus, the previous corollary implies that an isoperimetric inequality holds for such surfaces in $\mathbf{X}$.

Isoperimetric Inequality for general $\mathbf{X}$ for connected boundary surfaces

- **Meeks-Mira-Pérez-Ros** prove for $\mathbf{X}$ diffeomorphic to $\mathbf{R}^3$, compact immersed surfaces with absolute mean curvature function $|\mathbf{H}_{\mathbf{\Sigma}}|$ less than or equal to $\mathbf{H}(\mathbf{X})$ and $\mathbf{1}$ boundary curve of length at most $\mathbf{L}$ have a uniform bound on their intrinsic radii.
- This fact is deep and uses results concerning $\mathbf{H}$-spheres from Lecture $\mathbf{4}$.
- **Meeks-Mira-Pérez-Ros** also prove a similar result for minimal surfaces in $\mathbf{X}$ with at most two boundary curves.

Theorem (Isoperimetric Inequality, Meeks-Mira-Pérez-Ros)

- Let $\mathbf{X}$ be a metric Lie group diffeomorphic to $\mathbf{R}^3$.
- Given $\mathbf{L} > 0$, $\exists\ \mathbf{C} > 0$ such that $\forall$ compact immersed surfaces $\boldsymbol{\Sigma}$ with one boundary curve of total length at most $\mathbf{L}$ and absolute mean curvature function $|\mathbf{H}_{\boldsymbol{\Sigma}}|$ less than or equal to $\mathbf{H}(\mathbf{X})$, then

$$\mathbf{Area}(\boldsymbol{\Sigma}) \leq \mathbf{C} \cdot \mathbf{Length}(\partial\boldsymbol{\Sigma}).$$

Theorem (Minimal Isoperimetric Inequality, Meeks-Mira-Pérez-Ros)

- Let $\mathbf{X}$ be a metric Lie group diffeomorphic to $\mathbf{R}^3$.
- Given $\mathbf{L} > 0$, $\exists\ \mathbf{C} > 0$ such that $\forall$ compact immersed minimal surfaces $\boldsymbol{\Sigma}$ with one or two boundary curves of total length at most $\mathbf{L}$, then

$$\mathbf{Area}(\boldsymbol{\Sigma}) \leq \mathbf{C} \cdot \mathbf{Length}(\partial\boldsymbol{\Sigma}).$$

Theorem (DeChang Chen)

- Given $\mathbf{H}_0 \geq 0$, $\exists\ \mathbf{R}_0 > 0$ such that the following hold.
- Let $\mathbf{Y}$ be a simply connected Riemannian $\mathbf{3}$-manifold with absolute sectional curvature at most $\mathbf{1}$.
- $\forall$ compact immersed surfaces $\mathbf{\Sigma} \subset \mathbf{Y}$ with $|\mathbf{H}_{\mathbf{\Sigma}}| \leq \mathbf{H}_0$, then

$$\mathbf{Radius}(\mathbf{\Sigma}) \leq \mathbf{R}_0 \cdot \mathbf{Area}(\mathbf{\Sigma}).$$

Corollary (Meeks-Mira-Pérez-Ros)

- Let $\mathbf{X}$ be a metric Lie group diffeomorphic to $\mathbf{R}^3$.
- Given $\mathbf{L} > 0$, $\exists\ \mathbf{D_0}$ such that $\forall$ compact immersed surfaces $\mathbf{\Sigma}$ with one boundary curve of total length at most $\mathbf{L}$ and absolute mean curvature function $|\mathbf{H}_{\mathbf{\Sigma}}|$ less than or equal to $\mathbf{H}(\mathbf{X})$, then

$$\mathbf{Radius}(\mathbf{\Sigma}) < \mathbf{Diameter}(\mathbf{\Sigma}) \leq \mathbf{D_0}.$$

- Furthermore, this same result holds for minimal surfaces in $\mathbf{X}$ with at most $\mathbf{2}$ boundary components.

Theorem (Isoperimetric Inequality in $\mathbf{R}^3$)

- Given $\mathbf{L} > 0$, then $\forall$ compact immersed minimal surfaces $\boldsymbol{\Sigma} \subset \mathbb{R}^3$ with one boundary curves of total length at most $\mathbf{L}$, then

$$\mathbf{Area}(\boldsymbol{\Sigma}) \leq \frac{1}{4\pi} \cdot [\mathbf{Length}(\partial\boldsymbol{\Sigma})]^2.$$

- Furthermore, if one has equality in the above formula, then $\boldsymbol{\Sigma}$ is a round disk in a flat plane in $\mathbf{R}^3$.

Conjecture (Isoperimetric Inequality Conjecture in $\mathbf{R}^3$)

- Given $\mathbf{L} > 0$, then $\forall$ compact immersed minimal surfaces $\boldsymbol{\Sigma}$ with boundary $\partial\boldsymbol{\Sigma}$ of length at most $\mathbf{L} > 0$, then:

$$\mathbf{Area}(\boldsymbol{\Sigma}) \leq \frac{1}{4\pi} \cdot [\mathbf{Length}(\partial\boldsymbol{\Sigma})]^2.$$

- Furthermore, if one has equality in the above formula, then $\boldsymbol{\Sigma}$ is a round disk in a plane in $\mathbf{R}^3$.

Conjecture (Isoperimetric Inequality Conjecture, Meeks-Mira-Pérez-Ros)

Let $\mathbf{X}$ be a metric Lie group diffeomorphic to $\mathbf{R}^3$.

1. $\exists$ $\mathbf{C} > 0$ such that $\forall$ compact immersed surfaces $\mathbf{\Sigma}$ with boundary $\partial\mathbf{\Sigma}$ of length at most $\mathbf{L} > 0$ and absolute mean curvature function $|\mathbf{H}_{\mathbf{\Sigma}}|$ less than or equal to $\mathbf{H}(\mathbf{X})$, then
$$\mathbf{Area}(\mathbf{\Sigma}) \leq \mathbf{C} \cdot [\mathbf{Length}(\partial\mathbf{\Sigma})]^2.$$
2. Furthermore, if $\mathbf{X}$ is $\mathbb{H}^3$ with it usual metric, then the constant $\mathbf{C} = \dfrac{1}{4\pi}$ works in the above formula and if one has equality in the above formula, then $\mathbf{\Sigma}$ is a round disk in a horosphere in $\mathbb{H}^3$.

Remark

- Item **1** holds for minimal surfaces with at most **2** boundary components (see Lecture **4** for the proof).
- One important consequence of this conjecture is that complete embedded $\mathbf{H}$-surfaces of finite topology in any $\mathbf{X}$ would have bounded second fundamental form when $\mathbf{H} \in (0, \mathbf{H}(\mathbf{X}))$.
- This bounded curvature result will be proved in Lecture **4**.

Definition

- Let $\mathbf{Y}$ be a $\mathbf{3}$-dimensional homogeneous manifold and $\mathbf{\Gamma}$ be a 1-parameter subgroup of the isometry group of $\mathbf{Y}$.
- We say that a properly embedded surface $\mathbf{\Sigma} \subset \mathbf{Y}$ is an entire **$\mathbf{\Gamma}$-Killing graph** if **each orbit** of the left action of $\mathbf{\Gamma}$ on $\mathbf{Y}$ intersects $\mathbf{\Sigma}$ in exactly $\mathbf{1}$ point.

Example

If $\mathbf{H}$ is a $\mathbf{2}$-dimensional subgroup of $\mathbf{X}$, then $\mathbf{H}$ is a Killing graph with respect to some 1-parameter subgroup of $\mathbf{X}$.

A relationship of Killing graphs with the critical mean curvature

- If $\mathbf{\Sigma} \subset \mathbf{Y}$ is an entire **$\mathbf{\Gamma}$-Killing graph** with respect to a 1-parameter subgroup $\mathbf{\Gamma}$ and $\mathbf{\Sigma}$ is a noncompact $\mathbf{H}$-surface, then $\mathcal{F} = \{a\mathbf{\Sigma}) \mid a \in \mathbf{\Gamma}\}$ is an $\mathbf{H}$-foliation of $\mathbf{Y}$.
- If $\mathbf{Y}$ is noncompact, then every compact immersed surface $\mathbf{\Delta}$ in $\mathbf{Y}$ intersects and lies on the mean convex side of one of the leaves $b\mathbf{\Sigma}$ of $\mathcal{F}$.
- Let $p \in \mathbf{\Delta} \cap b\mathbf{\Sigma}$. By the mean curvature comparison principle, $\mathbf{max}(|\mathbf{H}_{\mathbf{\Delta}}|) \geq |\mathbf{H}_{\mathbf{\Delta}}|(p) > \mathbf{H}$.
- By definition of the critical mean curvature, $\mathbf{H}(\mathbf{Y}) \geq \mathbf{H}$.

Crucial in the proof of the equality $\mathrm{Ch}(\mathbf{Y}) = 2\mathbf{H}(\mathbf{Y})$ is the next theorem.

Theorem ($\mathbf{H}(\mathbf{X})$-Foliation Theorem, Meeks-Mira-Pérez-Ros)

Let $\mathbf{X}$ be diffeomorphic to $\mathbf{R}^3$ with $\mathrm{Ch}(\mathbf{X}) > 0$. Then:

- $\mathbf{X}$ contains a properly embedded $\mathbf{H}(\mathbf{X})$-surface $\mathbf{\Sigma}$ that is a $\mathbf{\Gamma}_1$-Killing graph for some 1-parameter subgroup $\mathbf{\Gamma}_1$.
- $\mathbf{\Sigma}$ is invariant under elements of a 1-parameter subgroup $\mathbf{\Gamma}_2$ and an infinite normal cyclic subgroup $\mathbb{Z} \not\subset \mathbf{\Gamma}_2$ whose elements commute with $\mathbf{\Gamma}_2$.
- $\mathbf{\Gamma}_2$ contains a cyclic subgroup $\mathbb{Z}'$ such that

  $$[\mathbf{\Sigma}/(\mathbb{Z} \times \mathbb{Z}')] \subset [\mathbf{Y} = \mathbf{X}/(\mathbb{Z} \times \mathbb{Z}')]$$

  is a torus.
- $\mathbf{\Sigma}/(\mathbb{Z} \times \mathbb{Z}')$ bounds a region of finite volume in $\mathbf{Y}$ and it is the unique solution to the isoperimetric problem in $\mathbf{Y}$ with this volume.
- Given any sequence of isoperimetric domains $\mathbf{\Omega}_n \subset \mathbf{X}$ with volumes tending to infinity, after left translations, the $\mathbf{\Omega}_n$ converge to the mean convex component of $\mathbf{X} - \mathbf{\Sigma}$ and $\mathbf{\Sigma} = \lim_{n\to\infty} \partial\mathbf{\Omega}_n$.

Remark (Related **CMC** foliations, Meeks-Pérez-Ros)

- Suppose $\mathbf{X} = \mathbb{H}^3 = \mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$, where $\mathbf{A}$ is the identity matrix and suppose $\mathcal{F}$ is a **CMC** foliation of $\mathbf{X}$.
- If every leaf of $\mathcal{F}$ has constant mean curvature at least $\mathbf{H}(\mathbf{X}) = 1$, then $\mathcal{F}$ is a foliation by horospheres.
- In this case the surface $\boldsymbol{\Sigma}$ in the previous theorem must be a horosphere, and so, it is unique up to ambient isometry.

Remark (Related **CMC** foliations, Meeks-Mira-Pérez-Ros)

- In the case of $\mathbf{X} = \mathbb{H} \times \mathbb{R}$ with the product metric, there are many vertical $\mathbf{H}(\mathbf{X})$-graphs over $\mathbb{H}$ (and so they are Killing graphs).
- But any complete, embedded doubly-periodic $\mathbf{H}(\mathbf{X})$-surface $\boldsymbol{\Sigma}'$ in $\mathbf{X}$ must be a leaf of the $\mathbf{H}(\mathbf{X})$-foliation arising from $\boldsymbol{\Sigma}$ and $\boldsymbol{\Gamma}_1$ in the previous theorem.

Theorem (Curvature Estimates for CMC Foliations, Meeks-Pérez-Ros)

- Suppose that $\mathcal{F}$ is a a **CMC** foliation of a homogeneous **3-dim Y**.
- Then the leaves of $\mathcal{F}$ have bounded second fundamental form and any leaf **L** of $\mathcal{F}$ with maximal mean curvature is strongly stable, i.e., it admits a positive Jacobi function.
- **Y** always admits a limit "weak" **CMC** foliation $\mathcal{F}'$ of some divergent sequence of translations of $\mathcal{F}$ such that $\mathcal{F}'$ has a leaf having constant mean curvature equal to the supremum of the absolute mean curvatures of the leaves of $\mathcal{F}$, and any such leaf is strongly stable.

Conjecture (Strong Stability Conjecture, Meeks-Mira-Pérez-Ros)

- A complete strongly stable **H**-surface in **X** with $\mathbf{H} \geq \mathbf{H}(\mathbf{X})$ is a Killing graph and so $\mathbf{H} = \mathbf{H}(\mathbf{X})$.
- In particular, if $\mathbf{H}(\mathbf{X}) = 0$, then any complete, strongly stable minimal surface $\boldsymbol{\Sigma}$ in **X** is a leaf of a minimal foliation of **X** and so, $\boldsymbol{\Sigma}$ is actually homologically area-minimizing in **X**.
- Hence, by the above theorem, any **CMC** foliation of an **X** isomorphic to $\mathbf{R}^3$, $\mathbf{Nil}_3$, $\widetilde{\mathbf{E}}(\mathbf{2})$ or $\mathbf{Sol}_3$ would be a minimal foliation.

Conjecture (Product CMC-foliation Conjecture, Meeks-Mira-Pérez-Ros)

Let $\mathcal{F}$ be a **CMC** foliation of a homogeneous **3**-dimensional **Y**.

- The constant mean curvatures of the leaves of $\mathcal{F}$ are at most $\mathbf{H}(\mathbf{Y})$.
- Topologically, $\mathcal{F}$ is a product foliation by planes or by spheres.
- If $\mathbf{Y} \approx \mathbf{R}^3$ and $\mathbf{p} \in \mathbf{Y}$, $\exists$ a product foliation of $\mathbf{Y} - \{\mathbf{p}\}$ by **H**-spheres.
- If $\mathbf{Y} \approx \mathbf{R}^3$ and $\mathcal{F}$ is an $\mathbf{H}(\mathbf{X})$-foliation, then:
    - Every leaf of $\mathcal{F}$ is some $\boldsymbol{\Gamma}$-Killing graph and $\mathcal{F}$ is the related "Killing"-foliation.
    - If $\mathrm{Ch}(\mathbf{X}) > 0$ and $\mathcal{F}$ has a leaf of quadratic area growth, then, up to ambient isometry, $\mathcal{F}$ is the foliation given in the $\mathbf{H}(\mathbf{X})$-Foliation Theorem of **Meeks-Mira-Pérez-Ros**.
    - If $\mathrm{Ch}(\mathbf{X}) = 0$ and $\mathcal{F}$ has a leaf of quadratic area growth, then, up to ambient isometry, $\mathcal{F}$ is the foliation of horizontal planes in a semidirect product structure $\mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$ for **X**.