Optimal convergence rates for Karsnoselskii-Mann fixed-point iterations

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Resumo/Abstract:
KEYWORDS: Nonexpansive maps, fixed points, Krasnoselskii-Mann iteration, asymptotic regularity

A popular method to compute a fixed point for a non-expansive map $T : C \to C$ is the following successive average iteration originally proposed by Krasnoselskii and Mann

$$(KM) \qquad x_{n+1} = (1 - \alpha_{n+1})x_n + \alpha_{n+1}Tx_n.$$

We show how optimal transport can be used to establish a recursive formula to estimate the distance between iterates $\|x_m - x_n\| \leq d_{mn}$. The recursive optimal transport d_{mn} induces a metric on the integers that allows to study the rate of convergence of (KM). As a result, we settle Baillon and Bruck's conjecture for the rate of convergence of the fixed point residuals: for every non-expansive map in any normed space the following estimate holds with $\kappa = 1/\sqrt{\pi}$

$$\|x_n - Tx_n\| \leq \kappa \frac{\text{diam}(C)}{\sqrt{\sum_{i=1}^{n} \alpha_k(1 - \alpha_k)}}.$$

The analysis exploits an unexpected connection with discrete probability and combinatorics, related to the Gambler's ruin for sums of non-homogeneous Bernoulli trials. We will also discuss the extent to which the constant $\kappa = 1/\sqrt{\pi}$ is sharp.

References

[1] R. Cominetti, J. Soto, J. Vaisman [2014], *On the rate of convergence of Krasnoselskii-Mann iterations and their connection with sums of Bernoullis*, Israel Journal of Mathematics 199(2), pp. 1–16.