Derived categories
of functors categories

ABSTRACT

Let $\mathcal{C}$ be small category and $\mathcal{A}$ an arbitrary category. Consider the category $\mathcal{C}(\mathcal{A})$ whose objects are functors from $\mathcal{C}$ in $\mathcal{A}$ and whose morphisms are natural transformations. Let $\mathcal{B}$ be other category, and again, consider the category $\mathcal{C}(\mathcal{B})$. Now, given a functor $F : \mathcal{A} \to \mathcal{B}$ we construct the induced functor $F_{\mathcal{C}} : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$.

Assuming $\mathcal{A}$ and $\mathcal{B}$ to be abelian categories we have the categories $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$ is also abelian. Therefore, it makes sense to talk about the derived category $D(\mathcal{C}(\mathcal{A}))$. Moreover, if $\mathcal{A}$ has enough injectives one can prove that $\mathcal{C}(\mathcal{A})$ also has enough injectives, which guarantees the existence of the derived functor R $(F_{\mathcal{C}}) : D(\mathcal{C}(\mathcal{A})) \to D(\mathcal{C}(\mathcal{B}))$.

In this work we have two main goals:

1. to find a relationship between $D(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(D(\mathcal{A}))$;
2. relate the functors $R(F_{\mathcal{C}})$ and $(RF)_{\mathcal{C}} : \mathcal{C}(D(\mathcal{A})) \to \mathcal{C}(D(\mathcal{B}))$.

Initially we show that $Kom(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(Kom(\mathcal{A}))$ are isomorphic categories, where $Kom(\mathcal{A})$ denotes the category of complexes of $\mathcal{A}$. We also show that if $\mathcal{Q}$ is a category generated by a quiver without relations, then $D(\mathcal{Q}(\mathcal{A}))$ is a full subcategory of $\mathcal{Q}(D(\mathcal{A}))$. And finally, we show that $D(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(D(\mathcal{A}))$ are equivalent categories if and only if $\mathcal{C} = \mathcal{Q}$, where $\mathcal{Q}$ is a category generated by a quiver without arrows.

Towards the second goal, we show that if the functor $(RF)_{\mathcal{Q}}$ is an equivalence of categories then $R(F_{\mathcal{Q}})$ is also an equivalence.

We use this results to prove a version of Mukai's Theorem for coherent Q-quasi-coherent sheaves.

Keywords: derived categories, functor categories, Q-quasi-coherent sheaves.