Minicurso 1: Diagonal arithmetics

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Abstract:
The purpose of this mini-course is to survey some of the recent advances concerning the (stable) rationality properties of algebraic varieties. An irreducible variety X over a field k is rational if it is k-birational to projective space, equivalently, if X is isomorphic to projective space away from a finite collection of proper closed subvarieties. We say that X is stably rational if a product of X with some projective space becomes rational and say that X is unirational if it is rationally dominated by a projective space. A classical question is to distinguish the properties of rationality and unirationality. In dimension 1, this is the Lüroth problem; in dimension 2 over the complex numbers, Castelnuovo's theorem provides a set of discrete invariants to determine rationality. In higher dimension, even over the complex numbers, the problem is very challenging. In the 1970s, three examples of unirational but not rational complex varieties were discovered: cubic threefolds (Clemens and Griffiths), some quartic threefolds (Iskovskikh and Manin), and some conic bundles over rational surfaces (Artin and Mumford). The example of Artin and Mumford is not stably rational, but it was not known if this property holds for other examples.

There are examples, due to Beauville, Colliot-Thélène, Sansuc, and Swinnerton-Dyer of varieties that are stably rational but not rational. In general, proving that a variety is not stably rational is harder than proving that it is not rational. In general, one wants to search for obstructions to rationality or stably rationality. Unramified cohomology groups, among them the torsion in the Picard group and the Brauer group, provide natural stable equivalence invariants that can be used to obstruct stable rationality. Also, the Chow group of zero-cycles up to rational equivalence is a stable invariant. A powerful theorem of Merkurjev explains how the Chow group of zero-cycles, considered not over the base field, but also over

all extensions F/k of the base field, in some sense controls all invariants of "unramified cohomology type." Starting from the work of Bloch and Srinivas, one investigates how the "universal Chow group of zero-cycles" is controlled by Chow theoretic decompositions ofthe diagonal.

In 2013, C. Voisin proved that a general quartic double solid with at most seven nodes is not stably rational. Following her work, joint with J.-L. Colliot-Thlne, we showed that a lot of quartic threefolds are not stably rational. Recently, B. Totaro managed to extend this result to a very general degree d hypersurface in a projective space of dimension $n+1$ with $d/2$ at least the smallest integer not less than $(n+2)/3$. Also a similar technique was recently used by Beauville to prove that a general sextic double solid is not stably rational. In this series of lectures we will discuss in detail the methods to establish the results above: properties of the decomposition of the diagonal in the Chow groups, the universal properties of the Chow group of zero cycles,. and the specialization techniques.