The Lê-Greuel formula for functions on analytic spaces

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Abstract

The classical Lê-Greuel formula, proved independently by G.-M. Greuel and Lê Dũng Tráng says that if $f_1, \cdots, f_k$ and g are holomorphic map germs $(\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}, 0)$ such that $f = (f_1, \cdots, f_k)$ and (f, g) define isolated complete intersection germs (ICIS for short), then the Milnor number $\mu(f, g)$ of (f, g) can be computed in terms of that of f by the formula: $\mu(f, g) + \mu(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+k,0}}{(f, Jac(f,g))}$, where $Jac(f, g)$ denotes the ideal generated by the determinants of all the $(k+1)$ minors of the corresponding Jacobian matrix. The term on the right can also be regarded as the number of critical points of a Morsification of g on a Milnor fibre of f.

In this work we extend this theorem to the rather general setting of a function $f = (f_1, \cdots, f_k) : X \to \mathbb{C}^k$ defined on a complex analytic variety X with arbitrary singular set, and another function $g : X \to \mathbb{C}$. We impose some conditions on these functions in order to have Milnor, or rather Milnor-Lê, type fibrations. We ask f to be generically a submersion with respect to some Whitney stratification on X, the dimension of its zero set $V(f)$ must be larger than 0, and f must have the Thom property with respect to this stratification. Then we ask g to have an isolated critical point in the stratified sense, both on X and on $V(f)$ of f. We show that the formula of Lê and Greuel extends to the general setting described above. We also restrict to the case when the singularities of X are all contained in V, so that the fibers F_f and $F_{f,g}$ are non-singular. We get, in particular, an extension of that formula to the case of complete intersection germs in $\mathbb{C}^{n+k}$ which does not require that f defines an ICIS, f can be an arbitrary complete intersection germ, provided it has the Thom property and g has an isolated singularity.