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Title: Specification of the S. Ulam's theorem on the topological conjugation of one-dimensional maps

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Abstract:

The conjugation of the maps $f,\, \widetilde{g}:\, [0,\, 1]\to [0,\, 1]$, where

$$f(x) = 1-|1-2x| \tag{1}$$

and $\widetilde{g}(x) = 4x(1-x)$, is the classical fact, which is mentioned in almost all books on one-dimensional dynamics. It appeared at first in S. Ulam's and J. von Neumann collaboration, which was devoted to the random generators (see for ex. [?] and [?]). Later, S. Ulam generalized the conjugation of f and $\widetilde{g}$ above as follows. He considered the map

$$g(x) = \begin{cases} g_l(x), & \text{if } \ 0\leq x< v,\\ g_r(x), & \text{if } \ v\leqslant x\leqslant 1, \end{cases} \tag{2}$$

where $v\in (0,\, 1)$, $g(0)=g(1)=0$, $g(v)=1$ and g_l, g_r are monotone functions, which make g continuous. [?, Appendix 1, §3] Let $f,\, g:\, [0,\, 1]\to [0,\, 1]$ be defined by (??) and (??), where g is convex. Then g is topologically conjugated to f if and only if the integer trajectory of 1 under g is dense in $[0,\, 1]$.

Noticing, that convexity of g is not used in the original proof of Theorem ??, we have obtained the construction, which led us to the following example.

There exist non-convex maps g of the form (??), which are conjugated to f.

Moreover, in our example both g and the conjugacy are piecewise linear. The following theorem illustrates the complicatedness of the class of functions, which are topologically conjugated to (??).

Let $f:\, [0,\, 1]\to [0,\, 1]$ be given by (??). For every $x_0\in [0,\, 1]$ and $\varepsilon>0$ there exists $g:\, [0,\, 1]\to [0,\, 1]$ of the form (??) such that:

1. $g(x)=f(x)$ for each $x\in [0,\, 1]\setminus (x_0-\varepsilon,\, x_0+\varepsilon)$;
2. f and g are not topologically conjugated.

References

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