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Topologically conjugated piecewise linear unimodal maps

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We study the topological conjugation of dynamical systems, generated by one-dimensional unimodal map of unit interval. This problem was introduced in the collaboration of S. Ulam and J. von Neumann, where it was established the topological conjugation of the maps $f, \widetilde{g} : [0, 1] \to [0, 1]$, where $f(x) = 1 - |1 - 2x|$ and $\widetilde{g}(x) = 4x(1 - x)$. S. Ulam proved the following characterization of topological conjugation of f and g, of the form

$$g(x) = \begin{cases} g_l(x), & \text{if } 0 \leq x < v, \\ g_r(x), & \text{if } v \leqslant x \leqslant 1, \end{cases} \tag{1}$$

where $v \in (0, 1)$.

Theorem 1. *[2, Appendix 1, §3] Let* $f, g\,[0,\ 1] \to [0, 1]$ *be unimodal maps such that* $f(x) = 1 - |1 - 2x|$ *and* g *is a convex of the form* (1)*. Then* g *is topologically conjugated to* f *if and only if the integer trajectory of* 1 *under* g *is dense in* $[0, 1]$.

In spite of that proof of Theorem 1, given in [2], is simple, the convexity of g is not neither used, or even commented there. We understand this as a fact, that S. Ulam, in fact, claimed that g, which is topologically conjugated to f, should be convex, but he could not to prove this claim properly. We have constructed an example of non-convex unimodal $g : [0, 1] \to [0, 1]$, which is topologically conjugated to f. This example was obtained during the study of topological conjugation of f via piecewise linear homeomorphism. We will present the following facts.

Theorem 2. *[1, Theorem 2] For every* $v \in (0, 1)$ *and arbitrary increasing piecewise linear* $g_1 : [0, v] \to [0, 1]$ *such that* $g_1(v) = 1$, $g_1'(0) = 2$ *there exists a mapping* g *of the form (1), which is conjugated with* f *via piecewise linear homeomorphism. Moreover, such* g *is uniquely defined by* g_1.

Theorem 3. *[1, Theorem 3] Let* $v \in (0, 1)$ *be arbitrary and* $g_2 : [v, 1] \to [0, 1]$ *be decreasing piecewise linear with the unique fixed point* x_0 *such that* $g_2(v) = 1$, $g_2(1) = 0$ *and* $(g_2^2)'(x_0) = 4$*. Then there exists a mapping* g *of the form (1), which is conjugated with* f *via piecewise linear homeomorphism. Moreover, such* g *is uniquely defined by* g_2.

Let g of the form (1) be piecewise linear. Denote the number of pieces of linearity of g_1 and g_2 by p and q respectively. We call the pair (p, q) the **type of piecewise linearity** of g. If for a pair (p, q) there exists a mapping g of the form (1), whose type of piecewise linearity is (p, q), then call this type **admissible**.

Theorem 4. *1. For any* $p \geq 2$ *and* $q \geq 2$ *the type of linearity* (p, q) *is admissible.*

2. A type of linearity $(p, 1)$ *and* $(1, q)$ *is admissible only if it is* $(1, 1)$*. In this case the mapping* g *coincides with* f.

References

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[2] P. R. Stein and S. M. Ulam, Non-linear transformation studies on electronic computers, Rozprawy Mat. 39 (1964), l-66. (also in Stanislaw Ulam: Sets, Numbers, and Universes, edited by W. A. Beyer, J, Mycielski, and G.-C. Rota. Cambridge, Massachusetts: The MIT Press, 1974 – pp. 401-484).