The Plateau Problem for Gaussian Curvature

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Introduction

1.1 Gaussian Curvature.

The purpose of this text is to solve a general Plateau problem for hypersurfaces of constant gaussian curvature in Euclidean Space. We first introduce the concept of gaussian curvature (also known as extrinsic curvature). We begin with the 2-dimensional case. Let S be a smooth, oriented, embedded surface in 3-dimensional Euclidean Space. We use the terminology of riemannian geometry (c.f. [6]). Let N be the unit, normal vector field over S compatible with the orientation. We denote by A the **shape operator** of S (also known as the **Weingarten operator**). We recall that at every point, x, of S, $A(x)$ defines a linear map from the tangent space of S at x to itself, and is expressed in terms on the derivative of N at x by the formula:

$$A(x)(U) = D\mathsf{N}(x)(U),$$

for any vector U which is tangent to S at x. We define the function $\kappa : S \to \mathbb{R}$ by:

$$\kappa = \mathrm{Det}(A),$$

and we refer to κ as the **gaussian curvature** of S.

The study of the gaussian curvature of surfaces is motivated by its relationship to intrinsic geometry. Indeed, since the ambient space is flat, by **Gauss' equation**, if we denote by R the Riemann curvature tensor of S, then for all vectors U, V and W tangent to S at x:

$$R(x)_{UV}W = \langle A(x)(V), W\rangle A(x)(U) - \langle A(x)(U), W\rangle A(x)(V).$$

Moreover, since S is 2-dimensional, the Riemann curvature tensor is determined entirely by the **scalar curvature**, which we denote Scal and which is given by the formula:

$$\mathrm{Scal}(x) = \langle R(x)_{e_1e_2}e_2, e_1\rangle,$$

where e_1, e_2 is any orthonormal basis of T_xS at x. In particular, Gauss' equation yields:

$$\mathrm{Scal} = \mathrm{Det}(A) = \kappa.$$

This leads to the following remarkable conclusion: that, despite being constructed via *extrinsic* data, the gaussian curvature is an *intrinsic* property. This is formally described by Gauss' celebrated **Theorema Egregium**:

Theorem 1.1, Gauss (1825/1827)

Let S_1 and S_2 be two smooth, oriented, embedded surfaces in Euclidean 3-space. Let κ_1 and κ_2 be the gaussian curvatures of S_1 and S_2 respectively. If there exists a distance preserving diffeomorphism $\Phi : S_1 \to S_2$, then for all points $x \in S_1$:

$$(\kappa_2 \circ \Phi)(x) = \kappa_1(x).$$

The difference between extrinsic and intrinsic properties may be quite subtle for the novice reader, and is perhaps best illustrated by considering the mean curvature H of the surface which is defined via the trace of the shape operator:

$$H = \frac{1}{2}\mathrm{Tr}(A).$$

Gauss' Theorema Egregium does not apply to H, and this has straightforward real-world consequences. Indeed, consider a sheet of paper. One may show that its gaussian curvature is equal to zero. Thus, whether it is laid flat on a desk, rolled into a cylinder, or even buckled in a smooth manner, the determinant of the resulting shape operator will vanish at every point. On the other hand, if this sheet of paper is laid flat on a desk, then the trace of the resulting shape operator vanishes, but if it is rolled up into a cylinder, the trace of the resulting shape operator can be made as high as we wish, and if it is buckled in a smooth manner, we can even ensure that the trace of the resulting shape operator varies from point to point. The gaussian curvature is thus a property intrinsic to the paper, wheras the mean curvature depends also on its current shape.

1.2 The Plateau Problem.

We now consider hypersurfaces of $(n+1)$-dimensional Euclidean Space, for arbitrary n. If S is a smooth, oriented, embedded hypersurface in Euclidean Space, and if A is its shape operator, then, as before, we define its **gaussian curvature**, κ to be equal to the determinant of A. Given a closed, codimension-2, embedded submanifold, C, of $\mathbb{R}^{n+1}$, and given a constant $k > 0$, the **Plateau problem** for gaussian curvature asks for the existence of a compact, smooth, embedded hypersurface of constant gaussian curvature equal to k whose boundary coincides with C.

Before formulating the Plateau problem more precisely, it is worth investigating certain of its geometric aspects. The first is the relationship between gaussian curvature and convexity. We recall that S is said to be **locally strictly convex** whenever its shape operator is everywhere positive definite. We now denote by $\text{Symm}(2, \mathbb{R}^n)$ the space of n-dimensional, symmetric matrices over $\mathbb{R}$ and consider the function $\text{Det} : \text{Symm}(2, \mathbb{R}^n) \rightarrow \mathbb{R}$. $\text{Det}^{-1}(\{0\})$ divides $\text{Symm}(2, \mathbb{R}^n)$ into $(n+1)$ many connected components. We denote these by $A_0, ..., A_n$, so that, for each k, A_k consists of those non-degenerate, symmetric matrices in $\mathbb{R}$ with k strictly negative real eigenvalues and $(n-k)$ strictly positive real eigenvalues. In particular, A_0 coincides with the cone Γ of positive-definite, symmetric matrices over $\mathbb{R}$. Thus, if we denote by $\mathcal{S}$ the set of embedded hypersurfaces of $\mathbb{R}^{n+1}$ whose gaussian curvature never vanishes, then S also decomposes into $(n+1)$ many (not necessarily connected) components. We denote these components by $\mathcal{S}_0, ..., \mathcal{S}_n$, so that, for each k, $\mathcal{S}_k$ consists of those hypersurfaces whose shape operator is at every point conjugate to an element of A_k. In particular, $\mathcal{S}_0$ is the set of locally strictly convex hypersurfaces. It turns out that the qualitative behaviour of an embedded hypersurface whose gaussian curvature never vanishes depends heavily on which of these components it belongs to. Moreover, for reasons related to the analytic concept of ellipticity, hypersurfaces in $\mathcal{S}_0$ are significantly more tractable than those in any other family, and it is for this reason in particular that we henceforth restrict attention to locally strictly convex hypersurfaces.

The second important geometric aspect of the Plateau problem for gaussian curvature is the importance of outer barriers. The situation is illustrated by the case where the boundary curve C is a circle of unit radius in 3-dimensional Euclidean Space. For all $k \in]0, 1[$, there exist exactly two strictly convex, embedded surfaces $S_k^\pm$ of constant gaussian curvature equal to k such that $\partial S_k^\pm = C$. In particular, since there are two solutions, the problem of finding surfaces of constant curvature equal to k whose boundary coincides with C is ill-posed in the sense of partial differential equations. Moreover, we observe that, for all k, $S_k^\pm$ is a portion of a sphere of radius $1/k$, and if we denote by Σ the unique sphere of unit radius containing C, then, exchanging S_k^+ and S_k^- if necessary, we may suppose that S_k^- lies on the inside of Σ whilst S_k^+ lies on its outside. This characterises an important qualitative difference between the small solution, S_k^-, and the big solution, S_k^+. Indeed, when we perturb the boundary curve C away from a circle, whilst it is relatively easy to ensure that the corresponding perturbation of the small solution remains within some compact set, it is not possible to obtain analogous control over the position of the big solution. For this reason, we consider Plateau problems where some sort of outer barrier is prescribed. One conveniant way of doing so is by prescribing a compact, convex subset, K, of $\mathbb{R}^{n+1}$ with smooth boundary such that C is contained in ∂K. We underline that this gives a relatively restricted version of the Plateau problem compared to what may be proven (c.f. [12] and [22]). However, it is nonetheless sufficiently general to state interesting theorems and discuss many of the techniques that arise in the study of this and more general problems.

We are now in a position to state the main result of this text:

Theorem 1.2

Choose $k > 0$. Let K be a compact, convex subset of $\mathbb{R}^{n+1}$ with smooth boundary. Let X be a closed subset of ∂K with C^2 boundary $C = \partial X$. If ∂K has gaussian curvature bounded below by k at every point of $(\partial K) \setminus X$, then there exists a compact, strictly convex, $C^{0,1}$ embedded hypersurface $S \subseteq \mathbb{R}^{n+1}$ with the properties that:

(1) $S \subseteq K$;

(2) $\partial S = C$; and

(3) $S \setminus \partial S$ is smooth and has constant gaussian curvature equal to k.

Remark: *In fact, if ∂X is smooth, then we can show that S is smooth up to the boundary (and not just over its interior). We shall not study this here.*

1.3 The Plateau Problem for Graphs.

A simpler version of the Plateau problem concerns finding hypersurfaces of constant gaussian curvature which are graphs over some affine hyperplane. This is referred to as the **non-parametric** problem. This case constitutes an important intermediate step in the proof of Theorem 1.2, forming the basis of a regularity result for weak solutions. The main advantage of the non-parametric case is that the condition of constant gaussian curvature now translates into a non-linear partial differential equation. Indeed:

Lemma 1.3

Let U be an open subset of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a smooth function. If we define $\kappa : U \to \mathbb{R}$ such that for all $x \in U$, $\kappa(x)$ is the gaussian curvature of the graph of f at the point $(x, f(x))$, then $\kappa(x)$ is given by:

$$\kappa(x) = \mathrm{Det}(D^2 f(x))/(1 + \|Dd(x)\|^2)^{(n+2)/2}.$$

Moreover, the graph of f is locally strictly convex if and only if f is strictly convex.

Remark: *We always orient graphs of functions such that the unit normal vector compatible with the orientation points downwards.*

Proof: We define $\hat{f} : U \to \mathbb{R}^{n+1}$ by $\hat{f}(x) = (x, f(x))$. The function $\hat{f}$ is thus a parametrisation of the graph of f. We define $\mathsf{N} : U \to \Sigma^n$ by:

$$\mathsf{N} = \frac{1}{\sqrt{1 + \|Df\|^2}}(Df, -1).$$

N is thus the unit normal vector field over the graph of f compatible with the orientation. Let $e_1, ..., e_n$ be an orthogonal basis of $\mathbb{R}^n$. Let g be the metric on $\mathbb{R}^{n+1}$. We denote by I_{ij} the matrix of $\hat{f}^* g$. For all i and for all j:

$$\begin{aligned} I_{ij} &= \langle D\hat{f}(e_i), D\hat{f}(e_j) \rangle \\ &= \langle (e_i, f_i), (e_j, f_j) \rangle \\ &= \delta_{ij} + f_i f_j. \end{aligned}$$

We denote by II_{ij} the matrix of the shape operator of the graph of f with respect to the basis $e_1, ..., e_n$. Observe that, for all i, $\langle \mathsf{N}, D\hat{f}(e_i)\rangle = 0$. Thus, for all i and for all j:

$$\begin{aligned} II_{ij} &= \langle D\mathsf{N}(e_i), D\hat{f}(e_j)\rangle \\ &= \langle (1 + \|Df\|^2)^{-1/2} D^2 f(e_i), 0), (e_j, f_j)\rangle \\ &= (1 + \|Df\|^2)^{-1/2} f_{ij}. \end{aligned}$$

Thus II is positive definite if and only if $D^2 f$ is positive definite. In other words, the graph of f is locally strictly convex if and only if f is strictly convex. Moreover, the gaussian curvature is given by $\kappa = \mathrm{Det}(II)/\mathrm{Det}(I)$, and so:

$$\kappa = \mathrm{Det}(D^2 f)/(1 + \|Dd\|^2)^{(n-2)/2},$$

as desired. □

The non-parametric Plateau problem is then solved using techniques from the theory of partial differential equations. Indeed, we denote by $\Gamma \subseteq \mathrm{Symm}(2, \mathbb{R}^n)$ the open cone of positive-definite, symmetric matrices over $\mathbb{R}^n$, and we define $F : \Gamma \to]0, \infty[$ by:

$$F(A) = \mathrm{Det}(A)^{1/n}.$$

We define $G_0 : \mathbb{R}^n \to \mathbb{R}$ by:

$$G_0(\xi) = (1 + \|\xi\|^2)^{\frac{n+2}{2n}}.$$

We leave the reader to verify that G_0 is convex and $G_0 \geqslant 1$. By Lemma 1.3, if $U \subseteq \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$ and if $f : U \to \mathbb{R}$ is a smooth, strictly convex function, then the graph of f has constant extrinsic curvature equal to k^n if and only if:

$$F(D^2 f)/G_0(Df) = k.$$

More generally, let $G : \mathbb{R}^n \to \mathbb{R}$ be any smooth, convex function such that $G \geqslant 1$. The non-parametric Plateau problem then follows from the following result:

Theorem 1.4

Let $\overline{\Omega}$ be a compact, convex subset of $\mathbb{R}^n$ with smooth boundary and non-trivial interior. Let $\phi \in C^\infty(\overline{\Omega})$ be a smooth, positive function. If there exists a strictly convex function $\hat{f} \in C_0^\infty(\overline{\Omega})$ such that:

$$F(D^2 \hat{f})/G(D\hat{f}) > \phi, \qquad \hat{f}|_{\partial\Omega} = 0,$$

then there exists a unique strictly convex function $f \in C_0^\infty(\overline{\Omega})$ such that:

$$F(D^2 f)/G(Df) = \phi, \qquad f|_{\partial\Omega} = 0.$$

1.4 Overview.

The proof of Theorem 1.2 leads us on a tour of many interesting features of the Plateau problem. First, we prove Theorem 1.4, and to do so, we develop an infinite dimensional topological degree theory which forms the content of Chapters 2 and 3. We recall that, in general, topological degree theories are only valid for proper mappings between topological spaces. For this reason, Chapter 2 is devoted to obtaining a compactness result for families of solutions to the partial differential equation given in Lemma 1.3. This compactness result is proven using the powerful (and general) techniques first developed by Caffarelli, Nirenberg and Spruck in [2]. Indeed, the compactness result itself follows from a-priori bounds on the derivatives of the solutions, where the main challenge lies in finding bounds of second-order, after which higher order bounds follow from general principles, and it is in obtaining the second-order bounds that the Caffarelli-Nirenberg-Spruck technique plays a central role. Chapter 3 is then devoted to developing the formal aspects of the degree theory. The main challenge here lies in showing how the tehcniques of finite-dimensional differential topology extend to the current infinite-dimensional case. This is acheived following the approach first described by Smale in [21] using the theory of smooth mappings between open subsets of Banach spaces.

Chapter 4 is devoted to studying the singularites that may arise in Hausdorff limits of families of strictly convex hypersurfaces of constant gaussian curvature. Indeed, by complementing Theorem 1.4 with an interior regularity estimate first used by Pogorelov in a slightly different context (c.f. [15]), we are able to obtain a regularity result for these Hausdorff limits which is expressed in Theorem 4.29. The proof of this result requires, in addition, an in-depth study of the structure of compact, convex subsets of Euclidean Space in order to derive the full geometric consequences of the above analytic results. This study forms the greater part of the content of this chapter, and we establish a number of properties of convex sets which are of independent interest, and which remain valid for convex subsets of any riemannian manifold.

With the regularity result of Theorem 4.29 at hand, we develop a theory of weak solutions which allows us to solve Theorem 1.2, and this forms the contents of Chapters 5 and 6. In Chapter 5 we study further properties of convex sets, though, whilst the results are used later, it is essentially tangential to the main flow of this text. In Chapter 6, we develop the concept of weak barriers as convex sets which are, more or less, Hausdorff limits of convex sets with smooth boundary and of gaussian curvature at least k. We show that the family of weak barriers is closed under Hausdorff limits and finite intersections, and this allows us to prove the existence of a unique volume minimising weak barrier. We then show that the family of weak barriers is closed under a certain excision operation, and this allows us to apply the regularity result of Theorem 4.29. We thus remove all singularities of the volume minimiser which then yields existence in Theorem 1.2.

2

The CNS Method

We study the Caffarelli-Nirenberg-Spruck (CNS) method for obtaining a-priori second order bounds of solutions to non-linear partial differential equations of Hessian type given the existence of an upper barrier. The most important result of this chapter is Theorem 2.30, which is a compactness result in the C^∞ sense for families of solutions to such non-linear partial differential equations, and in particular, for families of functions whose graphs are hypersurfaces of constant gaussian curvature.

The CNS method uses a barrier argument to obtain these bounds. Three points stand out in any discussion therof. The first is the importance of convexity which is repeatedly used throughout this chapter. The second is the concept of subharmonicity. Indeed, barrier arguments use the maximum principle which only applies to subharmonic functions. Consequently, as in Lemmas 2.15, 2.16 and 2.26, much of the work lies on determining functions which are subharmonic, or at least, whose laplacian is bounded below by terms which we are able to control. The third is the linearisation of the partial differential equation in question which presents the optimal choice of generalised laplacian with respect to which we determine subharmonicity.

In all situations where the CNS method may be applied, once a-priori second-order bounds have been obtained, higher order bounds follow from general results. First, the Krylov technique (c.f. Theorem 2.29) allows us to obtain a-priori $C^{2+\alpha}$ bounds for solutions. Then the Schauder technique (c.f. Theorem 2.28) allows us to obtain a-priori C^k bounds for all k, the compactness result then follows by applying the classical Arzela-Ascoli theorem. Detailled proofs of Theorems 2.28 and 2.29 would take us too far afield, and we therefore state them without proof, referring the interested reader to [11] for more details.

2.1 The Framework.

We henceforth consider a setting which is sufficiently general to illustrate the main concepts without introducing excessive complexity. In the sequel, we advise the reader to focus rather on the ideas than on the results themselves. We will highlight what we believe to be the key points as they arise.

Let Symm $:=$ Symm$(2,\mathbb{R}^n)$ be the space of real-valued, symmetric matrices of order n. Let $\Gamma\subseteq$ Symm be the open cone of positive-definite, symmetric matrices. We define $F:\Gamma\rightarrow]0,\infty[$ by:

$$F(A)=\mathrm{Det}(A)^{1/n}.$$

Let $G:\mathbb{R}^n\rightarrow\mathbb{R}$ be a smooth, convex function bounded below by 1. Let Ω be a compact, convex subset of $\mathbb{R}^n$ with smooth boundary. For any smooth function $\phi\in C^\infty(\overline{\Omega},]0,\infty[)$, we now consider smooth, strictly convex functions $f\in C^\infty(\overline{\Omega})$ which satisfy the following non-linear, partial differential equation with boundary condition:

$$F(D^2f)/G(Df)=\phi,\qquad f|_{\partial\Omega}=0. \tag{A}$$

We leave the reader to verify that hypersurfaces of constant gaussian curvature present a special case of this problem (c.f. Lemma 1.3).

We are interested in studying the problem given the existence of a lower barrier. This is a smooth, strictly convex function $\hat{f}\in C^\infty(\overline{\Omega})$ which satisfies the following non-linear partial differential inequation with boundary condition:

$$F(D^2\hat{f})/G(D\hat{f})>\phi,\qquad \hat{f}|_{\partial\Omega}=0.$$

We define the quantity $\delta(\hat{f})>0$ by:

$$\delta(\hat{f})=\operatorname*{Inf}_{x\in\overline{\Omega}}(F(D^2\hat{f}(x))-\phi(x)G(D\hat{f}(x))).$$

This quantity will be of use in the sequel. We are interested in solutions f of (A) which lie above the lower barrier. That is, with the property that $f\geqslant\hat{f}$. These will be obtained using degree theory, which is a generalisation of the continuity method and will be discussed in Section 3. In order to apply this technique, we require compactness results for any family of solutions of (A) bounded below by a corresponding family of lower barriers. By the classical Arzela-Ascoli theorem, such compactness results are equivalent to a-priori bounds for the norms of the k'th derivatives of solutions for all k. A-priori C^0 and C^1 maybe obtained without any further prerequisites:

Lemma 2.1

If $f\geqslant\hat{f}$, then:

$$\|f\|_{L^\infty}\leqslant\|\hat{f}\|_{L^\infty}.$$

Proof: Let Δ be the standard Laplacian on $\mathbb{R}^n$. Since f is smooth and strictly convex $\Delta f>0$. It follows from the maximum principle (Theorem 3.1 of [11]) that:

$$\operatorname*{Sup}_{x\in\overline{\Omega}}f(x)=\operatorname*{Sup}_{x\in\partial\Omega}f(x)=0.$$

Moreover, by definition:

$$\operatorname*{Inf}_{x\in\overline{\Omega}} f(x) = \operatorname*{Inf}_{x\in\overline{\Omega}} \hat{f}(x) \leqslant \|\hat{f}\|_{L^\infty}.$$

It follows that $\|f\|_{L^\infty} \leqslant \|\hat{f}\|_{L^\infty}$ as desired. □

Lemma 2.2

Choose $\epsilon > 0$ *and let* $a, b, c :]-\epsilon, \epsilon[\to \mathbb{R}$ *be continuous functions which are differentiable at* 0 *such that* $a(0) = b(0) = c(0)$. *If there exists a sequence* $(t_m)_{m\in\mathbb{N}} \in]-\epsilon, \epsilon[$ *converging to* 0 *such that for all* m:

$$a(t_m) \leqslant b(t_m) \leqslant c(t_m).$$

Then:

$$a'(0) \leqslant b'(0) \leqslant c'(0).$$

Proof: For all $m \in \mathbb{N}$:

$$\frac{1}{t_m}(b(t_m) - b(0)) \leqslant \frac{1}{t_m}(c(t_m) - c(0)).$$

Taking limits, we find that $b'(0) \leqslant c'(0)$. In like manner, we show that $a'(0) \leqslant b'(0)$, as desired. □

Lemma 2.3

If $f \geqslant \hat{f}$ *then:*

$$\|Df\|_{L^\infty} \leqslant \|D\hat{f}\|_{L^\infty}.$$

Proof: We first claim that:

$$\|Df\|_{L^\infty} = \operatorname*{Sup}_{x\in\partial\Omega} \|Df(x)\|.$$

Indeed, let x be any point of Ω. Denote $V = Df(x)/\|Df(x)\|$ and define $\gamma : \mathbb{R} \to \mathbb{R}^n$ by $\gamma(t) = x + tV$. Since Ω is compact and convex, $\gamma^{-1}(\overline{\Omega})$ is a closed interval, $[a, b]$ say, containing 0. We define $g : [a, b] \to \mathbb{R}$ by $g(t) = (f \circ \gamma)(t) = f(x + tV)$. Since f is convex, so is g. In particular, g' is monotone, and so without loss of generality we may assume that $g'(b) \geqslant g'(0) = \|Df(x)\|$. However, bearing in mind the Cauchy/Schwarz inequality:

$$\|Df(x)\| \leqslant g'(b) = \langle Df(\gamma(b)), V\rangle \leqslant \|Df(\gamma(b))\|\|V\| = \|Df(\gamma(b))\| \leqslant \operatorname*{Sup}_{y\in\partial\Omega} \|Df(y)\|,$$

The assertion now follows by taking the supremum over all x.

Observe that, as in the proof of Lemma 2.1, $f \leqslant 0$. Choose $x \in \partial\Omega$. Upon applying an affine isometry, we may assume that $x = 0$ and that the tangent space to $\partial\Omega$ at x coincides with the space spanned by the vectors $e_1, ..., e_{n-1}$. There exists $r > 0$ and a smooth function $B'_r(0) \to]-r, r[$ such that $(\partial\Omega) \cap (B'_r(0) \times]-r, r[)$ coincides with the graph of ω. In particular $D\omega(0) = 0$. Consider the function $g(x') = f(x', \omega(x'))$. Observe that

g vanishes identically. Thus, bearing in mind the chain rule, for all $1 \leqslant i \leqslant (n-1)$ we obtain:

$$0 = g_i(0) = f_i(0) + f_n(0)\omega_i(0) = f_i(0).$$

Define $\gamma : \mathbb{R} \to \mathbb{R}^n$ by $\gamma(t) = te_n$. There exists $\epsilon > 0$ such that for all $t \in [0, \epsilon[$, $\gamma(t) \in \overline{\Omega}$. We define $g, \hat{g} : [0, \epsilon[\to \mathbb{R}$ by $g(t) = (f \circ \gamma)(t) = f(te_n)$ and $\hat{g}(t) = (\hat{f} \circ \gamma)(t) = \hat{f}(te_n)$. Observe that both g and $\hat{g}$ extend to C^1 functions defined over the interval $]-\epsilon, \epsilon[$. Moreover, for all $t \in [0, \epsilon[$, $\hat{g}(t) \leqslant g(t) \leqslant 0$, and it follows from Lemma 2.2 that $\hat{g}'(0) \leqslant g'(0) \leqslant 0$. Since $\hat{g}'(0) = \hat{f}_n(0)$ and $g'(0) = f_n(0)$, and since $f_i(0) = 0$ for all $1 \leqslant i \leqslant (n-1)$, this yields:

$$\|Df(x)\| = |f_n(0)| \leqslant \left|\hat{f}_n'(0)\right| \leqslant \|D\hat{f}(0)\| \leqslant \operatorname*{Sup}_{y \in \partial\Omega} \|D\hat{f}(y)\|.$$

The result now follows by taking the supremum over all $x \in \partial\Omega$. □

Remark: *Operators of the type discussed here fall into the class of what are known as Monge-Ampère operators. The approach described in the sequel extends to the following more general framework (c.f. [4]). Let* $\mathrm{O}(n)$ *be the group of orthogonal matrices of order* n*. We recall that* $\mathrm{O}(n)$ *acts on* Symm *by conjugation. That is, for* $M \in \mathrm{O}(n)$ *and* $A \in$ Symm*:*

$$M(A) := M^{-1}AM \in \mathrm{Symm}.$$

We now denote the open cone of positive-definite, symmetric matrices by Γ_0*, and observe that every element of* $\mathrm{O}(n)$ *maps* Γ *bijectively to itself. We then consider any other cone* $\Gamma \subseteq$ Symm *centred on the origen which is convex, invariant under the action of* $\mathrm{O}(n)$ *on* Symm*, and in addition satisfies the property that for all* $x \in \Gamma$*,* $x + \Gamma_0 \subseteq \Gamma$*.*

Given such a cone Γ*, we consider any function* $F \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma})$ *which is concave, homoegeneous of order 1, invariant under the action of* $\mathrm{O}(n)$ *on* Symm *and which in addition satisfies the property that for all* $A \in \Gamma$ *and for all* $B \in \overline{\Gamma}_0 \setminus \{0\}$*,* $DF(A)(B) > 0$*.*

In particular, consider the symmetric polynomials $(\sigma_k)_{0 \leqslant k \leqslant n} : \mathrm{Symm} \to \mathbb{R}$ *defined such that for all* $A \in$ Symm *and for all* $t \in \mathbb{R}$*:*

$$\mathrm{Det}(\mathrm{Id} + tA) = \sum_{i=0}^{n} t^i \sigma_i(A).$$

For all $0 \leqslant k \leqslant n$*, we define* $\Gamma_k \subseteq$ Symm *by:*

$$\Gamma_k = \{A \mid \sigma_0(A), ..., \sigma_k(A) > 0\}.$$

For all $1 \leqslant k \leqslant n$*, we define* $F_k : \Gamma_k \to [0, \infty[$ *by* $F_k(A) = (\sigma_k(A))^{1/k}$*. It can be shown that for all* k*, the pair* (Γ_k, F_k) *satisfies the desired properties. In particular, when* $k = n$*,* $F_k = \mathrm{Det}^{1/n}$ *and when* $k = 1$*,* $F_k = \mathrm{Tr}$*, and we see that both the Monge-Ampère operator (studied here) and the Laplacian are covered by this framework.*

2.2 Basic Properties of F.

In order to obtain higher order a-priori bounds, it is necessary to understand better the differential properties of the function F. Let $\mathrm{End}(n)$ be the space of linear endomorphisms of $\mathbb{R}^n$. We recall that the canonical inner product of $\mathrm{End}(n)$ can be written in the form:

$$\langle A, B\rangle = \mathrm{Tr}(A^tB).$$

Observe that for $A, B \in \mathrm{Symm}(n)$, the above formula becomes:

$$\langle A, B\rangle = \mathrm{Tr}(AB).$$

If $\alpha : \mathrm{End}(n) \to \mathbb{R}$ is a linear map, then we identify it with a matrix $A \in \mathrm{End}(n)$ via this inner product. Let $\mathrm{GL}(n)$ be the group of invertible endomorphisms of $\mathbb{R}^n$. We recall the following version of Schurr's Lemma:

Lemma 2.4

If $\alpha : \mathrm{End}(n) \to \mathbb{R}$ is a linear map such that for all $M \in \mathrm{GL}(n)$ and for all $B \in \mathrm{End}(n)$:

$$\alpha(M^{-1}BM) = \alpha(B),$$

then there exists $\lambda \in \mathbb{R}$ such that for all $A \in \mathrm{End}(n)$:

$$\alpha(B) = \lambda\mathrm{Tr}(B).$$

Proof: Choose $A \in \mathrm{End}(n)$ such that for all $B \in \mathrm{End}(n)$:

$$\alpha(B) = \mathrm{Tr}(A^tB).$$

For all $M \in \mathrm{GL}(n)$ and for all $B \in \mathrm{End}(n)$:

$$\begin{aligned}\mathrm{Tr}((M^{-1}AM)^tB) &= \mathrm{Tr}(M^tA^t(M^t)^{-1}B)\\ &= \mathrm{Tr}(A^t(M^t)^{-1}BM^t)\\ &= \alpha((M^t)^{-1}BM^t)\\ &= \alpha(B)\\ &= \mathrm{Tr}(A^tB).\end{aligned}$$

Since $B \in \mathrm{End}(n)$ is arbitrary, we deduce that $M^{-1}AM = A$. Differentiating this relation at Id yields for all $M \in \mathrm{End}(n)$:

$$MA = AM.$$

Choose $v \in \mathbb{R}^n \setminus \{0\}$ and let $P \in \mathrm{End}(n)$ be the orthogonal projection onto the space spanned by v. Then:

$$Av = APv = PAv,$$

and so v is an eigenvector of A. However, the only matrices with the property that all vectors are eigenvectors are the scalar multiples of Id, and it follows that $A = \lambda\mathrm{Id}$ as desired. $\square$

We identify the derivative of F at any point with a symmetric matrix. We obtain:

Lemma 2.5

For all $A \in \Gamma$:

$$DF(A) = \frac{1}{n}F(A)A^{-1}.$$

Proof: Since every element of Γ is invertible, $\Gamma \subseteq \mathrm{GL}(n)$, the set of invertible linear maps of $\mathbb{R}^n$. Denote $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, and define the function $H : \mathrm{GL}(n) \to \mathbb{R}$ by:

$$H(M) = \mathrm{Det}(M).$$

We recall that H is a group homomorphism. In particular, since $\mathbb{R}^*$ is abelian, H is invariant under the action of conjugation by elements of $\mathrm{GL}(n)$.

Since $\mathrm{GL}(n)$ is an open subset of $\mathrm{End}(n)$, its tangent space at Id naturally coincides with $\mathrm{End}(n)$. Since H is invariant under the action of conjugation by elements of $\mathrm{GL}(n)$, and since conjugation fixes Id, upon differentiating, we obtain, for all $A \in \mathrm{End}(n)$ and for all $M \in \mathrm{GL}(n)$:

$$DH(\mathrm{Id})(M^{-1}AM) = DH(\mathrm{Id})(A).$$

Since $DH(\mathrm{Id})$ is linear, it follows from Lemma 2.4 that for all $A \in \mathrm{End}(n)$:

$$DH(\mathrm{Id})(A) = \lambda\mathrm{Tr}(A),$$

for some $\lambda \in \mathbb{R}$. By explicit calculation at $A = \mathrm{Id}$, we deduce that $\lambda = 1$.

Choose $M \in \mathrm{GL}(n)$. Using the homomorphism property of H along with the preceding calculation, we obtain for all $A \in \mathrm{End}(n)$:

$$\begin{aligned} DH(M)(A) &= \tfrac{d}{dt}H(M + tA)|_{t=0} \\ &= \tfrac{d}{dt}H(M)H(\mathrm{Id} + tM^{-1}A)|_{t=0} \\ &= H(M)\mathrm{Tr}(M^{-1}A). \end{aligned}$$

The result now follows by the chain rule, since $F = H^{1/n}$. □

Corollary 2.6

For all $A \in \Gamma$:

$$DF(A)(A) = F(A).$$

Remark: *This relation in fact follows directly from the homogeneity of F.*

Proof: Indeed:

$$DF(A)(A) = \frac{1}{n}F(A)\mathrm{Tr}(A^{-1}A) = \frac{1}{n}F(A)\mathrm{Tr}(\mathrm{Id}) = F(A),$$

as desired. □

Lemma 2.7

Suppose that $A \in \Gamma$ is diagonal and let $0 < \lambda_1 \leqslant ... \leqslant \lambda_n$ be the eigenvalues of A. Then, for all $B \in \mathrm{Symm}(n)$:

$$D^2F(A)(B,B) \leqslant -\frac{1}{n}F(A)\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}B_{ij}B_{ij}.$$

Remark: *an analogous relation may be deduced for more general F using the properties of concavity and ellipticity (c.f.* [19] *for details).*

Proof: Differentiating Lemma 2.5 yields:

$$D^2F(A)(B,B) = \frac{1}{n}F(A)\left(\frac{1}{n}\mathrm{Tr}(A^{-1}B)^2 - \mathrm{Tr}(A^{-1}BA^{-1}B)\right).$$

Since A is diagonal, this yields:

$$D^2F(A)(B,B) = \frac{1}{n}F(A)\left(\frac{1}{n}\sum_{i,j}\frac{1}{\lambda_i\lambda_j}B_{ii}B_{jj} - \sum_{i,j}\frac{1}{\lambda_i\lambda_j}B_{ij}^2\right).$$

However, by the Cauchy/Schwarz inequality:

$$n\sum_{i=1}^n\frac{1}{\lambda_i^2}B_{ii}^2 = \left(\sum_{i=1}^n 1^2\right)\left(\sum_{i=1}^n\frac{1}{\lambda_i^2}B_{ii}^2\right) \geqslant \left(\sum_{i=1}^n\frac{1}{\lambda_i}B_{ii}\right)^2.$$

The result follows by combining this with the preceeding relation. □

Corollary 2.8

F is concave over Γ.

We invite the reader to observe the frequency with which the concavity of F and the convexity of G are used throughout the sequel to remove awkward terms. This is the first key point of the CNS technique. Of these two properties, the concavity of F is perhaps more fundamental, as it is used to eliminate third order terms, wheras the convexity of G only eliminates second-order terms.

Remark: *In particular, in the more general framework alluded to at the end of the previous section, an explicit formula for DF is not necessary, and the results obtained in the sequel can be deduced from more general relations derived from the properties of concavity, homogeneity, ellipticity and $\mathrm{O}(n)$-invariance. See* [4] *for details.*

The concavity of F yields the following lower estimate:

Lemma 2.9

For all A in Γ:

$$\frac{1}{n}\mathrm{Tr}(A) \geqslant F(A).$$

Proof: By concavity:

$$DF(\mathrm{Id})(A - \mathrm{Id}) \geqslant F(A) - F(\mathrm{Id}).$$

Thus, by Lemma 2.5 and bearing in mind that $F(\mathrm{Id}) = 1$:

$$\begin{array}{rll} & \frac{1}{n}\mathrm{Tr}(A - \mathrm{Id}) & \geqslant F(A) - 1 \\ \Rightarrow & \frac{1}{n}\mathrm{Tr}(A) & \geqslant F(A), \end{array}$$

as desired. □

2.3 Linearisation.

Many estimates will be obtained using the maximum principle. For this reason, superharmonic functions play an important role in the theory. Importantly, the concept of "superharmonicity" requires an explicit choice of generalised Laplacian. The correct choice constitutes the second key point of the CNS technique. We define $\mathcal{L}_f : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ by:

$$\mathcal{L}_f g = DF(D^2 f)(D^2 g) - \phi DG(Df)(Dg).$$

The informed reader will see that $\mathcal{L}_f$ is precisely the linearisation at f of the partial differential operator Φ given by $\Phi(Dg, D^2 g) = F(D^2 g) - \phi G(Dg)$ (c.f. Section 3).

Lemma 2.10

$\mathcal{L}_f$ is a second-order, linear, elliptic, partial differential operator.

Proof: By definition that $\mathcal{L}_f$ is a second-order, linear, partial differential operator. It thus remains to show ellipticity. Let $\sigma_2(\mathcal{L}_f)$ be the principle symbol of $\mathcal{L}_f$ (c.f. [10]). We have to show that $\sigma_2(\mathcal{L}_f)$ is everywhere positive definite. Bearing in mind Lemma 2.5:

$$\begin{array}{rl} \sigma_2(\mathcal{L}_f)(\xi) & = DF(D^2 f)(\xi \otimes \xi) \\ & = \frac{1}{n} F(D^2 f)\mathrm{Tr}((D^2 f)^{-1}(\xi \otimes \xi)) \\ & = \frac{1}{n} F(D^2 f)\langle \xi, (D^2 f)^{-1}\xi \rangle. \end{array}$$

Since f is strictly convex, $D^2 f$ is positive definite at every point, and therefore so is its inverse. The principal symbol of $\mathcal{L}_f$ is thus also everywhere positive definite as desired. □

The following result is an important source of superharmonic functions in the theory:

Theorem 2.11, Superharmonic Functions

If $\delta \geqslant 0$ is a non-negative real number and $g, h \in C^\infty(\overline{\Omega})$ are smooth, strictly convex functions such that:

$$F(D^2 g) = \phi G(Dg), \qquad F(D^2 h) \geqslant \phi G(Dh) + \delta,$$

then:

$$\mathcal{L}_g(g - h) \leqslant -\delta.$$

Proof: By definition:

$$F(D^2 g) - \phi G(Dg) = 0 \leqslant F(D^2 h) - \phi G(Dh) - \delta.$$

By concavity of F:

$$DF(D^2 g)(D^2 g - D^2 h) \leqslant F(D^2 g) - F(D^2 h).$$

By positivity of ϕ and convexity of G:

$$\phi DG(Dg)(Dh - Dg) \leqslant \phi G(Dh) - \phi G(Dg).$$

Combining the above relations and recalling the definition of $\mathcal{L}_g$ yields:

$$\mathcal{L}_g(g - h) \leqslant -\delta,$$

as desired. □

This allows us in particular to prove the strong maximum principle in the non-linear setting:

Lemma 2.12, Strong Maximum Principle

Let $g, h \in C^\infty(\Omega)$ be strictly convex functions such that $F(D^2 h)/G(Dh) \geqslant F(D^2 g)/G(Dg)$ and let p be a point in $\overline{\Omega}$ where $g - h$ attains its minimum value. If $D(g - h) = 0$ at p, then $(g - h)$ is constant.

Remark: *In particular, if p is an interior point, then $D(g - h) = 0$ at p. Thus if $(g - h)$ attains its minimum value at any interior point, then it is constant. This is the usual formulation of the strong maximum principle.*

Proof: Suppose the contrary, and so $g - h$ is non-constant. We define the function ψ by $\psi = F(D^2 g)/G(Dg) > 0$. Then:

$$F(D^2 h) - \psi G(Dh) \geqslant 0 = F(D^2 g) - \psi G(Dg).$$

Thus, by Theorem 2.11 applied with $\phi = \psi$:

$$\mathcal{L}_g(g - h) \leqslant 0.$$

It follows from Hopf's maximum principle (c.f. Lemma 3.4 of [11]) that $D(g - h) \neq 0$ at p. This is absurd by hypothesis, and it follows that $g - h$ is constant as desired. □

2.4 The CNS Technique.

Let X be a smooth vector field in $\mathbb{R}^n$ which is tangential to $\partial\Omega$. We define the function $g \in C^\infty(\overline{\Omega})$ by:

$$g = Xf = Df(X).$$

Observe that, by definition, g vanishes along the boundary. In this section, we control $\mathcal{L}_f g$. It is not possible to obtain absolute a-priori bounds for this term. Instead, a-priori bounds turn out to be best expressed in terms of a function which depends on the data. The correct choice of which is the third key point of the CNS technique. We define the function $\Lambda(f)$ by:

$$\Lambda(f) = DF(D^2f)(\mathrm{Id}) = \mathrm{Tr}(DF(D^2f)) = \frac{1}{n}F(D^2f)\mathrm{Tr}((D^2f)^{-1}).$$

The following lemma ensures that fixed multiples of $\Lambda(f)$ also bound terms that are already known to be bounded by constants:

Lemma 2.13

For all f:

$$\Lambda(f) \geqslant 1.$$

Remark: *Observe that this holds for any concave F homogeneous of order 1 and having in addition the property that $F(\mathrm{Id}) = 1$ (c.f. the remark following Corollary 2.6.*

Proof: By Corollary 2.6, for all $A \in \Gamma$:

$$DF(A)(A) = F(A).$$

Since F is concave:

$$DF(A)(\mathrm{Id} - A) \geqslant F(\mathrm{Id}) - F(A) = 1 - F(A).$$

Thus, by linearity:

$$\begin{aligned}\mathrm{Tr}(DF(A)) &= DF(A)(\mathrm{Id}) \\ &= DF(A)(\mathrm{Id} - A) + DF(A)(A) \\ &\geqslant 1 - F(A) + F(A) \\ &= 1,\end{aligned}$$

as desired. □

We now control $\mathcal{L}_f g$:

Lemma 2.14

There exists $C > 0$ which only depends on $\|\phi\|_1$, $\|X\|_2$ and $\|f\|_1$ such that if $g = Xf$, then:

$$|\mathcal{L}_f g| \leqslant C\Lambda(f).$$

Remark: *Observe that this result only requires the homogeneity of F. The point is that since f is a solution of (A), then any derivative of f should satisfy the linearisation of this relation (c.f. Section 3) modulo lower order terms, which should already have been bounded. The non-triviality arises from the fact that $\mathcal{L}_f g$ actually involves terms which are of second-order in f and which are not therefore bounded a-priori. These terms are removed using Corollary 2.6, which as remarked previously, also follows directly from the homogeneity of F.*

Proof: By definition of $\mathcal{L}_f$, differentiating in the direction of e_i and recalling that $D^i f$ is symmetric for all k, we obtain:

$$\mathcal{L}_f f_i = \phi_i G(Df)$$

Thus, using the summation convention:

$$X^i \mathcal{L}_f f_i = X^i \phi_i G(Df).$$

There therefore exists $C_1 > 0$ which only depends on $\|\phi\|_1$ and $\|f\|_1$ such that:

$$\left|X^i \mathcal{L}_f f_i\right| \leqslant C_1.$$

We wish to move X^i to the other side of $\mathcal{L}_f$. First, define $\mathcal{L}_f^1$ such that for all g:

$$\mathcal{L}_f^1 g = -\phi DG(Df)(Dg).$$

In other words $\mathcal{L}_f^1$ is the first order component of $\mathcal{L}_f$. We calculate:

$$\mathcal{L}_f^1(X^i f_i) - X^i \mathcal{L}_f^1(f_i) = f_i \mathcal{L}_f^1(X^i).$$

There therefore exists $C_2 > 0$ which only depends on $\|\phi\|_0$, $\|f\|_1$ and $\|X\|_1$ such that:

$$\left|\mathcal{L}_f^1(X^i f_i) - X^i \mathcal{L}_f^1(f_i)\right| \leqslant C_2.$$

We denote $B^{ij} = \frac{1}{n} F(D^2 f)(D^2 f^{-1})^{ij}$. In other words, B is the matrix of $DF(D^2 f)$. By Lemma 2.7, we obtain:

$$\begin{aligned} X^i DF(D^2 f)(D^2 f_i) - DF(D^2 f)(D^2(Xf)) &= X^i B^{pq} f_{ipq} - B^{pq}(X^i f_i)_{pq} \\ &= -B^{pq} X^i{}_p f_{iq} - B^{pq} X^i{}_q f_{ip} - B^{pq} X^i{}_{pq} f_i. \end{aligned}$$

However, by bearing in mind that f is a solution of (A):

$$B^{pq} f_{ip} = \frac{1}{n} F(D^2 f)\delta^q{}_i = \frac{1}{n}\phi G(Df)\delta^q{}_i.$$

This allows us to eliminate the terms on the right hand side which are of second-order in f. We remark in passing that it is at this step that the homogeneity of F is required to ensure that the matrix B has the appropriate form. There therefore exists $C_3 > 0$, which only depends on $\|\phi\|_0$, $\|X\|_2$ and $\|f\|_1$ such that

$$\left|X^i DF(D^2 f)(D^2 f_i) - DF(D^2 f)(D^2(Xf))\right| \leqslant C_3\Lambda(f).$$

Combining the above relations, bearing in mind Lemma 2.13 and using the triangle inequality, we deduce that:

$$\begin{aligned} |\mathcal{L}_f(Xf)| &\leqslant \left|DF(D^2 f)(D^2(Xf)) - X^i DF(D^2 f)(D^2 f_i)\right| \\ &\qquad + \left|\mathcal{L}^1_f(X^i f_i) - X^i\mathcal{L}^1_f(f_i)\right| + \left|X^i(\mathcal{L}_f f_i)\right| \\ &\leqslant C_1 + C_2 + C_3\Lambda(f) \\ &\leqslant (C_1 + C_2 + C_3)\Lambda(f), \end{aligned}$$

as required. □

We now introduce the first main ingredient of the barrier function used in the CNS technique to control g. It is a perturbation of $f - \hat{f}$. By definition, this function is non-negative, and by Theorem 2.11, it is superharmonic with respect to $\mathcal{L}_f$. We perturb this function to be strictly negative away from a given point on the boundary whilst being careful to maintain its analytic properties. Thus, for all $p \in \partial\Omega$ and for all $\epsilon > 0$, we define $\hat{f}_{p,\epsilon}$ by:

$$\hat{f}_{p,\epsilon}(x) = \hat{f}(x) - \epsilon\|x - p\|^2.$$

Lemma 2.15

There exists $\epsilon_0 > 0$, which only depends on $\|\phi\|_0$, $\delta(\hat{f})$ and $\|\hat{f}\|_2$ such that for all $p \in \partial\Omega$ and for all $\epsilon < \epsilon_0$:

$$\mathcal{L}_f(f - \hat{f}_{p,\epsilon}) \leqslant 0.$$

Proof: By compactness, there exists $\epsilon_0 > 0$ which only depends on $\|\phi\|_0$, $\delta(\hat{f})$ and $\|\hat{f}\|_2$ such that for all $p \in \partial\Omega$ and for all $\epsilon < \epsilon_0$:

$$F(D^2\hat{f}_{p,\epsilon}) \geqslant \phi G(D\hat{f}_{p,\epsilon}).$$

The result now follows by Theorem 2.11. □

For all $p \in \partial\Omega$, we define the function $d_p : \Omega \to \mathbb{R}$ by:

$$d_p(x) = \|x - p\|.$$

This is the second ingredient of the CNS barrier function:

Lemma 2.16

There exists $r > 0$ which only depends on $\|\phi\|_0$ and $\|f\|_1$ such that for all $p \in \partial\Omega$:

$$\mathcal{L}_f d_p^2 \geqslant \Lambda(f),$$

over $\Omega \cap B_r(p)$.

Proof: By definition, for all g:

$$\mathcal{L}_f g = DF(D^2 f)(D^2 g) - \phi DG(Df)(Dg).$$

When $g = d_p^2$, for all x:

$$\|Dg(x)\| = 2d_p(x).$$

There therefore exists $r > 0$ which only depends on $\|\phi\|_0$ and $\|f\|_1$ such that for $d_p(x) < r$, bearing in mind Lemma 2.13:

$$|\phi(x)DG(Df(x))(Dg(x))| \leqslant 1 \leqslant \Lambda(f)(x).$$

However, for all x:

$$DF(D^2 f(x))(D^2 d_p^2(x)) = DF(D^2 f(x))(2\mathrm{Id}) = 2\Lambda(f)(x).$$

The result now follows by subtracting these two relations. $\square$

Lemma 2.17

There exists $C > 0$ which only depends on $\|\phi\|_1$, $\|X\|_2$, $\delta(\hat{f})$, $\|\hat{f}\|_2$ and $\|f\|_1$ such that if $f \geqslant \hat{f}$, then for all $p \in \partial\Omega$:

$$\|D(Xf)(p)\| \leqslant C.$$

Proof: Let r be as in Lemma 2.16 and let A be the constant given by Lemma 2.14. Then, by Lemma 2.16, for all $p \in \partial\Omega$, throughout $\Omega \cap B_r(p)$:

$$-\mathcal{L}_f(Ad_p^2) \leqslant \mathcal{L}_f(Xf) \leqslant \mathcal{L}_f(Ad_p^2),$$

Let ϵ_0 be as in Lemma 2.15. Observe that for all $p \in \partial\Omega$:

$$f - \hat{f}_{p,\epsilon_0} \geqslant \epsilon_0 d_p^2.$$

Thus, by Lemma 2.15, for all $p \in \partial\Omega$ and for all $B > 0$, throughout $\Omega \cap B_r(p)$:

$$\mathcal{L}_f(B(f - \hat{f}_{p,\epsilon_0}) - Ad_p^2) \leqslant \mathcal{L}_f(Xf) \leqslant \mathcal{L}_f(-B(f - \hat{f}_{p,\epsilon_0}) + Ad_p^2).$$

For $p \in \partial\Omega$, $\partial(\Omega \cap B_r(p))$ consists of two components, being $B_r(p) \cap \partial\Omega$ and $\Omega \cap \partial B_r(p)$. Choose $B > A/\epsilon_0 + \|X\|_1\|f\|_1/\epsilon_0 r^2$. For all $p \in \partial\Omega$ and for all $x \in \partial\Omega$:

$$\begin{aligned}
(B(f - \hat{f}_{p,\epsilon_0}) - Ad_p^2)(x) &\quad \geqslant 0 \quad = (Xf)(x) \\
-(B(f - \hat{f}_{p,\epsilon_0}) - Ad_p^2)(x) &\quad \leqslant 0 \quad = (Xf)(x).
\end{aligned}$$

Likewise, for all $p \in \partial\Omega$ and for all $x \in \Omega \cap \partial B_r(p)$:

$$\begin{array}{ll} (B(f-\hat{f}_{p,\epsilon_0}) - Ad_p^2)(x) & \geqslant (Xf)(x), \\ -(B(f-\hat{f}_{p,\epsilon_0}) - Ad_p^2)(x) & \leqslant (Xf)(x). \end{array}$$

We conclude that for all $p \in \partial\Omega$ and for all $x \in \partial(\Omega \cap B_r(p))$:

$$(B(f-\hat{f}_{p,\epsilon_0}) - Ad_p^2)(x) \geqslant (Xf)(x) \geqslant -(B(f-\hat{f}_{p,\epsilon_0}) - Ad_p^2)(x).$$

It thus follows by the maximum principle that for all $p \in \partial\Omega$:

$$B(f-\hat{f}_{p,\epsilon_0}) - Ad_p^2 \geqslant Xf \geqslant -B(f-\hat{f}_{p,\epsilon_0}) + Ad_p^2,$$

throughout $\Omega \cap B_r(p)$. However, by definition:

$$(B(f-\hat{f}_{p,\epsilon_0}) - Ad_p^2)(p) = -(B(f-\hat{f}_{p,\epsilon_0}) - Ad_p^2)(p) = 0 = (Xf)(p).$$

Thus, by Lemma 2.2:

$$\begin{array}{ll} \|D(Xf)(p)\| & \leqslant B\|D(f-\hat{f}_{p,\epsilon_0})(p)\| \\ & = B\|D(f-\hat{f})(p)\| \\ & \leqslant B(\|f\|_{C^1} + \|\hat{f}\|_{C^1}), \end{array}$$

as desired. □

Choose $p \in \partial\Omega$. Upon applying an isometry of $\mathbb{R}^n$, we may suppose that $p = 0$ and that the tangent space to $\partial\Omega$ at p is spanned by the vectors $e_1, ..., e_{n-1}$. We thus obtain:

Corollary 2.18

There exists $C > 0$ which only depends on $\|\phi\|_1$, $\delta(\hat{f})$, $\|\hat{f}\|_2$ and $\|f\|_1$ such that if $f \geqslant \hat{f}$ then for all $(i,j) \neq (n,n)$:

$$|f_{ij}(0)| \leqslant C.$$

Proof: Since $D^2 f$ is symmetric, we may suppose that $j < n$. For all r, we denote by $B'_r(0)$ the ball of radius r about 0 in $\mathbb{R}^{n-1}$. There exists $r > 0$ and a smooth function $\omega : B'_r(0) \to]-r, r[$ such that $\partial\Omega \cap (B'_r(0) \times]-r, r[)$ coincides with the graph of ω. Let $\chi \in C_0^\infty(B'_r(0) \times]-r, r[)$ be a smooth function of compact support equal to 1 near 0. We define the vector field X_j by:

$$X_j(x', t) = \chi(x', t)(e_j, \omega_j(x')).$$

Observe that X_j is tangent to $\partial\Omega$ and that $\|X_j\|_2$ is controlled by the geometry of Ω. Moreover, $(D\omega)(0) = 0$ and so $X_j(0,0) = e_j$. For all $1 \leqslant i \leqslant n$:

$$|f_{ij}(0)| \leqslant \|D(X_j f)(0)\| + |f(0)(\partial_i X_j)(0)| = \|D(X_j f\|(0)\|,$$

and the result now follows by Lemma 2.17. □

2.5 The Double Normal Derivative.

By Corollary 2.18, it only remains to control the second derivative in the double normal direction. This term typically presents a serious difficulty in applications of the CNS technique, and often requires a further, lengthy barrier argument (c.f. [3]). In the current case, however, a straightforward ad-hoc argument allows this term to be controlled. In this section, we continue to use the notation of Corollary 2.18, and we thus aim to control $|f_{nn}(0)|$.

Lemma 2.19

Let M be an $n \times n$ matrix, let M' be the $(n-1)\times(n-1)$ matrix given by $M'_{ij} = M_{ij}$ for all $1 \leqslant i, j \leqslant (n-1)$, and let $B > 0$ be such that:

(1) $|M_{ij}| < B$ for all $(i,j) \neq (n,n)$ and;

(2) $|\mathrm{Det}(M')| > (1/B)$.

Then:

$$|M_{nn}| \leqslant B(\mathrm{Det}(M) + B^n(n-1)(n-1)!).$$

Proof: Let $\mathrm{Adj}(M)$ be the adjugate matrix of M. For all k, $\mathrm{Adj}(M)_{nk}$ is a polynomial in the elements of M not including M_{nn}. In particular:

$$|\mathrm{Adj}(M)_{nk}| \leqslant (n-1)!B^{n-1}.$$

Moreover:

$$|\mathrm{Adj}(M)_{nn}| = |\mathrm{Det}(M')| \geqslant B^{-1}.$$

Finally, we recall :

$$\begin{array}{lll} & \mathrm{Det}(M) & = \sum_{k=1}^{n}\mathrm{Adj}(M)_{nk}M_{kn} \\ \Rightarrow & M_{nn}\mathrm{Adj}(M)_{nn} & = \mathrm{Det}(M) - \sum_{k=1}^{n-1}\mathrm{Adj}(M)_{nk}M_{nk} \\ \Rightarrow & |M_{nn}| & \leqslant B(\mathrm{Det}(M) + B^n(n-1)(n-1)!), \end{array}$$

as desired. □

It thus suffices to obtain lower bounds for the absolute value of the determinant of $(f_{ij})_{1\leqslant i,j\leqslant(n-1)}$.

Lemma 2.20

Suppose there exists $\epsilon > 0$ such that $f_n < -\epsilon$. Then there exists $\delta > 0$, which only depends on ϵ such that:

$$\left|\mathrm{Det}((f_{ij})_{1\leqslant i,j\leqslant(n-1)})\right| > \delta.$$

Proof: For sufficiently small r, there exists $\omega : B'_r(0) \to \mathbb{R}$ such that the intersection of $\partial\Omega$ with $B'_r(0)\times]-r,r[$ coincides with the graph of ω. Observe that, by definition, $D\omega(0) = 0$.

Moreover, since Ω is strictly convex, so is ω. Consider the function $f'(x) = f(x, \omega(x))$. Since this function is identically 0, by the chain rule, for all $1 \leqslant i, j \leqslant (n-1)$:

$$f_{ij} + f_n\omega_{ij} = f'_{ij} = 0,$$

and so:

$$\left|\mathrm{Det}((f_{ij})_{1\leqslant i,j\leqslant (n-1)})\right| > \epsilon^{n-1}\left|\mathrm{Det}(\omega_{ij})\right|.$$

There therefore exists $\delta > 0$ which only depends on ϵ and the geometry of Ω such that:

$$\left|\mathrm{Det}((f_{ij})_{1\leqslant i,j\leqslant (n-1)})\right| > \delta,$$

as desired. □

We now employ an ad-hoc barrier argument in order to bound f_n from above:

Lemma 2.21

There exists $\delta > 0$ which only depends on $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$ such that $f_n(p) < -\delta$.

Proof: We decompose $\mathbb{R}^n$ as the product $\mathbb{R}^{n-1} \times \mathbb{R}$. There exists $r > 0$, which only depends on the geometry of Ω such that $B_r((0, r))$ is contained within Ω. For all $\delta > 0$, we define $h_\delta \in C^\infty(B_r((0, r)))$ by:

$$h_\delta(x) = \delta\|x - (0, r)\|^2 - \delta r^2.$$

Observe that $F(D^2h_\delta) = 2\delta$. Thus, for $2\delta < \mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$:

$$F(D^2h_\delta) - \phi G(Dh_\delta) \leqslant 0 = F(D^2f) - \phi G(Df).$$

Moreover, for all $x \in \partial B_r((0, r))$:

$$(h_\delta - f)(x) = -f(x) \geqslant 0.$$

Thus, by the maximum principle (Lemma 2.12), $h_\delta - f \geqslant 0$ throughout $B_r((0, r))$. However:

$$h_\delta(0) = 0 = f(0),$$

and so, by Lemma 2.2:

$$f_n(0) \leqslant (\partial_n h_\delta)(0) = -2\delta r,$$

as desired. □

We conclude:

Theorem 2.22

There exists $C > 0$ which only depends on $\|\phi\|_1$, $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$, $\delta(\hat{f})$, $\|\hat{f}\|_2$ and $\|f\|_1$ such that if $f \geqslant \hat{f}$ then, for all $x \in \partial\Omega$:

$$\|D^2 f(x)\| \leqslant C.$$

Proof: Upon applying an affine isometry, we may suppose that $x = 0$ and that the tangent space to $\partial\Omega$ at 0 is spanned by the vectors $e_1, ..., e_n$. By Corollary 2.18, there exists $C_1 > 0$ which only depends on $\|\phi\|_1$, $\delta(\hat{f})$, $\|\hat{f}\|_2$ and $\|f\|_1$ such that if $f \geqslant \hat{f}$ then, for all $(i,j) \neq (n,n)$:

$$|f_{ij}(0)| \leqslant C_1.$$

By Lemma 2.21, there exists $\delta_1 > 0$ such that:

$$f_n(0) \leqslant -\delta_1.$$

By Lemma 2.20 there exists $\delta_2 > 0$ which only depends on δ_1 and C_1 such that:

$$\left|\mathrm{Det}((f_{ij}(0))_{1\leqslant i,j\leqslant(n-1)})\right| \geqslant \delta_2.$$

By Lemma 2.19, there exists $C_2 > C_1$, which only depends on δ_2 and C_1 such that:

$$|f_{nn}(0)| \leqslant C_2.$$

We conclude that $\|Df(0)\| \leqslant C_2$ as desired. □

2.6 Boundary to Global.

Global second-order bounds are obtained by applying the maximum principle with the help of an auxiliary function, the existence or otherwise of which often determines over which domains the Plateau Problem may be solved.

Let $\lambda_1, ..., \lambda_n : \Omega \to \mathbb{R}$ be such that, for all x, $0 < \lambda_1(x) \leqslant ... \leqslant \lambda_n(x)$ are the eigenvalues of $D^2 f(x)$. The functions $\lambda_1, ..., \lambda_n$ are continuous, but they are not necessarily smooth. In order to apply the maximum principle, we therefore introduce the following definition:

Definition 2.23

Let $U \subseteq \mathbb{R}^n$ be an open set, let L be a second-order, linear, partial differential operator defined over U, let $f : U \to \mathbb{R}$ be a continuous function and let $g : U \times \mathbb{R}^n \to \mathbb{R}$ be any other function. We say that $(Lf)(x) > g(x, Df(x))$ in the weak sense whenever for all $x \in U$, there exists a smooth function φ such that:

(1) $\varphi(x) = f(x)$;

(2) $\varphi \leqslant f$; and

(3) $(L\varphi)(x) > g(x, D\varphi(x))$.

Remark: *The informed reader with notice similarities with the concept of viscosity supersolutions (c.f. [8]). Definition 2.23 however yields a stronger property, since the latter does not assume the existence of smooth test functions at every point.*

The weak maximum principle applies to functions which are subharmonic in the weak sense:

Theorem 2.24

Let L be a second-order, elliptic, linear, partial differential operator over Ω with vanishing zeroeth order coefficient. Let $f : \overline{\Omega} \to \mathbb{R}$ be a continuous function. If $Lf > 0$ in the weak sense over Ω, then:

$$\mathop{\mathrm{Sup}}_{x\in\Omega} f(x) \leqslant \mathop{\mathrm{Sup}}_{x\in\partial\Omega} f(x).$$

Remark: *Observe that we make no assumption concerning the regularity of the coefficients of L. Stronger maximum principles can be shown in the usual manner provided further regularity on these coefficients. This is however not necessary for our purposes.*

Proof: L is of the form:

$$Lg(x) = a^{ij}(x)D^2g(x)_{ij} + b^i(x)Dg(x)_i,$$

where, by hypothesis, $a^{ij} > 0$. Since $\overline{\Omega}$ is compact, f attains its maximum at some point $x_0 \in \overline{\Omega}$ say. We claim that $x_0 \in \partial\Omega$. Indeed, suppose the contrary. Since $Lf > 0$ in the weak sense, there exists a smooth function $\varphi : \Omega \to \mathbb{R}$ such that $\varphi(x) \leqslant f(x)$ for all $x \in \Omega$, $\varphi(x_0) = f(x_0)$ and:

$$(L\varphi)(x_0) > 0.$$

However, since f attains its maximum at x_0, so does φ. Thus:

$$D^2\varphi(x_0) \leqslant 0.$$

Since L is elliptic, this implies that $(Lg)(x_0) \leqslant 0$ which is absurd and the result follows. $\square$

We denote $B^{ij} = \frac{1}{n}F(D^2f)(D^2f^{-1})^{ij}$. We recall:

Lemma 2.25

If $g : \overline{\Omega} \to \mathbb{R}$ is a smooth, positive function. Then:

$$\mathcal{L}_f\mathrm{Log}(g) = \frac{1}{g}\mathcal{L}_f g - B^{ij}\partial_i\mathrm{Log}(g)\partial_j\mathrm{Log}(g).$$

Proof: By the chain rule:

$$D\mathrm{Log}(g) = \frac{1}{g}Dg, \qquad (D^2\mathrm{Log}(g))_{ij} = \frac{1}{g}(D^2g)_{ij} - (D\mathrm{Log}(g))_i(D\mathrm{Log}(g))_j.$$

Since $DF(D^2f)$ and $DG(Df)$ are linear, this yields:

$$\begin{aligned}\mathcal{L}_f\mathrm{Log}(g) &= DF(D^2f)(D^2\mathrm{Log}(g)) - \phi DG(Df)(D\mathrm{Log}(g)) \\ &= \tfrac{1}{g}DF(D^2f)(D^2g) - \tfrac{1}{g}\phi DG(Df)(Dg) - DF(D^2f)^{ij}(D\mathrm{Log}(g))_i(D\mathrm{Log}(g))_j \\ &= \mathcal{L}_f\mathrm{Log}(g) - DF(D^2f)^{ij}(D\mathrm{Log}(g))_i(D\mathrm{Log}(g))_j,\end{aligned}$$

as desired. $\square$

Lemma 2.26

There exists $C > 0$, which only depends on $\|\phi\|_2$, $\|f\|_1$ and $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$ such that if $x \in \overline{\Omega}$ and if e_n coincides with the eigenvector of $D^2f(x)$ corresponding to the eigenvalue λ_n, then:

$$\mathcal{L}_f\mathrm{Log}(\lambda_n) + \frac{2}{n\lambda_n}(\partial_n\mathrm{Log}(\lambda_n))^2 \geqslant -C + B^{ij}(\partial_i\mathrm{Log}(\lambda_n))(\partial_j\mathrm{Log}(\lambda_n)),$$

in the weak sense.

Remark: *Observe that the second term on the left-hand side is of first order in $\mathrm{Log}(l)$, and, in addition, that its coefficient becomes small as λ_n becomes large. These properties also hold in the study of more general curvature functions. Observe in addition the positivity of the second term on the right hand side, which is the opposite of what normally follows upon calculating the laplacian of the logarithm of a given function, and is essentially a consequence of the convexity and ellipticity of F (c.f. the remark following Lemma 2.7). We invite the reader to observe at what stages these two properties become important in the sequel.*

Proof: Choose $x \in \Omega$. By applying an isometry, we may assume that $e_1, ..., e_n$ are the eigenvectors of $D^2f(x)$ corresponding to the eigenvalues $\lambda_1(f)(x), ..., \lambda_n(f)(x)$ respectively. We define the function $l = f_{nn}$. Observe that l is smooth, $l \leqslant f_{nn}$ and $l(x) = f_{nn}(x)$. It thus suffices to prove the desired relation for l at x. Differentiating (A) twice in the e_n direction at x yields:

$$\begin{aligned}
&DF(D^2f(x))(\partial_n\partial_nD^2f(x)) + D^2F(D^2f(x))(\partial_nD^2f(x), \partial_nD^2f(x)) \\
&\quad = \phi_{nn}(x)G(Df(x)) + 2\phi_n(x)DG(Df(x))(\partial_nDf(x)) \\
&\qquad +\phi(x)DG(Df(x))(\partial_n\partial_nDf(x)) + \phi(x)D^2g(Df(x))(\partial_nDf(x), \partial_nDf(x)).
\end{aligned}$$

Since the derivatives of f are symmetric, since G is convex and since ϕ is positive, recalling the definition of $\mathcal{L}_f$, this simplifies to:

$$\begin{aligned}
(\mathcal{L}_fl)(x) \quad &\geqslant \phi_{nn}(x)G(Df(x)) + 2\phi_n(x)DG(Df(x))(Df_n(x)) \\
&\quad -D^2F(D^2f(x))(D^2f_n(x), D^2f_n(x)).
\end{aligned}$$

We denote $\eta = \mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$. Bearing in mind Lemma 2.9 and the definition of G:

$$l = \lambda_n(f) \geqslant \frac{1}{n}\mathrm{Tr}(D^2f) \geqslant F(D^2f) = \phi G(Df) \geqslant \phi \geqslant \eta.$$

There therefore exists $C_1 > 0$ which only depends on $\|\phi\|_2$ and $\|f\|_1$ such that:

$$\phi_{nn}(x)G(Df(x)) \geqslant -C_1 = -(C_1/l(x))l(x) \geqslant -(C_1/\eta)l(x).$$

Observe that $\|D^2f(x)\| \leqslant \lambda_n(x) = l(x)$. There therefore exists C_2 which only depends on $\|\phi\|_1$ and $\|f\|_1$ such that:

$$2\phi_n(x)DG(Df(x))(Df_n(x)) \geqslant -C_2l(x).$$

In summary, there exists $C_3 > 0$ which only depends on $\|\phi\|_2$, $\|f\|_1$ and $\mathrm{Inf}_{x\in\overline{\Omega}}\phi(x)$ such that:

$$(\mathcal{L}_f l)(x) \geqslant -C_3 l(x) - D^2F(D^2f(x))(D^2f_n(x), D^2f_n(x)).$$

By Lemma 2.7, since all the eigenvalues of $D^2f(x)$ are positive:

$$\begin{array}{rl} -(D^2F)(D^2f(x))(D^2f_n(x), D^2f_n(x)) & \geqslant \frac{1}{n}F(A)\sum_{i\neq j}\frac{1}{\lambda_i(x)\lambda_j(x)}(f_{ijn})(x)^2 \\ & \geqslant \frac{2}{n}F(A)\lambda_n(x)\sum_{i=1}^{n-1}\frac{1}{\lambda_i(x)\lambda_n^2(x)}(f_{inn})(x)^2 \\ & = 2\lambda_n(x)B^{ij}(\partial_i\mathrm{Log}(f_{nn})(x))(\partial_j\mathrm{Log}(f_{nn})(x)) \\ & \quad -(2/n)(\partial_n\mathrm{Log}(f_{nn})(x))^2 \\ & = 2l(x)B^{ij}(\partial_i\mathrm{Log}(l)(x))(\partial_j\mathrm{Log}(l)(x)) \\ & \quad -(2/n)(\partial_n\mathrm{Log}(l)(x))^2. \end{array}$$

It follows that:

$$\mathcal{L}_f l(x) \geqslant -C_3 l(x) - (2/n)(\partial_n\mathrm{Log}(l)(x))^2 + 2l(x)B^{ij}(x)(\partial_i\mathrm{Log}(l)(x)(\partial_j\mathrm{Log}(l)(x)),$$

and so, by Lemma 2.25:

$$\mathcal{L}_f\mathrm{Log}(l)(x) \geqslant -C_3 - \frac{2}{n\lambda_n}(\partial_n\mathrm{Log}(l)(x))^2 + B^{ij}(x)(\partial_i\mathrm{Log}(l)(x)(\partial_j\mathrm{Log}(l)(x)),$$

as desired. □

Theorem 2.27

There exists $C > 0$ which only depends on $\|\phi\|_2$, $\mathrm{Inf}_{x\in\overline{\Omega}}\phi(x)$, $\delta(\hat{f})$, $\|\hat{f}\|_2$ and $\|f\|_1$ such that if $f \geqslant \hat{f}$ then:

$$\|f\|_2 \leqslant C.$$

Proof: For $A > 0$, consider the function $\varphi_A : \overline{\Omega} \to \mathbb{R}$ given by:

$$\varphi_A = \mathrm{Log}(\lambda_n) - A(f - \hat{f}).$$

It suffices to prove that $\varphi_A \leqslant C$ for some A and some C which both depend only on $\|\phi\|_2$, $\mathrm{Inf}_{x\in\overline{\Omega}}\phi(x)$, $\delta(\hat{f})$, $\|\hat{f}\|_2$ and $\|f\|_1$. By compactness of $\overline{\Omega}$, φ_A assumes its maximum at some point $x \in \overline{\Omega}$. Let C_1 be as in Theorem 2.22. Observe that C_1 only depends on $\|\phi\|_1$, $\mathrm{Inf}_{x\in\overline{\Omega}}\phi(x)$, $\delta(\hat{f})$, $\|f\|_2$ and $\|f\|_1$. If x is a boundary point of Ω, then:

$$\varphi_A(x) = \mathrm{Log}(\lambda_n(x)) = \mathrm{Log}(\|D^2f(x)\|) \leqslant \mathrm{Log}(C_1).$$

Now suppose that x is an interior point of Ω. We will show that $\mathcal{L}_f(\varphi_A)(x)$ is positive for sufficiently high values of $\varphi_A(x)$. Bounds on φ_A then follow by the maximum principle. Upon applying an isometry, we may suppose that the eigenvector of $D^2f(x)$ with eigenvalue $\lambda_n(x)$ coincides with e_n. We define $\tilde{\varphi}_A \in C^\infty(\overline{\Omega})$ by:

$$\tilde{\varphi}_A = \mathrm{Log}(f_{nn}) - A(f - \hat{f}).$$

Observe that $\tilde{\varphi}_A \leqslant \varphi_A$ and $\tilde{\varphi}_A(x) = \varphi_A(x)$. In particular, x is a local maximum of $\tilde{\varphi}_A$. Let C_2 be as in Lemma 2.26 and choose $A > (C_2+1)/\delta(\hat{f})$. Observe that A only depends on $\|\phi\|_2$, $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$, $\delta(\hat{f})$ and $\|f\|_1$. By Theorem 2.11:

$$\mathcal{L}_f(f-\hat{f}) \leqslant -\delta(\hat{f}).$$

Thus:

$$\mathcal{L}_f(\tilde{\varphi}_A)(x) > 1 - \frac{2}{n\lambda_n(x)}(\partial_n \mathrm{Log}(\lambda_n)(x))^2,$$

in the weak sense. However, since x is an local maximum of $\tilde{\varphi}_A$, $D\tilde{\varphi}_A(x) = 0$, and so, in particular:

$$|\partial_n \mathrm{Log}(\lambda_n)(x)| = A\left|f_n(x) - \hat{f}_n(x)\right| \leqslant A(\|f\|_1 + \|\hat{f}\|_1).$$

There therefore exists $C_3 > 0$ which only depends on $\|\phi\|_2$, $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$, $\delta(\hat{f})$, $\|\hat{f}\|_1$ and $\|f\|_1$ such that:

$$\mathcal{L}_f(\tilde{\varphi}_A)(x) > 1 - e^{C_3 - \tilde{\varphi}_A(x)}.$$

However, since x is a local maximum of $\tilde{\varphi}_A$:

$$\mathcal{L}_f(\tilde{\varphi}_A)(x) \leqslant 0.$$

We conclude that $\varphi_A(x) = \tilde{\varphi}_A(x) \leqslant C_3$, as desired. $\square$

2.7 Higher Order Bounds.

We observe that, up to this point, we have obtained a-priori C^2 bounds for solutions of (A) satisfying $f \geqslant \hat{f}$. We describe in this section the general principals required to obtain a-priori bounds of arbitrary order.

We first show how a-priori C^k bounds are obtained for all k provided we have already obtained a-priori $C^{2+\alpha}$ bounds for some $\alpha > 0$ (Hölder spaces and Hölder norms will be introduced and discussed in more detail in Section 3.4). Let M be a compact manifold with boundary, let U be an open subset of $\oplus_{i=1}^2\mathrm{Symm}(i,\mathbb{R}^n)$ and let $\Phi : M\times\overline{\Omega}\times U \to \mathbb{R}$ be a smooth function. We consider the manifold M as the parameter space for a smooth family of functions from $\overline{\Omega}\times U$ into $\mathbb{R}$. For all $\xi \in \oplus_{i=0}^1\mathrm{Symm}(i,\mathbb{R}^n)$, we define $U_\xi \subseteq \mathrm{Symm}(2,\mathbb{R}^n)$ by:

$$U_\xi = \{A \in \mathrm{Symm}(2,\mathbb{R}^n)\ |\ (\xi, A) \in U\},$$

and for all $(p,x,\xi) \in M\times\overline{\Omega}\times\oplus_{i=0}^1\mathrm{Symm}(i,\mathbb{R}^n)$, we define $\Phi_{p,x,\xi} : U_\xi \to \mathbb{R}$ by $\Phi_{x,p,\xi}(A) = \Phi(p,x,\xi,A)$. As in Section 2.2, for all $(p,x,\xi,A) \in M\times\overline{\Omega}\times U\times\Gamma$, we identify $D\Phi_{p,x,\xi}(A)$ with an element of $\mathrm{Symm}(2,\mathbb{R}^n)$. We say that Φ is **elliptic** whenever $D\Phi_{x,p,\xi}(A)$ is positive-definite for all $(p,x,\xi,A) \in M\times\overline{\Omega}\times U$. The following result encapsulates much of classical Schauder Theory (c.f. [11]):

Theorem 2.28

If Φ is elliptic, then for every compact subset $K \subseteq U$, for every $\varphi \in C^\infty(\partial\Omega)$, for all $B > 0$, for all $\alpha \in]0,1]$ and for all $k \in \mathbb{N}$, there exists $C > 0$ such that if p is a point in M and if $g : \overline{\Omega} \to \mathbb{R}$ is a smooth function with the properties that $\|g\|_{2+\alpha} \leqslant B$, $g|_{\partial\Omega} = \varphi$, $J^2(g)(x) \in K$ for all $x \in \overline{\Omega}$, and $\Phi(p, x, J^2 g(x)) = 0$ for all $x \in \overline{\Omega}$, then $\|g\|_k \leqslant C$.

We now show how a-priori $C^{2+\alpha}$ bounds are obtained from a-priori C^2 bounds. Let M be a compact manifold with boundary, let U be an open subset of $\oplus_{i=0}^{1}\mathrm{Symm}(i, \mathbb{R}^n)$, and let $\Phi : M \times \overline{\Omega} \times U \times \Gamma \to \mathbb{R}$ be a smooth function. As before, for all $(p, x, \xi) \in M \times \overline{\Omega} \times U$, we define $\Phi_{p,x,\xi} : \Gamma \to \mathbb{R}$ by $\Phi_{p,x,\xi}(A) = \Phi(x, \xi, A)$. The following result is a special case of Theorem 1 of [3]:

Theorem 2.29

Suppose that Φ is elliptic and that for all $(p, x, \xi) \in M \times \overline{\Omega} \times U$, $\Phi_{p,x,\xi}$ is concave. Then for every $\varphi \in C^\infty(\partial\Omega)$, for all $B > 0$, there exists $\alpha \in]0,1]$ and $C > 0$ such that if p is a point in M and if $g : \overline{\Omega} \to \mathbb{R}$ is a smooth function with the properties that $\|g\|_2 \leqslant B$, $g|_{\partial\Omega} = \varphi$, $J^2 g(x) \in U \times \Gamma$ for all $x \in \overline{\Omega}$, and $\Phi(p, x, J^2 g(x)) = 0$ for all $x \in \overline{\Omega}$, then $\|g\|_{2+\alpha} \leqslant C$.

We now return to the case where:

$$\Phi(p, x, (t, \xi, A)) = F(A) - \phi(p, x)G(p, \xi),$$

where $\phi > 0$, $G \geqslant 1$ and $\xi \mapsto G(p, \xi)$ is concave for all p.

Theorem 2.30

Let $(p_m)_{m\in\mathbb{N}}$ be a sequence of points in M and let $(f_m)_{m\in\mathbb{N}}, (\hat{f}_m)_{m\in\mathbb{N}} \in C_0^\infty(\overline{\Omega})$ be strictly convex functions such that for all m, $f_m \geqslant \hat{f}_m$ and for all $x \in \overline{\Omega}$:

$$\Phi(p_m, x, J^2\hat{f}_m) \geqslant 0 = \Phi(p_m, x, J^2 f_m).$$

If there exists $x_0 \in M$ towards which $(x_m)_{m\in\mathbb{N}}$ converges and $\hat{f}_\infty \in C_0^\infty(\overline{\Omega})$ towards which $(\hat{f}_m)_{m\in\mathbb{N}}$ converges in the C^∞ sense, then there exists $f_\infty \in C_0^\infty(\overline{\Omega})$ towards which $(f_m)_{m\in\mathbb{N}}$ subconverges in the C^∞ sense.

Proof: By Lemma 2.1 and 2.3, there exists $C_1 > 0$ such that for all m, $\|f_m\|_1 \leqslant C_1$. By Theorem 2.22, there exists $C_2 > 0$ such that, for all m, $\|f_m\|_2 \leqslant C_2$. By Lemma 2.5, Φ is convex. By Corollary 2.8, $\Phi_{(p,x,(t,\xi)}$ is concave for all $(p, x, (t, \xi))$. There, by Theorem 2.29, there exists $\alpha > 0$ and $C_{2+\alpha} > 0$ such that, for all m, $\|f_m\|_{2+\alpha} \leqslant C_{2+\alpha}$. By Theorem 2.28, for all $k \in \mathbb{N}$, there exists $C_k > 0$ such that, for all m, $\|f_m\|_k \leqslant C_k$. It follows by the classical Arzela-Ascoli theorem (c.f. Theorem 11.28 of [17]) that there exists $f_\infty \in C_0^\infty(\overline{\Omega})$ towards which $(f_m)_{m\in\mathbb{N}}$ subconverges as desired. $\square$

3

Degree Theory

We develop a topological degree theory for smooth mappings between open subsets of Banach spaces. Used in conjunction with Theorem 2.30, this yields the first main result of this text, being Theorem 1.4, which proves the existence of unique solutions to the Plateau problem for gaussian curvature in the case of graphs. This result itself is an important ingredient of our stated objective, being the proof of Theorem 1.2, which solves a general Plateau problem for hypersurfaces of constant gaussian curvature in Euclidean Space, and which we will return to in Chapter 5.

The topological degree theory we use dates back to Smale (c.f. [21]) and requires a fairly in-depth detour into functional analysis. A complete exposition of the required background material would take us too far afield, and we therefore quote a number of results without proof. We hope that this will not obscure too much the main ideas, and we refer the interested reader to the numerous excellent introductions to functional analysis (c.f. for example [1], [17] and [18]) for more information. Of particular interest is Theorem 3.7, which constructs a $\mathbb{Z}_2$ valued differential-topological degree for the zero set of a given smooth function between Banach spaces. We underline that although we do not appeal directly to [21], our argument, which uses first a finite dimensional reduction, and then the classical Sard's Theorem, follows exactly Smale's reasoning. We encourage the novice reader to study in detail this approach, as it clarifies the main ideas of Smale's result which, moreover, is often too specific as stated to be applied in many settings of interest in present-day mathematics.

3.1 Smooth Mappings and Differential Operators.

Let E and F be two normed vector spaces. We denote by $\mathrm{Lin}(E, F)$ the space of *bounded* linear maps from E into F. Observe that $\mathrm{Lin}(E,F)$ is also a normed vector space with norm given by:

$$\|A\| = \mathop{\mathrm{Sup}}_{x\in E\setminus\{0\}} \frac{\|Ax\|}{\|x\|},$$

for all $A \in \mathrm{Lin}(E,F)$. Let U be an open subset of E and let Φ be a mapping from U into F. For $x \in U$, we say that Φ is **differentiable** at x whenever there exists a *bounded* linear map $A : E \to F$ such that:

$$\mathop{\mathrm{Lim}}_{y\to 0} \frac{1}{\|y\|}\|\Phi(x+y) - \Phi(x) - A(y)\| = 0,$$

where y varies over all vectors in $E \setminus \{0\}$ having the property that $x + y \in U$. We refer to A as the derivative of Φ at x. Whenever the derivative exists, it is unique, and we denote it by $D\Phi(x)$.

Remark: *This definition only differs from the finite dimensional version by the insistence that the derivative be bounded. Importantly, since* $\mathrm{Lin}(E,F)$ *is also a normed vector space, this allows us to iterate the concept of differentiability and thus consider derivatives of arbitrarily high order. In the finite dimensional case, this condition is unnecessary, since it follows from local compactness that all linear maps between finite dimensional vector spaces are bounded anyway.*

We say that a function $\Phi : U \to F$ is C^1 whenever $D\Phi$ exists at every point of U and defines a continuous function from U into $\mathrm{Lin}(E,F)$. We define inductively the notion of higher order differentiability, and we say that Φ is C^k whenever $D\Phi$ exists at every point of U and defines a C^{k-1} function from U into $\mathrm{Lin}(E,F)$. We say that Φ is **smooth** whenever it is C^k for all $k \in \mathbb{N}$.

As in the finite dimensional case, smooth maps are constructed using the following rules:

Chain Rule: Let E_1, E_2 and E_3 be normed vector spaces. Let U_1 and U_2 be open subsets of E_1 and E_2 respectively and let $\Phi : U_1 \to U_2$ and $\Psi : U_2 \to E_3$ be smooth mappings.

Theorem 3.1

$\Psi \circ \Phi$ *is smooth and its first derivative is given by:*

$$D(\Psi\circ\Phi)(x)(y) = D\Psi(\Phi(x))(D\Phi(x)(y)).$$

Direct sums: Let E, F_1,...,F_n be normed vector spaces. Let U be an open subset of E and for $1 \leqslant i \leqslant n$, let $\Phi_i : U \to F_i$ be a smooth mapping.

Theorem 3.2

$\Phi = (\Phi_1, ..., \Phi_n)$ *defines a smooth mapping from* E *into* $F_1 \oplus ... \oplus F_n$ *and its first derivative is given by:*

$$D\Phi = (D\Phi_1, ..., D\Phi_n).$$

Multilinear forms: Let $E_1, ..., E_n$ and F be normed vector spaces. Let $\Phi : E_1 \oplus ... \oplus E_n \to F$ be a bounded, multilinear map.

Theorem 3.3

Φ is smooth and its first derivative is given by:

$$D\Phi(x_1,...,x_n)(V_1,...,V_n) = \Phi(V_1,x_2,...,x_n) + ... + \Phi(x_1,...,x_{n-1},V_n).$$

Remark: *Observe in particular that the product rule is a corollary of Theorems 3.1, 3.2 and 3.3.*

3.2 Banach Spaces.

Let E be a normed vector space. We say that E is a **Banach space** whenever it is complete. This extra hypothesis yields the inverse function theorem (c.f. [17]):

Theorem 3.4, Inverse Function Theorem

Let E and F be Banach spaces. Let U be an open subset of E and let Φ be a smooth mapping from U to F. If $D\Phi(x)$ is invertible at some point $x \in U$, then there exist neighbourhoods V of x in U, W of $\Phi(x)$ in F and a smooth mapping $\Psi : W \to V$ such that $W = \Phi(V)$ and:

$$\Psi \circ \Phi = \mathrm{Id}, \qquad \Phi \circ \Psi = \mathrm{Id}.$$

Let E be a Banach space. Let X be a subset of E. For $n \in \mathbb{N}$, we say that X is an n-dimensional submanifold of E whenever there exists a Banach space F with the property that for all $x \in X$, there exist neighbourhoods U of x in E and V of $(0,0)$ in $\mathbb{R}^n \times F$ and a smooth mapping $\Phi : U \to V$ with smooth inverse such that $\Phi(X \cap U) = (\mathbb{R}^n \times \{0\}) \cap V$. We refer to the triplet (Φ, U, V) as a **trivialising chart** of X about x.

Remark: *We make no assumption concerning the separability neither of E nor of X.*

Lemma 3.5

Let E be a Banach space. Let X be a finite-dimensional submanifold of E and let $e : X \to E$ be the canonical embedding. If X is separable, then X is a smooth, finite-dimensional manifold, and $e : X \to E$ is a smooth mapping.

Proof: Since X is a subset of a Hausdorff space, it is also Hausdorff. Moreover, by hypothesis, it is separable. It thus suffices to construct a smooth atlas of charts for X. Choose $x \in X$ and let $(\Phi, \tilde{U}, \tilde{V})$ be a trivialising chart of X about x. We denote $\varphi = \Phi|_{X \cap \tilde{U}}$, $U = X \cap \tilde{U}$ and $V = (\mathbb{R}^n \times \{0\}) \cap \tilde{V}$. We see that (φ, U, V) defines a homeomorphism from an open subset of X to an open subset of $\mathbb{R}^n$. We claim that the family of all such charts constitutes a smooth atlas for X. Indeed, choose $x' \in X$, let $(\Phi', \tilde{U}', \tilde{V}')$ be a trivialising chart of X about x' and denote $\varphi' = \Phi'|_{X \cap \tilde{U}'}$, $U' = X \cap \tilde{U}'$ and $V' = (\mathbb{R}^n \times \{0\}) \cap \tilde{V}'$. Observe that $U \cap U' = \tilde{U} \cap \tilde{U}' \cap X$ and:

$$\varphi' \circ (\varphi^{-1})|_{\varphi(U \cap U')} = \Phi' \circ (\Phi^{-1})|_{\Phi(\tilde{U} \cap \tilde{U}' \cap X)}.$$

In particular, the transition map is smooth. Since x and x' are arbitrary points in X, we deduce that the set of all such charts constitutes a smooth atlas as desired. Finally, in the chart (φ, U, V), the canonical immersion coincides with φ^{-1}. However:

$$\varphi^{-1} = \Phi^{-1}|_V,$$

and this map is smooth, as desired. □

Let E and F be two Banach spaces. Recall that a bounded linear mapping $A \in \mathrm{Lin}(E, F)$ is said to be **Fredholm** whenever it has closed image and both its kernel and its cokernel are finite dimensional. We define the **index** of a Fredholm mapping by:

$$\mathrm{Ind}(A) = \mathrm{Dim}(\mathrm{Ker}(A)) - \mathrm{Dim}(\mathrm{Coker}(A)).$$

Let U be an open subset of E and let Φ be a smooth mapping from U into F. We say that Φ is **Fredholm** whenever $D\Phi(x)$ is a Fredholm mapping for all $x \in U$. Recall that the linear Fredholm mappings constitute an open subset of $\mathrm{Lin}(E, F)$ and that two linear Fredholm mappings in the same connected component have the same index. It follows that if U is connected, then $\mathrm{Ind}(D\Phi(x))$ is independant of x in U and we therefore refer to it as the index of the mapping Φ. In addition, we recall that the set of surjective, linear Fredholm mappings also constitutes an open subset of $\mathrm{Lin}(E, F)$. This is relevant to situations where we apply the following submersion theorem:

Theorem 3.6

Let E and F be two Banach spaces. Let U be an open subset of E and let $\Phi : U \to F$ be a smooth, Fredholm map. If $D\Phi$ is surjective for all $x \in \Phi^{-1}(\{0\})$, then $\Phi^{-1}(\{0\})$ is a smooth $\mathrm{Ind}(\Phi)$*-dimensional submanifold of E.*

Proof: Indeed, choose $x_0 \in \Phi^{-1}(\{0\})$. Let $\mathrm{Ker}(D\Phi(x_0))$ be the kernel of $D\Phi(x_0)$. Since $D\Phi(x_0)$ is Fredholm and surjective, $\mathrm{Dim}(\mathrm{Ker}(D\Phi(x_0)))$ is equal to $\mathrm{Ind}(\Phi)$. By the Hahn-Banach theorem (Theorem 5.16 of [17]), the identity map $\mathrm{Id} : \mathrm{Ker}(D\Phi(x_0)) \to \mathrm{Ker}(D\Phi(x_0))$ extends to a bounded, linear projection π from E onto $\mathrm{Ker}(D\Phi(x_0))$. We define the mapping $\hat{\Phi} : U \to \mathrm{Ker}(D\Phi)(x_0) \times F$ by $\hat{\Phi}(x) = (\pi(x - x_0), \Phi(x))$. By Theorems 3.2 and 3.3, $\hat{\Phi}$ is also smooth, and for all vectors $y \in E$, $D\hat{\Phi}(x_0)(y) = (\pi(y), D\Phi(x_0)(y))$. In particular, $D\hat{\Phi}(x_0)$ is bijective. By the open mapping theorem (c.f. Theorems 5.9 and 5.10 of [17]), $D\hat{\Phi}(x_0)$ is invertible with bounded, linear inverse. It follows from Theorem 3.4 that there exist neighbourhoods U of x_0 in E and V of $(0, 0)$ in $\mathrm{Ker}(D\Phi(x_0)) \times F$ and a smooth mapping $\Psi : V \to U$ such that $\hat{\Phi}(U) = V$, $\hat{\Phi} \circ \Psi = \mathrm{Id}$ and $\Psi \circ \hat{\Phi} = \mathrm{Id}$. We now denote $X = \Phi^{-1}(\{0\})$, and to conclude, it suffices to show that $\hat{\Phi}(X \cap U) = (\mathrm{Ker}(D\Phi(x_0)) \times \{0\}) \cap V$. However, if $x \in X \cap U$ then $\Phi(x) = 0$ and so $\hat{\Phi}(x) \in (\mathrm{Ker}(D\Phi(x_0)) \times \{0\}) \cap V$. Conversely, if $(y, 0) \in (\mathrm{Ker}(D\Phi)(x_0) \times \{0\}) \cap V$, then, denoting $x = \Psi(y, 0)$, we find that $\hat{\Phi}(x) = \hat{\Phi} \circ \Psi(y, 0) = (y, 0)$. In particular, $\Phi(x) = 0$ and so $x \in X$. It follows that $(y, 0) \in \hat{\Phi}(X \cap U)$ and the two sets therefore coincide as desired. □

3.3 Degree Theory.

Let E be a finite dimensional vector space and let F_1 and F_2 be Banach spaces. Let U and V be open subsets of E and F_1 respectively, and let $\Phi : U \times V \to F_2$ be a smooth Fredholm mapping whose index is equal to the dimension of E. Let $\Pi : E \times F_1 \to E$ be the projection onto the first factor. By Theorem 3.3, Π is a smooth mapping. We define the **solution space** of Φ by:

$$\mathcal{Z} = \{(x,y) \in E \times F_1 \mid \Phi(x,y) = 0\}.$$

We denote by $\Pi_{\mathcal{Z}}$ the restriction of the mapping Π to $\mathcal{Z}$. Observe that if $D\Phi(x,y)$ is surjective for all $(x,y) \in \mathcal{Z}$ then, by Theorem 3.6, $\mathcal{Z}$ is a smooth, finite-dimensional submanifold of $E \times F_1$ of dimension equal to $\mathrm{Ind}(\Phi) = \mathrm{Dim}(E)$. We consider the topological degree of the mapping $\Pi_{\mathcal{Z}}$.

Theorem 3.7, Differential Topological Degree

If Φ is smooth and Fredholm, if $D\Phi(x,y)$ is surjective for all $(x,y) \in \mathcal{Z}$, and if $\Pi_{\mathcal{Z}}$ is proper, then there exists an open, dense subset $U' \subseteq U$ with the property that, for all $x \in U'$, $\Pi_{\mathcal{Z}}^{-1}(\{x\})$ is finite. Moreover, for all $x, y \in U'$:

$$\left|\Pi_{\mathcal{Z}}^{-1}(\{x\})\right| = \left|\Pi_{\mathcal{Z}}^{-1}(\{y\})\right| \ \textit{Mod}\ 2.$$

Proof: Denote $n = \mathrm{Dim}(E) = \mathrm{Ind}(\Phi)$. Since Φ is smooth and Fredholm, and since $D\Phi$ is surjective at every point of $\mathcal{Z} = \Phi^{-1}(\{0\})$, by Theorem 3.6, $\mathcal{Z} = \Phi^{-1}(\{0\})$ is a smooth, n-dimensional submanifold of $E \times F_1$. Since U is an open subset of a finite dimensional vector space, U is separable. Since $\Pi_{\mathcal{Z}} : \mathcal{Z} \to U$ is proper, $\mathcal{Z}$ is also separable. By Lemma 3.5, $\mathcal{Z}$ is therefore a smooth, n-dimensional manifold and the canonical embedding $e : \mathcal{Z} \to E$ is a smooth mapping. In particular, $\Pi_{\mathcal{Z}} = \Pi \circ e$ is also smooth.

We now apply standard differential topological techniques to the mapping $\Pi_{\mathcal{Z}}$. We use the terminology of [13]. We define U' to be the set of regular values of $\Pi_{\mathcal{Z}}$. If $x \in U'$, then $\Pi_{\mathcal{Z}}^{-1}(\{x\})$ is discrete, and since $\Pi_{\mathcal{Z}}$ is proper, this set is compact and therefore finite. Moreover, since $\Pi_{\mathcal{Z}}$ is a smooth, proper mapping, by Sard's Theorem (c.f. [13]), U' is open and dense, and the first assertion follows.

Now choose $x, y \in U'$ and let $\gamma : [0,1] \to U$ be any smooth, embedded curve such that $\gamma(0) = x$ and $\gamma(1) = y$. By genericity (c.f. [13]), there exists another smooth, embedded curve $\gamma' : [0,1] \to U$ which we may choose as close to γ as we wish in the C^∞ sense with the property that $\gamma'(0) = \gamma(0) = x$, $\gamma'(1) = \gamma(1) = y$ and γ' is transverse to $\Pi_{\mathcal{Z}}$. If we denote by $\Gamma' \subseteq U$ the image of γ', then, by transversality, $\Pi_{\mathcal{Z}}^{-1}(\Gamma')$ is a smooth, 1-dimensional, embedded, submanifold of $\mathcal{Z}$ with boundary given by:

$$\partial \Pi_{\mathcal{Z}}^{-1}(\Gamma') = \Pi_{\mathcal{Z}}^{-1}(\{x\}) \cup \Pi_{\mathcal{Z}}^{-1}(\{y\}).$$

Since $\Pi_{\mathcal{Z}}^{-1}$ is proper, $\Pi_{\mathcal{Z}}^{-1}(\Gamma')$ is compact, and therefore has an even number of boundary points. Thus:

$$\left|\Pi_{\mathcal{Z}}^{-1}(\{x\})\right| + \left|\Pi_{\mathcal{Z}}^{-1}(\{y\})\right| = \left|\partial \Pi_{\mathcal{Z}}^{-1}(\Gamma')\right| = 0 \text{ Mod } 2,$$

as desired. □

3.4 Hölder Spaces and Hölder Norms.

Let E be a finite dimensional normed vector space. For $\alpha \in]0,1]$ we denote by $[\cdot]_\alpha$ the **Hölder semi-norm** over $C^0(\overline{\Omega}, E)$ of order α. That is, for all $f \in C^0(\overline{\Omega}, E)$:

$$[f]_\alpha = \mathop{\mathrm{Sup}}_{x \neq y \in \overline{\Omega}} \frac{\|f(x) - f(y)\|}{\|x - y\|^\alpha}.$$

In the case where f is continuously differentiable, the convexity of $\overline{\Omega}$ yields an explicit formula for $[f]_1$:

Lemma 3.8

For all continuously differentiable $f \in C^0(\overline{\Omega}, E)$:

$$[f]_1 = \|Df\|_{L^\infty}.$$

Remark: *In general, for a compact set Ω with rectifiable boundary, $[f]_1 < C(\Omega)\|Df\|_{L^\infty}$, where $C \geqslant 1$ depends on the geometry of Ω. In fact, convex sets are characterised amongst all compact sets with rectifiable boundary by the property that $C(\overline{\Omega}) = 1$.*

Proof: Choose $x \neq y \in \overline{\Omega}$. Since $\overline{\Omega}$ is convex $(1-t)x + ty \in \overline{\Omega}$ for all $t \in [0,1]$. Thus, bearing in mind the Cauchy-Schwarz inequality:

$$\begin{aligned} \|f(y) - f(x)\| &= \| \textstyle\int_0^1 Df((1-t)x + ty)(y - x)dt\| \\ &\leqslant \textstyle\int_0^1 \|Df((1-t)x + ty)(y - x)\|dt \\ &\leqslant \textstyle\int_0^1 \|Df((1-t)x + ty)\|\|y - x\|dt \\ &\leqslant \|Df\|_{L^\infty}\|y - x\|. \end{aligned}$$

Thus, for all $x \neq y \in \overline{\Omega}$:

$$\frac{\|f(y) - f(x)\|}{\|y - x\|} \leqslant \|Df\|_{L^\infty}.$$

It follows upon taking the supremum over all $x \neq y \in \overline{\Omega}$ that $[f]_1 \leqslant \|Df\|_{L^\infty}$. Conversely, for all $x \in \Omega$ and for any vector $X \in \mathbb{R}^n$ there exists $\epsilon > 0$ such that $x + tX \in \Omega$ for all $t \in]-\epsilon, \epsilon[$. Thus, bearing in mind that $x \mapsto \|x\|$ is a continuous function of $\mathbb{R}^n$:

$$\begin{aligned} \|Df(x)(X)\| &= \| \mathrm{Lim}_{t \to 0} \tfrac{1}{t}(f(x + tX) - f(x)\| \\ &= \mathrm{Lim}_{t \to 0} \|\tfrac{1}{t}(f(x + tX) - f(x))\| \\ &\leqslant [f]_1\|X\|. \end{aligned}$$

Thus:

$$\|Df(x)\| = \mathop{\mathrm{Sup}}_{X \in \mathbb{R}^n \setminus \{0\}} \|Df(x)(X)\|/\|X\| \leqslant [f]_1.$$

By continuity, this relation also holds when x is a boundary point of Ω, and it now follows by taking the supremum over all $x \in \overline{\Omega}$ that $\|Df\|_{L^\infty} \leqslant [f]_1$. We conclude that $[f]_1 = \|Df\|_{L^\infty}$ as desired. □

For all $\lambda = k + \alpha \in]0, \infty[$, where $k \in \mathbb{N}$ and $\alpha \in]0, 1]$, we denote by $\| \cdot \|_\lambda$ the **Hölder norm** over $C^k(\overline{\Omega})$ of order λ. That is, for all $f \in C^k(\overline{\Omega})$:

$$\|f\|_\lambda = \sum_{i=0}^{k} \|D^i f\|_{L^\infty} + [D^k f]_\alpha.$$

For all $\lambda = k + \alpha \in]0, \infty[$ where $k \in \mathbb{N}$ and $\alpha \in]0, 1]$, we denote by $C^\lambda(\overline{\Omega})$ the space of all functions $f \in C^k(\overline{\Omega})$ such that $\|f\|_\lambda < \infty$. We refer to $C^\lambda(\overline{\Omega})$ as the space of λ times Hölder differentiable functions over $\overline{\Omega}$.

We recall the classical Arzela-Ascoli theorem which we state in the following form (c.f. [17]):

Theorem 3.9, Arzela-Ascoli

Choose $\lambda \in]0, \infty[$ and let $(f_m)_{m\in\mathbb{N}}$ be a sequence of functions in $C^\lambda(\overline{\Omega})$. If there exists $B > 0$ such that $\|f_m\|_\lambda < B$ for all m, then there exists $f_\infty \in C^\lambda(\overline{\Omega})$ such that $\|f_\infty\|_\lambda < B$ and $(f_m)_{n\in\mathbb{N}}$ subconverges to f_∞ in the C^μ norm for all $\mu \in]0, \lambda[$.

In particular, this yields:

Lemma 3.10

For all $\lambda \in]0, \infty[$, $(C^\lambda(\overline{\Omega}), \| \cdot \|_\lambda)$ is a Banach space.

Proof: Choose $\lambda > 0$ and let $(f_m)_{m\in\mathbb{N}}$ be a Cauchy sequence of functions in $C^\lambda(\overline{\Omega})$. We need to show that $(f_m)_{n\in\mathbb{N}}$ converges in $C^\lambda(\overline{\Omega})$. For all $i \in \mathbb{N}$, we define the subset F_i of $C^\lambda(\overline{\Omega})$ by:

$$F_i = \{f_m \mid m \geqslant i\},$$

and for all i, we define d_i to be the diameter of F_i. Since $(f_m)_{m\in\mathbb{N}}$ is a Cauchy sequence, the sequence $(d_i)_{i\in\mathbb{N}}$ converges to 0.

By Theorem 3.9, there exists a subsequence $(m_i)_{i\in\mathbb{N}}$ and a function f_∞ in $C^\lambda(\overline{\Omega})$ such that $(f_{m_i})_{i\in\mathbb{N}}$ converges to f_∞ in the C^μ norm for all $\mu < \lambda$. Moreover, we may assume that $m_i \geqslant i$ for all i. Consequently, for all i and for all $j > i$, $f_{m_j} \in F_i$ and so:

$$\|f_{m_j} - f_i\| \leqslant d_i.$$

Choose $i \in \mathbb{N}$. By Theorem 3.9, there exists $g \in C^\lambda(\overline{\Omega})$ such that $\|g\|_\lambda \leqslant d_i$ and $(f_{m_j} - f_i)_{j\in\mathbb{N}}$ subconverges to g in the C^μ norm for all $\mu < \lambda$. However, by definition of $(m_j)_{j\in\mathbb{N}}$, $(f_{m_j} - f_i)_{j\in\mathbb{N}}$ also converges to $(f_\infty - f_i)$ in the C^μ norm for all $\mu < \lambda$. It follows that $g = f_\infty - f_i$. In particular, $\|f_\infty - f_i\|_\lambda = \|g\|_\lambda \leqslant d_i$ and we conclude that $(f_i)_{i\in\mathbb{N}}$ converges to f_∞ as desired. □

For all $\lambda \in]0, \infty[$, we denote by $C^\lambda_0(\overline{\Omega})$ the linear subspace of $C^\lambda(\overline{\Omega})$ consisting of those functions which vanish along the boundary.

Lemma 3.11

For all $\lambda \in]0,\infty[$, $C^\lambda_0(\overline{\Omega})$ is a closed subspace of $C^\lambda(\overline{\Omega})$. In particular, it is a Banach subspace.

Proof: Choose $\lambda \in]0,\infty[$ and let $(f_m)_{m\in\mathbb{N}}$ be a sequence of functions in $C^\lambda_0(\overline{\Omega})$ converging to the function f_∞ in $C^\lambda(\overline{\Omega})$. Observe that $(f_m)_{m\in\mathbb{N}}$ converges to f_∞ pointwise, and thus, for all $x \in \partial\Omega$:

$$f_\infty(x) = \operatorname*{Lim}_{m\rightarrow\infty} f_m(x) = 0,$$

as desired. □

We leave the reader to verify that for all $\lambda \leqslant \mu$, $C^\mu(\overline{\Omega})$ (resp. $C^\mu_0(\overline{\Omega})$) canonically embeds as a subspace of $C^\lambda(\overline{\Omega})$ (resp. $C^\lambda_0(\overline{\Omega})$). Moreover, this embedding is continuous and if $\lambda < \mu$, it is also a compact mapping. In addition:

$$C^\infty(\overline{\Omega}) = \bigcap_{\lambda>0} C^\lambda(\overline{\Omega}), \qquad C^\infty_0(\overline{\Omega}) = \bigcap_{\lambda>0} C^\lambda_0(\overline{\Omega}),$$

and, moreover, a sequence $(f_m)_{m\in\mathbb{N}}$ in $C^\infty(\overline{\Omega})$ (resp. $C^\infty_0(\overline{\Omega})$) converges to a limit f_∞ in $C^\infty(\overline{\Omega})$ (resp. $C^\infty_0(\overline{\Omega})$) if and only if it converges to f_∞ in the C^λ-norm for all $\lambda > 0$.

3.5 Smooth Mappings of Hölder Spaces.

Let E be a finite dimensional vector space and let U be an open subset of E. For all $\lambda > 0$, we denote by $C^\lambda(\overline{\Omega}, U)$ the open subset of $C^\lambda(\overline{\Omega}, E)$ consisting of those functions g having the property that $g(x) \in U$ for all $x \in \overline{\Omega}$. Let F be another finite dimensional vector space and let $\phi : \overline{\Omega} \times U \rightarrow F$ be a smooth mapping. We define the mapping $\Phi : C^0(\overline{\Omega}, U) \rightarrow C^0(\overline{\Omega})$ by $\Phi(f)(x) = \phi(x, f(x))$. Together with Theorems 3.1, 3.2 and 3.3, the following result allows us apply the techniques of the preceeding sections to almost every function that we may encounter.

Theorem 3.12

For all $\lambda > 0$ and for all $g \in C^\lambda(\overline{\Omega}, U)$, $\Phi(g) \in C^\lambda(\overline{\Omega})$. Moreover, Φ defines a smooth mapping from $C^\lambda(\overline{\Omega}, U)$ into $C^\lambda(\overline{\Omega})$, and:

$$D\Phi(g)(h) = D_2\phi(g)(h),$$

where $D_2\phi$ is the partial derivative of ϕ with respect to the second component.

Remark: *Importantly, whilst post-composition by a smooth mapping defines a smooth mapping of Hölder spaces, precomposition by a smooth mapping does not. Indeed, if $\phi : \overline{\Omega} \rightarrow \overline{\Omega}$ is a smooth diffeomorphism, then although $g \circ \phi \in C^\lambda(\overline{\Omega})$ for all $g \in C^\lambda(\overline{\Omega})$, the mapping $g \mapsto g \circ \phi$ is not even continuous with respect to the C^λ norm.*

For all k and for all λ, consider the function $\mathcal{J}^k : C^{\lambda+k}(\overline{\Omega}) \rightarrow C^\lambda(\overline{\Omega}, \oplus_{i=0}^k \mathrm{Symm}(i, \mathbb{R}^n))$ given by $\mathcal{J}^k(f)(x) = (J^k f)(x)$, where we recall that $J^k f$ is the k-jet of f (c.f. Appendix A).

Lemma 3.13

For all k and for all λ, $\mathcal{J}^k$ is a smooth mapping and $D\mathcal{J}^k(g)(h) = J^k h$.

Remark: *At first glance this result often surprises, as one usually expects the derivative of the derivative mapping to yield the second derivative. We underline that in the context of Banach spaces, the derivative mapping is a bounded linear map in its own right.*

Proof: Indeed, $\mathcal{J}^k$ is bounded and linear. The result now follows from Lemma 3.3. □

Let U be an open subset of $\oplus_{i=0}^2\mathrm{Symm}(i,\mathbb{R}^n)$. For all $\lambda \geqslant 2$, we define $\mathcal{U}_0^\lambda(\overline{\Omega})$ to be the set of all functions g in $C_0^\lambda(\overline{\Omega})$ having the property that $J^2 g \in U$. Let E be any finite dimensional vector space and let $F : E \times \overline{\Omega} \times U \to \mathbb{R}$ be a smooth function. We define the mapping $\mathcal{F} : E \times \mathcal{U}_0^2(\overline{\Omega}) \to C^0(\overline{\Omega})$ by $\mathcal{F}(p,g)(x) = F(p,x,J^2g(x))$.

Lemma 3.14

For all $\lambda \in]0,\infty[$ and for all $(p,g) \in E \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$, $\mathcal{F}(p,g) \in C^\lambda(\overline{\Omega})$. Moreover, $\mathcal{F}$ defines a smooth mapping from $E \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$ and its partial derivative with respect to the second component is given by:

$$D_2\mathcal{F}(p,g)(h) = D_3F(p,x,J^2g)(J^2h),$$

where D^3F is the partial derivative of F with respect to the third component.

Proof: Choose $(p,g) \in E \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$. Then $(p,J^2g) \in E \times C^\lambda(\overline{\Omega}, \oplus_{i=0}^2\mathrm{Symm}(i,\mathbb{R}^n))$. Thus, by Theorem 3.12, $\mathcal{F}(p,g) = F(p,x,J^2g)$ is an element of $C^\lambda(\overline{\Omega})$ as desired. By Theorem 3.12, Lemma 3.13 and the chain rule (Theorem 3.1), $\mathcal{F}$ defines a smooth mapping from $E \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$ and:

$$D_2\mathcal{F}(p,g)(h) = D_3F(p,x,J^2g)(J^2h),$$

as desired. □

Thus, for all $\lambda > 0$, we think of $\mathcal{F}$ as a smooth family of mappings sending $\mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$ which is parametrised by E.

Observe that for all $(g,h) \in E \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$, $D_2\mathcal{F}(g,h)$ is a second-order, linear, partial differential operator from $C_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$.

As in Section 2.7, for all $\xi \in \oplus_{i=0}^1\mathrm{Symm}(i,\mathbb{R}^n)$, we define $U_\xi \subseteq \mathrm{Symm}(2,\mathbb{R}^n)$ by:

$$U_\xi = \{A \in \mathrm{Symm}(2,\mathbb{R}^n) \mid (\xi,A) \in U\},$$

and for all $(p,x,\xi) \in E \times \overline{\Omega} \times \oplus_{i=0}^1\mathrm{Symm}(i,\mathbb{R}^n)$, we define $F_{p,x,\xi} : U_\xi \to \mathbb{R}$ by:

$$F_{p,x,\xi}(A) = F(p,x,\xi,A).$$

We say that F is **elliptic** whenever $DF_{p,x,\xi}(A)$ is positive-definite for all $(p,x,\xi,A) \in E \times \overline{\Omega} \times U$.

Lemma 3.15

If F is elliptic, then $D_2\mathcal{F}(p,g)$ is an elliptic operator for all $(p,g) \in E \times \mathcal{U}_0^2(\overline{\Omega})$.

Proof: We denote by D_3F the partial derivative of F with respect to the third factor. By Lemma 3.14, for $(p,g) \in E \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$, for $h \in C_0^{\lambda+2}(\overline{\Omega})$ and for $x \in \overline{\Omega}$:

$$D_2\mathcal{F}(p,g)(h)(x) = D_3F(p,x,J^2g(x))(J^2h(x)) = DF_{p,x}(J^2g(x))(J^2h(x)).$$

Let $\sigma_2(D_2\mathcal{F}(p,g))(x)$ be the principal symbol of the operator $D_2\mathcal{F}(p,g)$ at the point x (c.f. [10]). If we denote by $D_3F_{p,x}$ the partial derivative of $F_{p,x}$ with respect to the third component, then, for all $\xi \in \mathbb{R}^n$:

$$\sigma_2(D_2\mathcal{F}(p,g))(x)(\xi,\xi) = (D_2F_{p,x}(J^2g(x)))^{ij}\xi_i\xi_j.$$

By definition of ellipticity for F, $(D_2F_{p,x}(J^2g(x)))$ is positive definite. The principal symbol of $D_2\mathcal{F}(p,g)(x)$ is therefore also positive definite, and since $x \in \overline{\Omega}$ is arbitrary, we deduce that $D_2\mathcal{F}(p,g)$ is an elliptic operator as desired. □

Observe that the reasoning of the previous chapter only applies to solutions which are already known to be smooth. In particular, it does not apply to arbitrary solutions which are only known to be Hölder differentiable of some finite order. However, the following regularity result, derived by inductively applying the classical Schauder estimates to difference quotients (c.f. [11]) allows us to also apply these estimates to solutions in $C^{2+\lambda}$ for all $\lambda \in]0,\infty[$.

Theorem 3.16, Regularity

Suppose that F is elliptic. Choose $\lambda > 0$ and let g be a function in $\mathcal{U}^{2+\lambda}(\overline{\Omega})$. If F is elliptic and if $F(x,J^2g) = 0$, then g is a smooth function.

In order to apply the degree theory described in Sections 3.1, 3.2 and 3.3, we require that $\mathcal{F}$ be a Fredholm mapping. This follows from classical elliptic theory (c.f. [11]):

Lemma 3.17

If $\mathcal{F}$ is elliptic, then for all $\lambda \notin \mathbb{N}$, $\mathcal{F}$ defines a Fredholm mapping from $E \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$. Moreover, $\mathrm{Ind}(\mathcal{F}) = \mathrm{Dim}(E)$.

Remark: *Firstly, we observe that $\mathcal{F}$ is only Fredholm for non-integer values of λ. This is a very important limitation of elliptic theory in Hölder spaces. Secondly, we draw the reader's attention to the fact that the calculation of the index follows from general considerations. Indeed, if an elliptic operator sends sections of a bundle E_1 into sections of a bundle E_2, then the index of the operator only depends on the topology of $\mathrm{Hom}(E_1,E_2)$. In particular, in the case at hand, we may show that any elliptic operator sending $C_0^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$ has index 0. We refer the interested reader to [1], [11] and [18] for more details.*

Proof: Let $D_1\mathcal{F}$ and $D_2\mathcal{F}$ be the partial derivatives of $\mathcal{F}$ with respect to the first and second components respectively. By Lemma 3.15, for all $(p,g) \in E \times \mathcal{U}_0^{\lambda+2}$, $D_2\mathcal{F}(p,g)$ is elliptic. By classical elliptic theory, for all such (p,g), $D_2\mathcal{F}(p,g)$ is a Fredholm operator from $C_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$. Since $D_2\mathcal{F}(p,g)$ acts on real valued functions, $\mathrm{Ind}(D_2\mathcal{F}(p,g)) = 0$. Let $\pi_1 : E \times C_0^{\lambda+2}(\overline{\Omega})$ and $\pi_2 : E \times C_0^{\lambda+2}(\overline{\Omega})$ be the canonical projections onto the first and second factors respectively. Trivially, π_2 is Fredholm of index $\mathrm{Dim}(E)$. Since the composition of two Fredholm operators is Fredholm, $D_2\mathcal{F}(p,g)\circ\pi_2$ is also Fredholm, and since the index of the composition of two Fredholm operators is equal to the sum of their indices, $\mathrm{Ind}(D_2\mathcal{F}(p,g)\circ\pi_2) = \mathrm{Dim}(E)$. Since E is finite dimension, π_1 has finite rank, and therefore so too does $D_1\mathcal{F}(p,g)\circ\pi_1$. Since the sum of a Fredholm operator and a finite rank operator is also a Fredholm operator of the same index, it follows that $D\mathcal{F}(p,g) = D_1\mathcal{F}(p,g)\circ\pi_1 + D_2\mathcal{F}(p,g)\circ\pi_2$ is also Fredholm of Fredholm index equal to $\mathrm{Dim}(E)$, as desired. □

3.6 Existence.

We now recall the construction of Section 2. Let $\Gamma \subseteq \mathrm{Symm}(2,\mathbb{R}^n)$ be the open cone of positive-definite, symmetric matrices, and denote $U = (\oplus_{i=0}^{1}\mathrm{Symm}(i,\mathbb{R}^n))\times\Gamma$. Let $G : \mathbb{R}^n \to \mathbb{R}$ be a smooth, convex function bounded below by 1. Let $\phi \in C^\infty([0,1]\times\overline{\Omega},]0,\infty[)$ be a smooth family of smooth, positive functions over $\overline{\Omega}$. Let E be a finite dimensional subspace of $C^\infty(\overline{\Omega})$ and let $r > 0$ be a positive number, both of which we will chose presently (c.f. Lemma 3.21). We denote by B_r the ball of radius r about 0 in E and we define $F : [0,1]\times B_r\times\overline{\Omega}\times U \to \mathbb{R}$ by:

$$F(s,g,x,(t,\xi,A)) = \mathrm{Det}(A) - (\phi_s(x) + g(x))(sG(\xi) + (1-s)).$$

Observe that, by compactness, there exists $r > 0$ such that for all $(s,g) \in [0,1]\times B_r$, $\phi_s + g > 0$.

Lemma 3.18

F is elliptic.

Proof: This follows from Lemma 2.5. □

For all $\lambda \notin \mathbb{N}$, we define the mapping $\mathcal{F} : [0,1]\times B_r\times\mathcal{U}_0^{\lambda+2}(\overline{\Omega}) \to C^\lambda(\overline{\Omega})$ by:

$$\mathcal{F}(s,g,h)(x) = F(s,g,x,J^2h(x)).$$

Lemma 3.19

$\mathcal{F}$ defines a smooth Fredholm mapping from $[0,1]\times B_r\times\mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$. Moreover, $\mathrm{Ind}(\mathcal{F}) = \mathrm{Dim}(E)$.

Proof: Choose $\lambda \notin \mathbb{N}$. By Lemma 3.14, $\mathcal{F}$ defines a smooth mapping from $[0,1]\times B_r\times\mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$. By Lemma 3.18, F is elliptic. Thus, by Lemma 3.15, $\mathcal{F}$ is elliptic and so, by Lemma 3.17, $\mathcal{F}$ defines a Fredholm mapping of Fredholm index equal to $\mathrm{Dim}(E)+1$, as desired. □

We now apply the degree theory of Section 3.3 to this mapping. We define the **solution space** $\mathcal{Z} \subseteq [0,1]\times B_r\times\mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ by:

$$\mathcal{Z} = \{(s,g,f) \mid \mathcal{F}(s,g,f) = 0\}.$$

Let $\Pi : [0,1] \times B_r \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega}) \to [0,1] \times B_r$ be the projection into the first factor, and let $\Pi_{\mathcal{Z}}$ be the restriction of Π to $\mathcal{Z}$. Let $\hat{f} \in C^\infty([0,1] \times \overline{\Omega})$ be a smooth family of strictly convex functions such that, for all s, $\hat{f}_s(x) = 0$ for all $x \in \partial\Omega$ and:

$$\mathcal{F}(s, 0, \hat{f}_s) > 0.$$

Observe that, by compactness, upon reducing r if necessary, we may suppose that for all $(s,g) \in [0,1] \times B_r$:

$$\mathcal{F}(s, g, \hat{f}_s) > 0.$$

We show that $\Pi_{\mathcal{Z}}$ is a proper mapping:

Lemma 3.20

Suppose that for all $(s,g) \in [0,1] \times B_r$, $\phi_s - g > 0$ and $\mathcal{F}(s,g,\hat{f}_s) > 0$. Then $\Pi_{\mathcal{Z}}$ is a proper mapping.

Proof: Let $(s_m, g_m)_{m\in\mathbb{N}}$ be sequence in $[0,1] \times B_r$ converging to a limit (s_∞, g_∞) in $[0,1] \times B_r$. Let $(f_m)_{m\in\mathbb{N}}$ be a sequence in $\mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ such that for all m, $(s_m, g_m, f_m) \in \mathcal{Z}$. Since F is elliptic, by Theorem 3.16, for all m, f_m is smooth. By Lemma 2.12, for all m, $f_m \geqslant \hat{f}_m$. Thus, by Theorem 2.30, there exists $f_\infty \in C^\infty(\overline{\Omega}) \subseteq C_0^{\lambda+2}(\overline{\Omega})$ towards which $(f_m)_{m\in\mathbb{N}}$ subconverges in the C^∞ sense. Since $\mathcal{F}$ is continuous, in particular, $\mathcal{F}(s_\infty, g_\infty, f_\infty) = 0$. It remains to show that $f_\infty \in \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$, that is, that f_∞ is strictly convex. Indeed, suppose the contrary. Since f_∞ is a limit of a sequence of convex functions, it is convex. Since it is not strictly convex, there exists a point $x \in \overline{\Omega}$ at which $D^2 f_\infty$ is degenerate. However, at this point $\mathrm{Det}(D^2 f_\infty(x)) = 0$, and so $F(s_\infty, g_\infty, x, J^2 f_\infty(x)) < 0$. This is absurd, and it follows that $f_\infty \in \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ as asserted. In particular, $(s_\infty, g_\infty, f_\infty) \in \mathcal{Z}$ and compactness follows. □

Lemma 3.21

There exists a finite dimensional subspace $E \subseteq C^\infty(\overline{\Omega})$ and $r > 0$ such that for all $(s,g,f) \in \mathcal{Z}$, $D\mathcal{F}(s,g,f)$ is surjective.

Proof: Choose $\lambda \notin \mathbb{N}$. We define $\mathcal{F}_0 : [0,1] \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega}) \to C^\lambda(\overline{\Omega})$ by:

$$\mathcal{F}_0(s,f)(x) = F(s, 0, x, J^2 h(x)) = \mathrm{Det}(D^2 f(x)) - \phi_s(x)(sG(Df(x)) + (1-s)).$$

Choose $(s_1, f_1) \in [0,1] \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ such that $\mathcal{F}_0(s_1, f_1) = 0$. By Lemma 3.19, $D\mathcal{F}_0(s_1, f_1)$ is a Fredholm operator. In particular, its cokernel is finite dimensional. Let E_1 be the dual space to $\mathrm{Im}(D\mathcal{F}_0(s_1, f_1))$ in $C^\lambda(\overline{\Omega})$ with respect to the L^2 norm. Observe that E_1 is finite dimensional and:

$$C^\lambda(\overline{\Omega}) = \mathrm{Im}(D\mathcal{F}_0(s_1, f_1)) \oplus E_1.$$

Since $C^\infty(\overline{\Omega})$ is a dense as a subset of $C^\lambda(\overline{\Omega})$ with respect to the L^2 norm, we may perturb E_1 to a subset E_1' of $C^\infty(\overline{\Omega})$ such that $C^\lambda(\overline{\Omega}) = \mathrm{Im}(D\mathcal{F}_0(s_1, f_1)) \oplus E_1'$. Since surjectivity of Fredholm mappings is an open property, there exists a neighbourhood U_1 of (s_1, f_1) in $[0,1] \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ such that for all $(s,f) \in U$:

$$C^\lambda(\overline{\Omega}) = \mathrm{Im}(D\mathcal{F}_0(s_1, f_1)) \oplus E_1.$$

By Lemma 3.20, there exist finitely many points $(s_i, f_i)_{1\leqslant i\leqslant n}$ in $[0,1]\times\mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ such that $\mathcal{F}_0^{-1}(\{0\})$ is contained in the union of the collection $(U_i)_{1\leqslant i\leqslant n}$. We therefore choose $E = E_1' + ... + E_n'$. For all $(s,f)\in\mathcal{F}_0^{-1}(\{0\})$, we obtain:

$$C^\lambda(\overline{\Omega}) = \mathrm{Im}(D\mathcal{F}_0(s_1,f_1))\oplus E\subseteq\mathrm{Im}(D\mathcal{F}(s_1,0,f_1).$$

Since surjectivity of Fredholm mappings is an open property, by Lemma 3.20 again, there exists $r>0$ such that if $g\in B_r$ and if $(s,g,f)\in\mathcal{Z}_r$, then $D\mathcal{F}(s,g,f)$ is surjective as desired. □

Theorem 1.4

Let $\overline{\Omega}$ be a compact, convex subset of $\mathbb{R}^n$ with smooth boundary and non-trivial interior. Let $\phi\in C^\infty(\overline{\Omega})$ be a smooth, positive function. If there exists a strictly convex function $\hat{f}\in C_0^\infty(\overline{\Omega})$ such that:

$$F(D^2\hat{f})/G(D\hat{f}) > \phi,\qquad \hat{f}|_{\partial\Omega} = 0,$$

then there exists a unique strictly convex function $f\in C_0^\infty(\overline{\Omega})$ such that:

$$F(D^2f)/G(Df) = \phi,\qquad f|_{\partial\Omega} = 0.$$

Proof: We first prove uniqueness. Let $f,f'\in C_0^\infty(\overline{\Omega})$ be strictly convex functions such that:

$$F(D^2f)/F(Df) = F(D^2f')/G(Df') = \phi,\qquad f|_{\partial\Omega} = f'|_{\partial\Omega} = 0.$$

It follows from Lemma 2.12 that both $f-f'$ and $f'-f$ attain their minimum values along the boundary. In particular, $f=f'$ and uniqueness follows.

For all $t\in[0,1]$, we denote $G_t = tG + (1-t)$. Choose $\alpha\in]0,1[$. Since $\hat{f}$ is strictly convex, so is $\alpha\hat{f}$. We denote $\phi_0 = F(D^2(\alpha f))$. Observe that $D^2\hat{f} > D^2(\alpha\hat{f})$, and so $F(D^2\hat{f}) > F(D^2(\alpha\hat{f})) = \phi_0$. By continuity, there exists $\delta_0>0$ such that $F(D^2\hat{f})/G_t(D\hat{f}) > (1-t/\delta_0)\phi_0$. Likewise, there exists $\delta_1>0$ such that $F(D^2\hat{f})/G_t(D\hat{f}) > (1-(1-t)/\delta_1)\phi$. Finally, by compactness, there exists $\epsilon>0$ such that $F(D^2\hat{f})/G_t(D\hat{f}) > \epsilon$ for all $t\in[0,1]$. For all $t\in[0,1]$, we define ϕ_t by:

$$\phi_t = \mathrm{Max}((1-t/\delta_0)\phi_0, (1-(1-t)/\delta_1)\phi, \epsilon).$$

Observe that ϕ_t is strictly positive for all t. Moreover, by construction, $\mathcal{F}(t,0,\hat{f})>0$ for all $t\in[0,1]$. Upon perturbing ϕ_t slightly, we may suppose, moreover, that ϕ_t is smooth. By Lemma 3.21, there exists a finite dimensional subspace $E\subseteq C^\infty(\overline{\Omega})$ and $r>0$ such that $D\mathcal{F}$ is surjective at every point of $\mathcal{Z}$. Upon reducing r further if necessary, we may suppose in addition that $\mathcal{F}(t,g,\hat{f})>0$ for all $(t,g)\in[0,1]\times B_r(0)$.

We define $f_0 = \alpha\hat{f}$. By construction, the function f_0 vanishes along the boundary. Moreover $\mathcal{F}(0,0,f_0)=0$. By uniqueness, it is the only function with these properties. It follows that $\Pi_{\mathcal{Z}}^{-1}(\{(0,0)\}) = \{(0,0,f_0)\}$. We claim that $(0,0)$ is a regular value of $\Pi_{\mathcal{Z}}$. Indeed, let

$D_3\mathcal{F}$ be the partial derivative of $\mathcal{F}$ with respect to the third component. By Lemmas 2.5 and 3.14:

$$D_2\mathcal{F}(0,0,f_0)(g) = \frac{1}{n}F(D^2 f_0)(D^2 f_0^{-1})^{ij}g_{ij}.$$

We claim that $D_2\mathcal{F}(0,0,f_0)$ is invertible. Indeed, it is an elliptic operator. Thus, by classical elliptic theory, it is Fredholm and since it acts on real valued functions, it has index 0. Choose $g \in \mathrm{Ker}(D_2\mathcal{F}(0,0,f_0))$. By the maximum principle, g attains its maximum and minimum values along $\partial\Omega$. Since g is an element of $C_0^{\lambda+2}(\overline{\Omega})$, it follows that $g = 0$. Since $g \in \mathrm{Ker}(D_2\mathcal{F}(0,0,f_0))$ was arbitrary, we deduce that this kernel is trivial. Since the index of $D_2\mathcal{F}(0,0,f_0)$ is equal to 0, if follows that $\mathrm{Coker}(D_2\mathcal{F}(0,0,f_0))$ is also trivial and so $D_2\mathcal{F}(0,0,f_0)$ is invertible as desired.

Now let (t,g) be any vector in $\mathbb{R}\times E$. Since $D_2\mathcal{F}(0,0,f_0)$ is invertible, there exists $h \in C_0^{\lambda+2}(\overline{\Omega})$ such that $D_2\mathcal{F}(0,0,f_0)(h) = -D\mathcal{F}(0,0,f_0)(t,g,0)$. Thus, by linearity, $D\mathcal{F}(0,0,f_0)(t,g,h) = 0$. In particular (t,g,h) is a tangent vector to $\mathcal{Z}$ at $(0,0,f_0)$. However:

$$D\Pi_{\mathcal{Z}}(0,0,f_0)(t,g,h) = D\Pi(0,0,f_0)(t,g,h) = (t,g).$$

Since $(t,g)\in\mathbb{R}\times E$ was arbitrary, and since $(0,0,f_0)$ is the only element of $\Pi_{\mathcal{Z}}^{-1}(\{(0,0)\})$, we conclude that $(0,0)$ is a regular value of $\Pi_{\mathcal{Z}}$, as desired. In particular, the degree of $\Pi_{\mathcal{Z}}$ is equal to 1 modulo 2.

By Theorem 3.7, there exists a sequence $(t_m,g_m)_{m\in\mathbb{N}}$ of regular values of $\Pi_{\mathcal{Z}}$ in $[0,1]\times B_r$ which converges to $(1,0)$. Since the degree of $\Pi_{\mathcal{Z}}$ is nonzero modulo 2, for all m, there exists a strictly convex function $f_m \in C_0^{\lambda+2}(\overline{\Omega})$ such that $\mathcal{F}(t_m,g_m,f_m) = 0$ for all m. By Theorem 3.16, f_m is smooth for all m. By Lemma 2.12, $f_m \geqslant \hat{f}_m$ for all m. Thus, by Theorem 2.30, there exists $f_\infty \in C_0^\infty(\overline{\Omega})$ towards with $(f_m)_\infty$ converges in the C^∞ sense. In particular $f_\infty|_{\partial\Omega} = 0$ and $\mathcal{F}(1,0,f_\infty) = 0$. In other words $f_\infty \in C_0^\infty(\overline{\Omega})$ and:

$$F(D^2 f_\infty)/G(Df_\infty) = \phi,$$

as desired. □

4

Singularities

We study the singularities that may arise in Hausdorff limits of smooth hypersurfaces of constant gaussian curvature. The most important result of this chapter is Theorem 4.29, which says that the singular points of such limits are precisely those points which possess the local geodesic property (which we define in Section 4.5). This result follows from interior a-priori estimates obtained using a technique dating back to Pogorelov (c.f. [15], but see also [5] and [19]). In order to deduce the full geometric consequences of these analytic estimates, it is necessary to perform an in-depth study of the structure of convex sets, and this forms the content of Sections 4.1 to 4.5 inclusive. The first most important result of these sections is Theorem 4.13, which shows how every convex set with non-trivial interior may locally be described as the graph of a convex, Lipschitz continuous function. We interpret this as a strong regularity result for the boundaries of convex sets. Moreover, Theorem 4.13 is very general, and we leave the enthusiastic reader to show how it adapts to convex subsets of any riemannian manifold. The second most important result of these sections lies in Theorems 4.19 and 4.20 which describe the relationship between the local geodesic property and the concept of convex hull. These results often allow us to apply ad-hoc geometric arguments to exclude all singularities and thus obtain smoothness in Theorem 4.29. However, in contrast to Theorem 4.13, these results are deeply tied into the geometry of Euclidean Space, and only extend to convex subsets of spaces of constant sectional curvature.

4.1 The Hausdorff Topology.

Let X and Y be two compact subsets of $\mathbb{R}^{n+1}$, we recall that $d_H(X,Y)$, the **Hausdorff distance** between X and Y, is defined by:

$$d_H(X,Y) = \mathop{\mathrm{Sup}}_{x\in X}\mathop{\mathrm{Inf}}_{y\in Y}\|x-y\| + \mathop{\mathrm{Sup}}_{y\in Y}\mathop{\mathrm{Inf}}_{x\in X}\|x-y\|.$$

We leave the reader to verify that d_H defines a metric on the set of compact subsets of $\mathbb{R}^{n+1}$.

Theorem 4.1

For all $R>0$, the set of compact subsets of $\mathbb{R}^{n+1}$ contained in $\overline{B}_R(0)$ is compact in the Hausdorff topology.

Remark: *We leave the reader to verify that the proof generalises to the set of compact subsets of any given compact metric space.*

Proof: Choose $R>0$ and let $(X_m)_{m\in\mathbb{N}}\subseteq\overline{B}_R(0)$ be a sequence of compact sets. For all $k\in\mathbb{N}$, we denote by $Q_k\subseteq\mathbb{R}^{n+1}$ the closed cube of side length $1/2^k$ based on the origin:

$$Q_k = [0,2^{-k}]^{n+1}.$$

For every vector $\alpha\in\mathbb{Z}^{n+1}$, we then define:

$$Q_{k,\alpha} = Q_k + 2^{-k}\alpha.$$

For all m and for all k, we define $X_{m,k}\subseteq\mathbb{R}^{n+1}$ by:

$$X_{m,k} = \bigcup_{Q_{k,\alpha}\cap X_m\neq\emptyset} Q_{k,\alpha}.$$

For all k, every subsequence of $(X_{m,k})_{m\in\mathbb{N}}$ contains a subsubsequence converging in the Hausdorff sense to a compact limit in $\mathbb{R}^{n+1}$. Thus, by a diagonal argument, after extracting a subsequence, we may suppose that for all k, $(X_{m,k})_{m\in\mathbb{N}}$ converges to a compact limit $X_{\infty,k}$, say.

For all k and for all m, $X_{m,k+1}\subseteq X_{m,k}$. Taking limits, it follows that for all k, $X_{\infty,k+1}\subseteq X_{\infty,k}$. We thus denote:

$$X_\infty = \bigcap_{k\in\mathbb{N}} X_{\infty,k}.$$

Since X_∞ is the intersection of a countable, nested family of non-empty compact sets, it is also a non-empty compact set. It remains to show that $(X_m)_{m\in\mathbb{N}}$ converges to X_∞ in the Hausdorff sense. Indeed, for all k and for all m:

$$d_H(X_m,X_{m,k})\leqslant 2^{-k}\sqrt{n+1}.$$

Thus, by the triangle inequality, for all $k\leqslant l$ and for all m:

$$d_H(X_{m,k},X_{m,l})\leqslant 2^{1-k}\sqrt{n+1}.$$

Taking limits yields, for all $k\leqslant l$:

$$d_H(X_{\infty,k},X_{\infty,l})\leqslant 2^{1-k}\sqrt{n+1}.$$

Letting l tend to infinity yields, for all k:

$$d_H(X_{\infty,k}, X_\infty) \leqslant 2^{1-k}\sqrt{n+1}.$$

Now choose $\delta > 0$ and $k > 0$ such that $2^{-k}\sqrt{n+1} < 2^{1-k}\sqrt{n+1} < \delta/3$ and choose $M > 0$ such that for $m \geqslant M$, $d_H(X_{m,k}, X_{\infty,k}) < \delta/3$. Then, for $m \geqslant M$:

$$d_H(X_m, X_\infty) \leqslant d_H(X_m, X_{m,k}) + d_H(X_{m,k}, X_{\infty,k}) + d_H(X_{\infty,k}, X_\infty) < \delta.$$

Since δ may be chosen arbitrarily small, it follows that $(X_m)_{m\in\mathbb{N}}$ converges to X_∞ in the Hausdorff sense as desired. □

Lemma 4.2

For all $R > 0$, the set of closed, convex subsets of $\overline{B}_R(0)$ is a closed subset of the set of compact subsets of $\overline{B}_R(0)$ with respect to the Hausdorff topology.

Proof: Let $(K_m)_{m\in\mathbb{N}}$ be a sequence of compact, convex subsets of $\overline{B}_R(0)$ converging to a compact limit $K_\infty \subseteq \overline{B}_R(0)$ in the Hausdorff sense. Choose two points $x_\infty, y_\infty \in K_\infty$. Since $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense, there exist sequences $(x_m)_{m\in\mathbb{N}}$ and $(y_m)_{m\in\mathbb{N}}$ converging to x_∞ and y_∞ such that for all m, the points x_m and y_m are elements of K_m. Choose $t \in [0,1]$. For all m, since K_m is convex, $(1-t)x_m + ty_m \in K_m$. Taking limits, it follows that $(1-t)x_\infty + ty_\infty \in K_\infty$. Since $x_\infty, y_\infty \in K_\infty$ and $t \in [0,1]$ are arbitary, we deduce that K_∞ is convex as desired. □

4.2 Supporting Normals.

Let $\Sigma^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere. We recall that if K is a convex subset of $\mathbb{R}^{n+1}$, if x is a point of K, and if N is a vector in Σ^n, then N is said to be a **supporting normal** to K at x whenever every $y \in K$ satisfies:

$$\langle y - x, \mathsf{N}\rangle \leqslant 0.$$

Observe that supporting normals can only exist at boundary points of K. Supporting normals, when they exist, need not be unique. Hence, for any boundary point x of K, we denote by $\mathcal{N}(x;K) \subseteq \Sigma^n$ the set of supporting normals to K at x, and when there is no ambiguity, we denote $\mathcal{N}(x;K)$ merely by $\mathcal{N}(x)$. We show that for any boundary point x of K, $\mathcal{N}(x)$ is non-empty and compact. Moreover, we show that $\mathcal{N}(x)$ varies semi-continuously with x in a sense that will be made clear presently.

Lemma 4.3

Let $(K_m)_{m\in\mathbb{N}}$ and K_∞ be compact, convex subsets of $\mathbb{R}^{n+1}$ such that $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense. For all finite m, let x_m be a boundary point of K_m and let N_m be a supporting normal to K_m at x_m. Suppose, moreover, that $(x_m)_{m\in\mathbb{N}}$ and $(\mathsf{N}_m)_{m\in\mathbb{N}}$ converge to x_∞ and N_∞ respectively. Then x_∞ is a boundary point of K_∞ and N_∞ is a supporting normal to K_∞ at x_∞.

Proof: Indeed, choose $y \in K_\infty$. Since $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense, there exists a sequence $(y_m)_{m\in\mathbb{N}}$ in $\mathbb{R}^{n+1}$ converging to y such that y_m is an element of K_m for all m. For all m, since N_m is a supporting normal to K_m at x_m:

$$\langle y_m - x_m, \mathsf{N}_m\rangle \leqslant 0.$$

Taking limits yields:

$$\langle y - x_\infty, \mathsf{N}_\infty\rangle \leqslant 0.$$

Since $y \in K_\infty$ is arbitrary, we deduce that $x_\infty \in \partial K_\infty$ and that N_∞ is a supporting normal to K_∞ at x_∞ as desired. $\square$

This yields compactness of the set of supporting normals at any point:

Corollary 4.4

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. If x is a boundary point of K then $\mathcal{N}(x)$ is compact.

The supporting normals characterise the closest point to any exterior point of a given convex set in the following sense:

Lemma 4.5

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let x be a point in the complement of K. Then $y \in K$ minimises distance to x if and only if y is a boundary point of K and $(x-y)/\|x-y\|$ is a supporting normal to K at y.

Proof: Suppose that y is a boundary point and $(x-y)/\|x-y\|$ is a supporting normal to K at y. Choose $z \in K$. By definition of supporting normals, $\langle z-y, x-y\rangle \leqslant 0$. Thus:

$$\begin{aligned}\|z-x\|^2 &= \|(z-y)-(x-y)\|^2\\ &= \|z-y\|^2 - 2\langle z-y, x-y\rangle + \|y-x\|^2\\ &\geqslant \|z-y\|^2 + \|y-x\|^2.\end{aligned}$$

In particular, $\|z-x\|^2 \geqslant \|y-x\|^2$. Taking square roots yields $\|z-x\| \geqslant \|y-x\|$. Since $z \in K$ was arbitrary, it follows that y minimises distance to x in K as desired. Conversely, suppose that $y \in K$ minimises distance to x. We claim that $\langle z-y, x-y\rangle \leqslant 0$ for all $z \in K$. Indeed, suppose the contrary. There exists $z \in K$ such that $\langle z-y, x-y\rangle > 0$. For all $t \in [0,1]$, denote $z_t = (1-t)y + tz$. By convexity, $z_t \in K$ for all t. Moreover:

$$\partial_t\|z_t - x\|^2|_{t=0} = 2\langle z-y, y-x\rangle < 0.$$

Thus, for sufficiently small t, $\|z_t - x\|^2 < \|y-x\|^2$. This is absurd by minimality of y, and the assertion follows. We conclude that y is a boundary point of K and $(x-y)/\|x-y\|$ is a supporting normal to K at y as desired. $\square$

This yields existence of supporting normals at any point:

Lemma 4.6

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. For every boundary point x of K there exists a supporting normal N to K at x.

Proof: Indeed, choose $x \in \partial K$. Since x is a boundary point, there exists a sequence $(x_m)_{m\in\mathbb{N}}$ of points in the complement of K converging to x. Since K is compact, for all

m, there exists a point y_m in K minimising the distance to x_m. In particular, for all m, $d(x_m, y_m) \leqslant d(x_m, x)$ and so:

$$d(x, y_m) \leqslant d(x, x_m) + d(x_m, y_m) \leqslant 2d(x_m, x),$$

and it follows that $(y_m)_{m\in\mathbb{N}}$ also converges to x. For all m, we define $\mathsf{N}_m \in \Sigma^n$ by:

$$\mathsf{N}_m = \frac{x_m - y_m}{\|x_m - y_m\|}.$$

By Lemma 4.5 for all m, y_m is a boundary point of K and N_m is a supporting normal to K at y_m. After extracting a subsequence we may suppose that $(\mathsf{N}_m)_{m\in\mathbb{N}}$ converges to an element N in Σ^n. By Lemma 4.3, N is a supporting normal to K at x as desired. □

Corollary 4.7

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. If x is a boundary point of K, then $\mathcal{N}(x) \neq \emptyset$.

We now examine semi-continuity of $\mathcal{N}(x)$. For two non-empty subsets X and Y of Σ^n, we define $\delta(X, Y)$ by:

$$\delta(X, Y) = \mathop{\mathrm{Sup}}_{y\in Y} \mathop{\mathrm{Inf}}_{x\in X} d_\Sigma(x, y).$$

Observe, in particular, that δ is not symmetric. Moreover, by definition:

$$d_{H,\Sigma}(X, Y) = \delta(X, Y) + \delta(Y, X) \geqslant \delta(X, Y).$$

In other words, δ is weaker than the Hausdorff distance of the sphere.

Lemma 4.8

Let $(K_m)_{m\in\mathbb{N}}$ and K_∞ be compact, convex subsets of $\mathbb{R}^{n+1}$ such that $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense. For all m let x_m be a boundary point of K_m. For all $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that for $m \geqslant M$, $\delta(\mathcal{N}(x_\infty), \mathcal{N}(x_m)) < \epsilon$.

Proof: Suppose the contrary. Upon extracting a subsequence, we may suppose that there exists $\epsilon > 0$ such that $\delta(\mathcal{N}(x_\infty), \mathcal{N}(x_m)) \geqslant \epsilon$ for all m. For all m, there therefore exists $\mathsf{N}_m \in \mathcal{N}(x_m)$ such that $d(\mathcal{N}(x_\infty), \mathsf{N}_m) \geqslant \epsilon$. Upon extracting a further subsequence, we may suppose that there exists N_∞ towards which $(\mathsf{N}_m)_{m\in\mathbb{N}}$ converges. By Lemma 4.3, $\mathsf{N}_\infty \in \mathcal{N}(x_\infty)$. However, upon taking limits $d(\mathcal{N}(x_\infty), \mathsf{N}_\infty) \geqslant \epsilon$. This is absurd, and the result follows. □

Finally, we show that $\mathcal{N}(x)$ is in fact defined locally. Indeed, recall that if K and L are compact, convex sets, then so is their intersection.

Lemma 4.9

Let K and L be compact, convex subsets of $\mathbb{R}^{n+1}$. Let x be a boundary point of $K \cap L$ and let $\mathsf{N} \in \Sigma^n$ be a supporting normal to $K \cap L$ at x. If $x \in L^o$, then x is also a boundary point of K and N is also a supporting normal to K at x.

Proof: It suffices to show that for all $y \in K$:

$$\langle y - x, \mathsf{N} \rangle \leqslant 0.$$

Suppose the contrary. There exists $y \in K$ such that $\langle y - x, \mathsf{N} \rangle > 0$. For all $t \in [0, 1]$, denote $y_t = (1 - t)x + ty$. For all t:

$$\langle y_t - x, \mathsf{N} \rangle = t \langle y - x, \mathsf{N} \rangle > 0.$$

By convexity, y_t is an element of K for all t. Since x is an interior point of L, for sufficiently small t, y_t is also an element of L and is therefore an element of $K \cap L$. Thus, by definition of supporting normals, for sufficiently small t, $\langle y_t - x, \mathsf{N} \rangle \leqslant 0$. This is absurd, and the assertion follows. □

Corollary 4.10

Let K and L be compact, convex subsets of $\mathbb{R}^{n+1}$. Let x be a boundary point of $K \cap L$ and let $\mathsf{N} \in \Sigma^n$ be a supporting normal to $K \cap L$ at x. If $x \in L^o$, then $\mathcal{N}(x; K) = \mathcal{N}(x; L)$.

The following result allows us to include other types of convex subsets of $\mathbb{R}^{n+1}$ into the current framework:

Lemma 4.11

Let K be a closed, convex subset of $\mathbb{R}^{n+1}$. For every point $x \in \mathbb{R}^{n+1}$, there exists a point $y \in K$ minimising distance to x. Moreover, y is a boundary point of K and $\mathsf{N} = (x - y)/\|x - y\|$ is a supporting normal to K at y.

Proof: For all $r > 0$, we denote $K_r = K \cap \overline{B}_r(x)$. Observe that for all r, K_r is compact and convex. There exists r_0 such that K_{r_0} is non-empty. Since it is compact, there exists $y \in K_{r_0}$ minimising distance in K_{r_0} to x. We denote $\mathsf{N} = (x - y)/\|x - y\|$. Observe that if $z \in K \setminus K_{r_0}$, then $\|z - x\| > r_0 \geqslant \|y - x\|$, and it follows that y also minimises distance in K to x. In particular, for all $s > r_0$, y also minimises distance in K_s to x. Thus, by Lemma 4.5, N is a supporting normal to K_s at x. Choose $z \in K$. There exists $s > 0$ such that $z \in K_s$. By definition of supporting normals, $\langle z - y, \mathsf{N} \rangle \leqslant 0$. Since $z \in K$ is arbitrary, we conclude that N is a supporting normal to K at y, and, in particular, y is a boundary point of K as desired. □

4.3 Convex Sets as Graphs.

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let x be a boundary point of K and let N be a supporting normal to K at x. Upon applying an affine isometry, we may suppose that $x = 0$ and that N is any given unit vector in the sphere. The results which follow are therefore completely general. We decompose $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \mathbb{R}$. For $C, r > 0$, we say that ∂K is a C-Lipschitz **graph** over a radius r near 0 whenever there exists a C-Lipschitz function $f : B'_r(0) \to]-2Cr, 2Cr[$ such that the intersection of ∂K with $B'_r(0) \times]-2Cr, 2Cr[$ coincides with the graph of f over $B'_r(0)$. We refer to f as the **graph function** of K near 0.

Lemma 4.12

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Suppose that 0 is a boundary point of K and that every supporting normal N to K at 0 satisfies $\langle \mathsf{N}, e_{n+1}\rangle < 0$. Choose $C, r > 0$ and suppose that ∂K is a C-Lipschitz graph over a radius r near 0. If $f : B_r'(0) \to]-2Cr, 2Cr[$ is the graph function of ∂K near 0, then f is convex.

Proof: We define $\hat{K}$ by:

$$\hat{K} = \{(x', t) \in B_r'(0)\times]-2Cr, 2Cr[\mid t \geqslant f(x')\}.$$

We claim that $\hat{K}$ coincides with $K\cap(B_r'(0)\times]-2Cr, 2Cr[)$. Indeed, for all $x' \in B_r'(0)$, denote $L_{x'} = \{x'\}\times]-2Cr, 2Cr[$ and for any subset X of $L_{x'}$, denote by $\partial^r X$ the relative boundary of X as a subset of $L_{x'}$. For all x', $L_{x'}\cap K$ is convex. In particular, it is connected and thus consists of a single, relatively closed subinterval of $L_{x'}$. However, $\partial^r(L_{x'}\cap K)$ is contained within $L_{x'}\cap(\partial K)$, and so, by definition of f, it only consists of a single point, being $(x', f(x'))$. Thus, $L_{x'}\cap K$ coincides either with the relatively closed subinterval of $L_{x'}$ lying above the point $(x', f(x'))$ or with the relatively closed subinterval lying below that point.

We now show that $L_{x'}\cap K$ does not coincide with the relatively closed subinterval of L_x lying below the point $(x', f(x'))$. Observe that, since f is C-Lipschitz, for all $x' \in B_r'(0)$:

$$|f(x')| = |f(x') - f(0)| \leqslant C\|x'\| < Cr.$$

It thus suffices to show that $(x', -Cr) \in \mathbb{R}^{n+1}\setminus K$ for all $x' \in B_r'(0)$. When $x' = 0$, by hypothesis:

$$\langle (0, -Cr) - (0,0), \mathsf{N}\rangle = (-Cr)\langle e_{n+1}, \mathsf{N}\rangle > 0,$$

By definition of supporting normals, it follows that $(0, -Cr)$ lies in $\mathbb{R}^{n+1}\setminus K$ as desired. Now choose $x' \in B_r'(0)$. Let I be the set of all points $t \in [0,1]$ such that $(tx', -Cr) \in \mathbb{R}^{n+1}\setminus K$. Since I contains 0, it is non-empty. Since $\mathbb{R}^{n+1}\setminus K$ is open, I is an open subset of $[0,1]$. We claim that I is also a closed subset of $[0,1]$. Indeed, if $(tx', -Cr) \in K$, then, by hypothesis, since $f(x') > -Cr$, this point cannot be a boundary point of K, and it is therefore an interior point. It follows that $[0,1]\setminus I$ is also an open subset of $[0,1]$ and I is therefore a closed subset of $[0,1]$ as asserted. It follows by connectivity that $I = [0,1]$. In particular $(x', -Cr) \in \mathbb{R}^{n+1}\setminus K$, as desired. We conclude that for all $x' \in B_r'(0)$, $K\cap L_{x'}$ coincides with the subinterval of $L_{x'}$ lying above $(x', f(x'))$, and so:

$$K\cap L_{x'} = \{(x', t) \mid f(x') \leqslant t < 2Cr\}.$$

Thus $\hat{K} = K\cap(B_r'(0)\times]-2Cr, 2Cr[)$ as asserted. In particular, since it is the intersection of two convex sets, $\hat{K}$ is convex.

Choose $x', y' \in B_r'(0)$. By definition, $(x', f(x')), (y', f(y')) \in \hat{K}$. For all $s \in [0,1]$, we denote $x_s' = (1-s)x' + sy'$ and $t_s = (1-s)f(x') + sf(y')$. By convexity, $(x_s', t_s) \in \hat{K}$ for all s. It follows by definition of $\hat{K}$ that $f(x_s) \leqslant t_s$ for all s, and so:

$$f((1-s)x' + sy') \leqslant (1-s)f(x') + sf(y').$$

Since $x', y' \in B_r(0)$ and $s \in [0,1]$ are arbitrary, it follows that f is convex as desired. $\square$

Theorem 4.13

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$ and suppose that 0 is a boundary point of K. Choose $\theta \in [0, \pi/2[$ and $r > 0$ and suppose that for all $x \in \partial K \cap B_r(0)$ and for every supporting normal N to K at x:

$$\langle \mathsf{N}, -e_{n+1} \rangle \geqslant \cos(\theta).$$

Then, denoting $C = \tan(\theta)$ and $\rho = \frac{r}{\sqrt{1+4C^2}}$, there exists a function $f : B'_\rho(0) \to]-C\rho, C\rho[$ such that:

(1) $f(0) = 0$;

(2) f is convex and C-Lipschitz; and

(3) $(\partial K) \cap (B'_\rho(0) \times]-2C\rho, 2C\rho[)$ coincides with the graph of f.

Remark: *In other words, f is a C-Lipschitz graph over a radius r near 0.*

Proof: Choose $x, y \in (\partial K) \cap B_r(0)$. Decomposing $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, we denote $x = (x', s)$ and $y = (y', t)$. Let N be the supporting normal to K at x and let N' be its orthogonal projection onto $\mathbb{R}^n$. In particular:

$$\|\mathsf{N}'\|^2 = 1 - \langle \mathsf{N}, e_{n+1} \rangle^2 \leqslant 1 - \cos^2(\theta) = \sin^2(\theta).$$

Bearing in mind the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned} \langle y - x, \mathsf{N} \rangle &= \langle y - x, \langle \mathsf{N}, e_{n+1} \rangle e_{n+1} \rangle + \langle y - x, \mathsf{N}' \rangle \\ &= \langle \mathsf{N}, e_{n+1} \rangle (t - s) + \langle y' - x', \mathsf{N}' \rangle \\ &\geqslant \langle \mathsf{N}, e_{n+1} \rangle (t - s) - \|y' - x'\| \sin(\theta) \\ &= \langle \mathsf{N}, -e_{n+1} \rangle (s - t) - \|y' - x'\| \sin(\theta). \end{aligned}$$

However, by definition of the supporting normal, $\langle y - x, \mathsf{N} \rangle \leqslant 0$, and so:

$$(s - t) \leqslant \|y' - x'\| \frac{\sin(\theta)}{\langle \mathsf{N}, -e_{n+1} \rangle} \leqslant \tan(\theta) \|y' - x'\|.$$

By symmetry, we conclude that:

$$|s - t| \leqslant \tan(\theta) \|y' - x'\| = C \|y' - x'\|. \tag{B}$$

Observe that $B'_\rho(0) \times]-2C\rho, 2C\rho[\subseteq B_r(0)$. We claim that for all $x' \in B'_\rho(0)$, there exists a unique $t \in]-2C\rho, 2C\rho[$ such that $(x', t) \in \partial K$. We first prove uniqueness. Indeed, suppose there exists $s \neq t \in]-2C\rho, 2C\rho[$ such that $(x', s), (x', t) \in \partial K$. Then, in particular, $(x', s), (x', t) \in B_r(0)$ and so, by (B), $|s - t| = 0$. This is absurd, and uniqueness follows.

We now prove existence. For all $t \in [-2C\rho, 2C\rho]$, we denote $B'_t = B'_\rho(0) \times \{t\}$. Observe that $B'_t \subseteq B_r(0)$ for all t. We show that $B'_{C\rho}$ is contained in the interior of K. First we claim that $B'_{C\rho}$ does not intersect the boundary of K. Indeed, otherwise there exists

$(x', C\rho) \in (\partial K) \cap B_{C\rho} \subseteq (\partial K) \cap B_r(0)$. However, bearing in mind that $(0,0) \in \partial K$, by (B), $C\rho \leqslant C\|x'\| < C\rho$. This is absurd, and the assertion follows. It follows that $B_{C\rho} = (B_{C\rho} \cap K^0) \cup (B_{C\rho} \cap K^c)$ and since it is connected, one of these components must be empty.

We now claim that $(0, C\rho) \in K^0$. Denote $L_0 = \{0\} \times]-2C\rho, 2C\rho[$. Since $K \cap L_0$ is convex, it is connected, and thus consists of a single, relatively closed subinterval. Moreover, by hypothesis, $(0,0)$ is a boundary point of this subinterval. Since the relative boundary of $K \cap L_0$ is contained in $(\partial K) \cap L_0$, if follows by uniqueness that this subinterval has no other boundary point. Thus, either $L_0 = \{0\} \times]-2C\rho, 0]$ or $L_0 = \{0\} \times [0, 2C\rho[$. Suppose that $L_0 = \{0\} \times]-2C\rho, 0]$. In particular, since K is closed, $(0, -C\rho) \in K$. Let N_0 be a supporting normal to K at 0. By hypothesis, $\langle \mathsf{N}_0, -e_{n+1}\rangle \geqslant \cos(\theta)$ and so:

$$\langle (0, -C\rho) - (0,0), \mathsf{N}_0\rangle = 2C\rho\langle -e_{n+1}, \mathsf{N}_0\rangle \geqslant 2C\rho\cos(\theta) > 0.$$

However, by definition of a supporting normal, $\langle (0, -C\rho) - (0,0), \mathsf{N}_0\rangle \leqslant 0$. This is absurd, and it follows that $L_0 = \{0\} \times [0, 2C\rho[$. Moreover, by uniqueness, $(0, C\rho)$ is not a boundary point of K, and so $(0, C\rho) \in K^0$ as asserted.

We conclude that $B_{C\rho}$ is contained in the interior of K. In like manner, we show that $B_{-C\rho}$ is contained in the complement of K. Choose $x' \in B'_\rho(0)$. Since $(x', -C\rho) \notin K$ and $(x', C\rho) \in K^o$, there exists $t \in]-C\rho, C\rho[$ such that $(x', t) \in \partial K$, and existence follows.

There therefore exists a function $f : B'_\rho(0) \to]-C\rho, C\rho[$ such that $(\partial K) \cap (B'_\rho(0) \times]-2C\rho, 2C\rho[)$ coincides with the graph of f. Moreover, for all $x', y' \in B'_\rho(0)$, since the points $(x', f(x'))$ and $(y', f(y'))$ lie in $B_r(0)$, by (B):

$$|f(x') - f(y')| \leqslant C\|x' - y'\|,$$

and it follows that f is C-Lipschitz. Finally, by Lemma 4.12, f is convex, and this completes the proof. □

4.4 Convex Hulls.

Let X be a subset of $\mathbb{R}^{n+1}$. We define the **convex hull** of X to be the intersection of all open, convex subsets of $\mathbb{R}^{n+1}$ containing X. We denote this set by $\mathrm{Conv}(X)$. Observe, in particular, that $\mathrm{Conv}(X)$ is convex.

Lemma 4.14

If K is a convex set, then $\overline{K}$ and K^o are also convex.

Proof: Choose $x, y \in \overline{K}$. Let $(x_m)_{m\in\mathbb{N}}$, $(y_m)_{m\in\mathbb{N}}$ be sequences of points in K converging to x and y respectively. Choose $t \in [0,1]$. By convexity, $(1-t)x_m + ty_m \in K$ for all m. Taking limits, it follows that $(1-t)x + ty \in \overline{K}$. Since $x, y \in \overline{K}$ and $t \in [0,1]$ are arbitrary, it follows that $\overline{K}$ is convex as desired.

Now choose $x, y \in K^o$. Choose $\delta > 0$ such that $B_\delta(x), B_\delta(y) \subseteq K$. Choose $t \in [0,1]$. By convexity, for all $z \in B_\delta(0)$:

$$(1-t)x + ty + z = (1-t)(x+z) + t(y+z) \in K.$$

It follows that $B_\delta((1-t)x + ty) \subseteq K$ and so $(1-t)x + ty \in K^o$. Since $x, y \in K^o$ and $t \in [0,1]$ are arbitrary, it follows that K^o is convex as desired. □

Lemma 4.15

If X is compact, then $\mathrm{Conv}(X)$ *is compact.*

Proof: Since X is bounded, there exists $R > 0$ such that $X \subseteq B_R(0)$. Since $B_R(0)$ is also convex and open, by definition, $\mathrm{Conv}(X)$ is a subset of $B_R(0)$ and is therefore bounded. We claim that the complement of $\mathrm{Conv}(X)$ is open. Indeed, choose $y \in \mathbb{R}^{n+1} \setminus \mathrm{Conv}(X)$. There exists a bounded, open, convex set K containing X such that $y \in \mathbb{R}^{n+1} \setminus K$. Upon replacing K by $K \cap B_R(0)$, if necessary, we may suppose that K is bounded. If $y \notin \overline{K}$, then there exists $\delta > 0$ such that $B_\delta(0) \subseteq \mathbb{R}^{n+1} \setminus K \subseteq \mathbb{R}^{n+1} \setminus \mathrm{Conv}(X)$ as desired. Otherwise, $y \in \partial\overline{K}$. By Lemma 4.14, $\overline{K}$ is convex. By Lemma 4.6, there exists a supporting normal N to $\overline{K}$ at y. Since $X \subseteq K \subseteq \overline{K}^o$, by definition of supporting normals, $\langle x - y, \mathsf{N}\rangle < 0$ for all $x \in X$. By compactness, there exists $\epsilon > 0$ such that $\langle x - y, \mathsf{N}\rangle < -\epsilon$ for all $x \in X$. We define:

$$K' = \{x \in K \mid \langle x - y, \mathsf{N}\rangle < -\epsilon\}.$$

K' is open and convex and $X \subseteq K'$. Thus, by definition, $\mathrm{Conv}(X) \subseteq K'$. However, $B_\epsilon(y) \subseteq \mathbb{R}^{n+1} \setminus K' \subseteq \mathbb{R}^{n+1} \setminus \mathrm{Conv}(X)$, and it follows that y lies in the interior of $\mathbb{R}^{n+1}\mathrm{Conv}(X)$. Since $y \in \mathbb{R}^{n+1} \setminus \mathrm{Conv}(X)$ is arbitrary, it follows that $\mathbb{R}^{n+1} \setminus \mathrm{Conv}(X)$ is open as asserted. In particular, $\mathrm{Conv}(X)$ is closed. We conclude by the Heine-Borel Theorem (Theorem 2.41 of [16]) that $\mathrm{Conv}(X)$ is compact as desired. □

Lemma 4.16

If X is open, then $\mathrm{Conv}(X)$ *is open.*

Proof: Choose $x \in \partial\mathrm{Conv}(X)$. For all $r > 0$, we denote $K_r = \overline{B}_r(0) \cap \mathrm{Conv}(X)$. Observe that K_r is convex for all r. Observe that, for all $r > s$, $K_s = K_r \cap \overline{B}_s(0)$. Denote $r_0 = \|x\|$. By Lemma 4.6 there exists a supporting normal N to K_{2r_0} at x. By Lemma 4.9, for all $r > 2r_0$, N is also a supporting normal to K_r at x. Choose $y \in X \subseteq \mathrm{Conv}(X)$. There exists $r > 0$ such that $y \in K_r$. Thus, by definition of supporting normals $\langle y - x, \mathsf{N}\rangle \leqslant 0$. Since X is open, it follows that $\langle y - x, \mathsf{N}\rangle < 0$. Since $y \in X$ is arbitrary, if we define:

$$K' = \left\{y \in \mathbb{R}^{n+1} \mid \langle y - x, \mathsf{N}\rangle < 0\right\},$$

then $X \subseteq K'$. Observe that, in particular, K' is open and convex and $X \subseteq K'$. Thus, by definition, $\mathrm{Conv}(X) \subseteq K'$. However, $x \notin K'$ and so $x \notin \mathrm{Conv}(X)$. Since $x \in \partial\mathrm{Conv}(X)$ is arbitrary, it follows that:

$$\mathrm{Conv}(X) \cap \partial\mathrm{Conv}(X) = \emptyset,$$

and so $\mathrm{Conv}(X)$ is open as desired. □

Lemma 4.17

If K is compact and convex, then $\mathrm{Conv}(K) = K$.

Remark: *Observe that the analogous result for open convex sets follows immediately from our definition of the convex hull.*

Proof: By definition, $K \subseteq \mathrm{Conv}(K)$. Choose $x \in \mathbb{R}^{n+1} \setminus K$. Let $y \in K$ be a point minimising distance to x. Define $\mathsf{N} = (x-y)/\|x-y\|$. By Lemma 4.5, N is a supporting normal to K at y. By definition of supporting normals, for all $z \in K$, $\langle z-y, \mathsf{N}\rangle \leqslant 0$. Thus:

$$\langle z-x, \mathsf{N}\rangle = \langle (z-y)+(y-x), \mathsf{N}\rangle \leqslant -\|x-y\| < 0.$$

We define:

$$K' = \left\{ z \in \mathbb{R}^{n+1} \mid \langle z-x, \mathsf{N}\rangle < 0 \right\}.$$

K' is open and convex and $K \subseteq K'$. Thus $\mathrm{Conv}(K) \subseteq K'$. However, $x \notin K'$, and so $x \in \mathbb{R}^{n+1} \setminus \mathrm{Conv}(K)$. Since $x \in \mathbb{R}^{n+1} \setminus \mathrm{Conv}(K)$ is arbitrary, it follows that $\mathbb{R}^{n+1} \setminus K \subseteq \mathbb{R}^{n+1} \setminus \mathrm{Conv}(K)$. Taking complements yields $\mathrm{Conv}(K) \subseteq K$ and so the two sets coincide as desired. □

4.5 The Local Geodesic Property.

We define an open straight-line segment in $\mathbb{R}^{n+1}$ to be any set Γ of the form:

$$\Gamma = \{x + ty \mid a < t < b\}.$$

where x and y are points in $\mathbb{R}^{n+1}$ and $a < b$ are real numbers. Let K be a compact, convex subset of $\mathbb{R}^{n+1}$ and let x be any point of K. We say that K satisfies the **local geodesic property** at x whenever there exists an open straight-line segment Γ such that $x \in \Gamma \subseteq K$. Every interior point of K trivially satisfies the local geodesic property.

Lemma 4.18

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let x be a boundary point of K and let Γ be an open straight-line segment such that $x \in \Gamma \subseteq K$. If N is a supporting normal to K at x then Γ is contained in the hyperplane passing through x normal to N. In particular, Γ is contained in the boundary of K.

Proof: By definition, there exists $y \in \mathbb{R}^{n+1}$ and real numbers $a < 0 < b$ such that:

$$\Gamma = \{x + ty \mid a < t < b\}.$$

By definition of supporting normals, for all $t \in]a, b[$:

$$t\langle y, \mathsf{N}\rangle = \langle (x+ty) - x, \mathsf{N}\rangle \leqslant 0.$$

It follows that $\langle y, \mathsf{N}\rangle = 0$, and so $\langle (x+ty)-x, \mathsf{N}\rangle = 0$ for all $t \in]a, b[$ as desired. □

The local geodesic property characterises convex hulls in the following sense:

Theorem 4.19

Let K be a compact, convex set. Let X be a closed subset of ∂K. Let Y be the set of all points of ∂K satisfying the local geodesic property. If $X \cup Y$ is closed, then $Y \subseteq \mathrm{Conv}(X)$.

Proof: We prove this by induction on the dimension of the ambient space. First suppose that $n = 1$. Choose $y \in Y$, let H be a supporting line to K at y. We identify H with $\mathbb{R}$ and we denote $K' = K \cap H$, $Y' = Y \cap H$ and $X' = X \cap H$. Observe that $K' = [a, b]$ is a compact interval and $X' \cup Y'$ is closed. We claim that $\{a, b\} \subseteq X'$. We claim first that $\{a, b\} \cap Y = \emptyset$. Indeed, suppose the contrary. Without loss of generality, there exists an open straight-line segment Γ such that $a \in \Gamma \subseteq K$. By Lemma 4.18, $\Gamma \subseteq H$. In particular, $\Gamma \subseteq H \cap K = K' = [a, b]$. This is absurd, since a is an interior point of Γ and it follows that $\{a, b\} \cap Y = \emptyset$ as asserted. However $]a, b[\subseteq Y'$, and, by taking closures, it follows that $[a, b] \subseteq Y' \cup X'$. Thus $\{a, b\} \subseteq X'$ as asserted. Consequently $K' = \mathrm{Conv}(X')$ and so:

$$y \in Y' \subseteq K' \subseteq \mathrm{Conv}(X') \subseteq \mathrm{Conv}(X).$$

Since $y \in Y$ is arbitrary, we conclude that $Y \subseteq \mathrm{Conv}(X)$ as desired.

Now consider an ambient space of arbitrary dimension. Choose $y \in Y$ and let H be a supporting hyperplane to K at y. We denote $K' = K \cap H$, $Y' = Y \cap \partial K'$ and $X' = X \cap \partial K'$. Observe that K' is a compact, convex subset of H, X' is a closed subset of $\partial K'$ and $X' \cup Y'$ is closed. We claim that Y' coincides with the set of all boundary points of K' satisfying the local geodesic property. Indeed, if $z \in Y'$, then there exists an open straight-line segment Γ such that $z \in \Gamma \subseteq K$. Since H is a supporting hyperplane to K, By Lemma 4.18, $\Gamma \subseteq H$. It follows that $\Gamma \subseteq K'$ and so K' also satisfies the local geodesic property at z. Conversely, if z is a boundary point of K' and K' satisfies the local geodesic property at z, then z is also a boundary point of K and K also satisfies the local geodesic property at z. We conclude that these two sets coincide as asserted and it follows by the inductive hypothesis that $Y' \subseteq \mathrm{Conv}(X') \subseteq \mathrm{Conv}(X)$.

If $y \in Y'$, then $y \in \mathrm{Conv}(X)$, and we are done. Otherwise, suppose that $y \in Y \setminus Y'$. In particular, y lies in the relative interior of K'. Let V be any vector in H. Define $\gamma : \mathbb{R} \to H$ by $\gamma(t) = y + tV$. Denote $I = \gamma^{-1}(K')$. Since K' is compact and convex, I is a compact interval. Observe that, by definition, for all $t \in I^o$, K satisfies the local geodesic property at $\gamma(t)$. In other words $I^o \subseteq \gamma^{-1}(Y) \subseteq \gamma^{-1}(Y \cup X)$. Taking closures therefore yields, $I \subseteq \gamma^{-1}(Y \cup X)$. However, $\partial I \subseteq \gamma^{-1}(\partial K')$, and so $\partial I \subseteq \gamma^{-1}(Y' \cup X')$. By the discussion of the previous paragraph, it follows that $\partial I \subseteq \gamma^{-1}(\mathrm{Conv}(X))$ and so, by convexity, $I \subseteq \gamma^{-1}(\mathrm{Conv}(X))$. In particular, $y \in \mathrm{Conv}(X)$, and since $y \in Y$ is arbitrary, we conclude that $Y \subseteq \mathrm{Conv}(X)$ as asserted. $\square$

Conversely:

Theorem 4.20

Let X be a compact subset of $\mathbb{R}^{n+1}$. Then $\mathrm{Conv}(X)$ satisfies the local geodesic property at every point of $\mathrm{Conv}(X) \setminus X$.

Proof: We prove this by induction on the dimension. The result trivially holds when the ambient space is 1-dimensional. Now choose $x \in \mathrm{Conv}(X) \setminus X$. If x is an interior point of $\text{Conv}(X)$, then we are done. We therefore assume that x is a boundary point of $\text{Conv}(X)$. By Lemma 4.6, there exists a supporting normal N to $\text{Conv}(X)$ at x. Let H be the hyperplane normal to N passing through x. We denote $X' = H \cap X$. We claim that $\text{Conv}(X') = \text{Conv}(X) \cap H$. Indeed, X' is compact. By Lemma 4.15, $\text{Conv}(X')$ is compact. Choose $x' \in H \setminus \text{Conv}(X')$. Let $y' \in H$ be a point in $\text{Conv}(X')$ minimising the distance to x'. Denote $\mathsf{N}' = (x' - y')/\|x' - y'\|$. By Lemma 4.5, N' is a supporting normal to $\text{Conv}(X')$ at y'. In particular, for all $z' \in X' \subseteq \text{Conv}(X')$:

$$\langle z' - y', \mathsf{N}' \rangle \leqslant 0.$$

Thus:

$$\langle z' - x', \mathsf{N}' \rangle = \langle (z' - y') + (y' - x'), \mathsf{N}' \rangle \leqslant -\|x' - y'\| < 0.$$

However, for all $z \in X \setminus X' = X \setminus H$:

$$\langle z - y', \mathsf{N} \rangle < 0.$$

Thus:

$$\langle z - x', \mathsf{N} \rangle = \langle (z - y') + (y' - x'), \mathsf{N} \rangle < 0.$$

Combining these relations, we obtain for all $z \in X$:

$$\langle z - x, (\mathsf{N} + \mathsf{N}') \rangle < 0.$$

We define $K \subseteq \mathbb{R}^{n+1}$ by:

$$K = \{ z \mid \langle z - x, (\mathsf{N} + \mathsf{N}') \rangle < 0 \}.$$

K is open and convex, and $X \subseteq K$. Thus $\text{Conv}(X) \subseteq K$. However, $x' \notin K$ and so $x' \in H \setminus \text{Conv}(X)$. Since $x' \in H \setminus \text{Conv}(X)$ was arbitrary, we conclude that $H \setminus \text{Conv}(X') \subseteq H \setminus \text{Conv}(X)$ and so $\text{Conv}(X) \cap H \subseteq \text{Conv}(X')$. Conversely, let K be an open, convex set containing X. Then $K \cap H$ is also convex and relatively open. Since $K \cap H$ contains X', by definition, it also contains $\text{Conv}(X')$. Upon taking the intersection over all such open sets, it follows that $\text{Conv}(X') \subseteq \text{Conv}(X) \cap H$, and so the two sets coincide as asserted. It follows by the inductive hypothesis that $\text{Conv}(X) \cap H = \text{Conv}(X')$ satisfies the local geodesic property at x and therefore so does $\text{Conv}(X)$. Since $x \in \text{Conv}(X) \setminus X$ is arbitrary, the result follows. □

We now characterise the local geodesic property. Let $X \subseteq \Sigma^n$ be any closed subset. We say that X is **strictly contained in a hemisphere** whenever there exists a unit vector $\mathsf{N} \in \Sigma^n$ such that for all $\mathsf{N}' \in X$:

$$\langle \mathsf{N}, \mathsf{N}' \rangle < 0.$$

Lemma 4.21

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let x be a point in K and suppose that K satisfies the local geodesic property at x. Then for all sufficiently small $r > 0$, $K \cap (\partial B_r(x))$ is not strictly contained in a hemisphere.

Proof: Let Γ be the open straight-line segment passing through x contained in K. By definition, there exists a point $y \in \mathbb{R}^{n+1}$ and real numbers $a < 0 < b$ such that:

$$\Gamma = \{x + ty \mid a < t < b\}.$$

Choose $r < \mathrm{Min}(-a, b)$. Then $x \pm ry \in K \cap (\partial B_r(x))$. Suppose that there exists $\mathsf{N} \in \Sigma^n$ such that $\langle (x \pm ry) - x, \mathsf{N}\rangle < 0$. Then, $\pm\langle y, \mathsf{N}\rangle < 0$, which is absurd, and the assertion follows. □

Lemma 4.22

Let $X \subseteq \Sigma^n$ be a closed subset not strictly contained in a hemisphere. Then 0 is an element of $\mathrm{Conv}(X)$.

Proof: Suppose the contrary. By Lemma 4.15, $\mathrm{Conv}(X)$ is compact. let $x \in \mathrm{Conv}(X)$ be the point minimising distance in $\mathrm{Conv}(X)$ to 0. Denote $\mathsf{N} = -x/\|x\|$. By Lemma 4.5, N is a supporting normal to $\mathrm{Conv}(X)$ at x. By definition of supporting normals, for all $y \in X \subseteq \mathrm{Conv}(X)$, $\langle y - x, \mathsf{N}\rangle \leqslant 0$. Thus:

$$\langle y, \mathsf{N}\rangle = \langle (y - x) + x, \mathsf{N}\rangle \leqslant -\|x\| < 0.$$

Since $y \in X$ is arbitrary, it follows that X is strictly contained in a hemisphere. This is absurd, and it follows that 0 is an element of $\mathrm{Conv}(X)$ as desired. □

Lemma 4.23

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let x be a boundary point of K and let N be a supporting normal to K at x. Suppose that K does not satisfy the local geodesic property at x. Then, for all $r > 0$ there exists N' which we may choose as close to N as we wish with the property that for all $y \in K \cap (B_r(x))^c$:

$$\langle y - x, \mathsf{N}\rangle < 0.$$

Proof: Upon applying an affine isometry, we may suppose that $x = 0$. Choose $r > 0$. Since K does not satisfy the local geodesic property at 0, by the contrapositive of Lemma 4.20, 0 does not lie in the convex hull of $K \cap (\partial B_r(0))$. By the contrapositive of Lemma 4.22, $K \cap (\partial B_r(0))$ is strictly contained in a hemisphere. There therefore exists a unit vector $\mathsf{N}' \in \Sigma^n$ such that for all $y \in K \cap (\partial B_r(0))$:

$$\langle y, \mathsf{N}'\rangle < 0.$$

For all $\epsilon > 0$, we denote $\mathsf{N}_\epsilon = (\mathsf{N} + \epsilon\mathsf{N}')/\|\mathsf{N} + \epsilon\mathsf{N}'\|$. For all ϵ and for all $y \in K \cap (\partial B_r(0))$, bearing in mind the definition of supporting normals:

$$\begin{aligned}\|\mathsf{N} + \epsilon\mathsf{N}'\|\langle y, \mathsf{N}_\epsilon\rangle &= \langle y, \mathsf{N}\rangle + \epsilon\langle y, \mathsf{N}'\rangle \\ &\leqslant \epsilon\langle y, \mathsf{N}'\rangle \\ &< 0.\end{aligned}$$

By chosing ϵ as small as we wish, we may choose N_ϵ as close to N as we wish. We now claim that N_ϵ has the desired properties for all $\epsilon > 0$. Indeed, if $y \in K \cap (\partial B_s(0))$ for some $s > r$ then, by convexity:

$$(r/s)y \in K \cap (\partial B_r(0)),$$

and so:

$$\langle y, \mathsf{N}_\epsilon\rangle = (s/r)\langle (r/s), \mathsf{N}_\epsilon\rangle < 0,$$

as desired. □

4.6 Interior a-Priori bounds.

We return to the framework of Section 2. We thus consider functions $f \in C^\infty(\overline{\Omega})$ which are solutions of the equation:

$$F(D^2 f) = \phi G(Df).$$

We aim to study the structure of singularities of uniform limits of sequences of such functions. The following reasoning dates to Pogorelov (c.f. [15], but see also [5] and [19]). As in Section 2.6, we denote:

$$B^{ij}(x) = \frac{1}{n}F(D^2 f(x))(D^2 f(x)^{-1})^{ij}.$$

We complement Lemma 2.26 with the two following estimates:

Lemma 4.24

There exists $C > 0$ which only depends on $\|\phi\|_0$ such that:

$$\mathcal{L}_f \mathrm{Log}(-f) \geqslant -C(-f)^{-1} - B^{ij}(\partial_i \mathrm{Log}(-f))(\partial_j \mathrm{Log}(-f)).$$

Remark: *Observe that since f is strictly convex and vanishes along $\partial\Omega$, $f < 0$ throughout Ω and so the logarithm of $(-f)$ is well defined.*

Proof: By Lemma 2.6:

$$DF(D^2 f)(D^2 f) = F(D^2 f).$$

Since G is convex:

$$DG(Df)(Df) \geqslant G(Df) - G(0).$$

Since, ϕ is positive, by definition of $\mathcal{L}_f$:

$$\mathcal{L}_f f \leqslant F(D^2 f) - \phi G(Df) + \phi G(0) = \phi G(0).$$

There therefore exists $C > 0$ which only depends on $\|\phi\|_0$ such that:

$$\mathcal{L}_f(-f) \geqslant -C.$$

By Lemma 2.25:

$$\mathcal{L}_f \mathrm{Log}(-f) \geqslant -C(-f)^{-1} - B^{ij}(\partial_i \mathrm{Log}(-f))(\partial_j \mathrm{Log}(-f)),$$

as desired. □

Lemma 4.25

There exists $C > 0$ *which only depends on* $\|\phi\|_1$ *and* $\|f\|_1$ *such that:*

$$\mathcal{L}_f\|Df\|^2 \geqslant -C + \frac{2}{n}\phi\lambda_n(f).$$

Proof: By the product rule, for all i, using the summation convention:

$$\begin{array}{ll} \partial_i\|Df\|^2 & = 2f_{ik}f_k \\ \partial_i\partial_j\|Df\|^2 & = 2f_{ijk}f_k + 2f_{ik}f_{jk}. \end{array}$$

Thus, bearing in mind that the derivatives of f are symmetric:

$$\begin{array}{ll} \mathcal{L}_f\|Df\|^2 & = 2f_kDF(D^2f)(D^2f_k) + 2B^{ij}f_{ik}f_{jk} - 2f_k\phi DG(Df)(Df_k) \\ & = 2f_k\mathcal{L}_ff_k + \frac{2}{n}F(D^2f)f_{kk}. \end{array}$$

Since D^2f is positive definite:

$$f_{kk} = \mathrm{Tr}(D^2f) \geqslant \lambda_n(f).$$

Thus, since f satisfies (A) and since ϕ is positive and $G \geqslant 1$:

$$\mathcal{L}_f\|Df\|^2 \geqslant 2f_k\mathcal{L}_ff_k + \frac{2}{n}\phi\lambda_n(f).$$

However, differentiating (A) once in the e_k direction yields:

$$\mathcal{L}_ff_k = \phi_kG(Df).$$

Thus:

$$\mathcal{L}_f\|Df\|^2 \geqslant 2f_k\phi_kDG(Df) + \frac{2}{n}\phi\lambda_n(f).$$

There therefore exists $C > 0$ which only depends on $\|\phi\|_1$ and $\|f\|_1$ such that:

$$\mathcal{L}_f\|Df\|^2 \geqslant -C + \frac{2}{n}\phi\lambda_n(f),$$

as desired. □

Lemma 4.26

There exists $C > 0$ which only depends on $\|\phi\|_2$, $\|f\|_1$ and $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$ such that:

$$\mathop{\mathrm{Sup}}_{x\in\overline{\Omega}} |f(x)|^2 \, \|D^2 f(x)\| \leqslant C.$$

Remark: *Observe the important role played by the positivity of the $D\mathrm{Log}(l)$ term in Lemma 2.26. If this term were negative, as is usually the case when one calculates a generalised Laplacian of the logarithm of a function (c.f. Lemma 2.25), then we would not be able to factorise the second set of bad terms in the following proof. Pogorelov's trick is to then observe that all the bad terms are controlled modulo terms which vanish at a local maximum.*

Proof: For $\alpha \in]0,1[$, we define the function $\psi_\alpha : \Omega \to \mathbb{R}$ by:

$$\psi_\alpha = 2\mathrm{Log}(-f) + \mathrm{Log}(\lambda_n(f)) + \alpha^2 \|Df\|^2.$$

We claim that there exists $\alpha_0 \in]0,1[$ which only depends on $\|\phi\|_2$, $\|f\|_1$ and $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$ with the property that for all $\alpha \in]0,\alpha_0[$ there exists $C > 0$ which only depends on $\|\phi\|_2$, $\|f\|_1$, $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$ such that $\psi_\alpha \leqslant C$. We achieve this by showing that $\mathcal{L}_f(\psi_\alpha)$ is positive for sufficiently high values of ψ_α, and the assertion then follows by the maximum principle.

Fix α and observe that ψ_α is continuous and tends to $-\infty$ near the boundary of Ω. It therefore attains its maximum at some interior point $x \in \Omega$. Upon applying an isometry, we may suppose that $e_1, ..., e_n$ are the eigenvectors of $D^2 f(x)$ corresponding to the eigenvalues $\lambda_1, ..., \lambda_n$ respectively. Let $l : \Omega \to \mathbb{R}$ be a smooth function such that $l \leqslant \lambda_n(f)$ and $l(x) = \lambda_n(x)$. We define $\tilde{\psi}_\alpha$ by:

$$\tilde{\psi}_\alpha = 2\mathrm{Log}(-f) + \mathrm{Log}(l) + \alpha^2 \|Df\|^2.$$

Observe that $\tilde{\psi}_\alpha \leqslant \psi_\alpha$ and $\tilde{\psi}_\alpha(x) = \psi_\alpha(x)$. In particular, $\tilde{\psi}_\alpha$ attains a local maximum at x. By Lemmas 2.26, 4.24 and 4.25, and by definition of weak differential inequalities, there exists $C_1 > 0$ which only depends on $\|\phi\|_2$, $\|f\|_1$ and $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$ such that:

$$\begin{aligned}
\mathcal{L}_f \tilde{\psi}_\alpha \;&\geqslant -C_1(-f)^{-1} - 2(n\lambda_n)^{-1}(\partial_n \mathrm{Log}(l))^2 + \tfrac{2\alpha^2}{n}\phi\lambda_n(f) \\
&\qquad + B^{ij}(\partial_i \mathrm{Log}(l))(\partial_j \mathrm{Log}(l)) - 2B^{ij}(\partial_i \mathrm{Log}(-f))(\partial_j \mathrm{Log}(-f)) \\
&= -C_1(-f)^{-1} - 2(n\lambda_n)^{-1}(\partial_n \mathrm{Log}(l))^2 + \tfrac{2\alpha^2}{n}\phi\lambda_n(f) \\
&\qquad + B^{ij}(\partial_i \mathrm{Log}(l) - \sqrt{2}\partial_i \mathrm{Log}(-f))(\partial_j \mathrm{Log}(l) + \sqrt{2}\partial_j \mathrm{Log}(-f)).
\end{aligned}$$

Observe that the second and the fourth terms on the right hand side are of order 3 in f. We eliminate these terms as follows. First, since $\tilde{\psi}$ attains a local maximum at x, $D\tilde{\psi} = 0$, and so, for all i

$$2\partial_i \mathrm{Log}(-f) + \partial_i \mathrm{Log}(l) + \alpha^2 \partial_i \|Df\|^2 = 0.$$

Thus, at x, bearing in mind the definition of λ_n:

$$\begin{aligned}-\tfrac{1}{\lambda_n}(\partial_n\mathrm{Log}(l))^2 &= -\tfrac{C}{\lambda_n}(2\partial_n\mathrm{Log}(-f)+\alpha^2\partial_n(\|Df\|^2))^2\\ &= -\tfrac{1}{\lambda_n}(\tfrac{2}{(-f)}(-f_n)+2\alpha^2\lambda_n f_n)^2\\ &= -\tfrac{4}{\lambda_n(-f)^2}f_n^2+\tfrac{4\alpha^2 f_n^2}{(-f)}-4\alpha^4\lambda_n f_n^2\end{aligned}$$

Bearing in mind Lemma 2.9 and the definition of G:

$$\lambda_n \geqslant \frac{1}{n}\mathrm{Tr}(D^2f)\geqslant F(D^2f)=\phi G(Df)\geqslant \phi.$$

There therefore exists $C_2>0$, which only depends on $\|f\|_1$ and $\mathrm{Inf}_{x\in\overline{\Omega}}\phi(x)$ such that:

$$-\frac{1}{\lambda_n}(\partial_n\mathrm{Log}(l))^2\geqslant -\frac{C_2}{(-f)^2}-C_2\alpha^4\lambda_n.$$

This eliminates the first of the two terms that are of order 3 in f. Now, bearing in mind that B is positive definite:

$$\begin{aligned}&B^{ij}(\partial_i\mathrm{Log}(l)-\sqrt{2}\partial_i\mathrm{Log}(-f))(\partial_j\mathrm{Log}(l)+\sqrt{2}\partial_j\mathrm{Log}(-f))\\ &\quad= B^{ij}(\alpha^2\partial_i\|Df\|^2+(2+\sqrt{2})\partial_i\mathrm{Log}(-f))\\ &\qquad\times(\alpha^2\partial_j\|Df\|^2+(2-\sqrt{2})\partial_j\mathrm{Log}(-f))\\ &\quad\geqslant \tfrac{2\sqrt{2}\alpha^2}{(-f)}B^{ij}(-f)_i(\partial_j\|Df\|^2)\\ &\quad= \tfrac{4\sqrt{2}\alpha^2}{n(-f)}F(D^2f)\|Df\|^2\\ &\quad= \tfrac{4\sqrt{2}\alpha^2}{n(-f)}\phi G(Df)\|Df\|^2\\ &\quad\geqslant \tfrac{4\sqrt{2}\alpha^2}{n(-f)}\phi\|Df\|^2.\end{aligned}$$

There therefore exists $C_3>0$ which only depends on $\|\phi\|_0$ and $\|f\|_1$ such that:

$$B^{ij}(\partial_i(\mathrm{Log}(l)-2\mathrm{Log}(-f)))(\partial_j(\mathrm{Log}(l)+2\mathrm{Log}(-f)))\geqslant -\frac{C_3}{(-f)^2}.$$

This eliminates the second of the two terms which are of order 3 in f. We conclude that there exists $C_4>0$ which only depends on $\|\phi\|_2$, $\|f\|_1$ and $\mathrm{Inf}_{x\in\overline{\Omega}}\phi(x)$ such that:

$$\mathcal{L}_f\tilde{\psi}_\alpha(x)\geqslant -C_4(-f)^{-2}+\alpha^2(\frac{2\phi}{n}-C_4\alpha^2)\lambda_n(f).$$

We choose $\alpha_0\in]0,1[$ such that:

$$\alpha_0^2<\frac{2}{C_4 n}\mathop{\mathrm{Inf}}_{x\in\overline{\Omega}}\phi(x).$$

Choose $\alpha\in]0,\alpha_0[$. There exists C_5 which only depends on $\|\phi\|_2$, $\|f\|_1$, $\mathrm{Inf}_{x\in\overline{\Omega}}\phi(x)$ and α such that:

$$\mathcal{L}_f(\tilde{\psi}_\alpha)(x)\geqslant\frac{1}{(-f)^2}(-C_5+\frac{1}{C_5}e^{\tilde{\psi}_\alpha}).$$

However, since x is a local maximum of $\tilde{\psi}_\alpha$, $\mathcal{L}_f(\tilde{\psi}_\alpha)\leqslant 0$, and so:

$$\tilde{\psi}_\alpha\leqslant 2\mathrm{Log}(C_5).$$

Choose $\alpha<\alpha_0$. Choose $C>0$ such that $\psi_\alpha\leqslant C$ throughout Ω. Then, for all $x\in\Omega$:

$$|f(x)|^2\,\|D^2f(x)\|=e^{\tilde{\psi}_\alpha(x)}e^{\alpha^2\|Df\|^2},$$

as desired. □

4.7 The Structure of Singularities.

We conclude this section by characterising the singularities that arise upon taking limits. We first show some preliminary results:

Lemma 4.27

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. If K has non-trivial interior, then $\mathcal{N}(x)$ is strictly contained in a hemisphere for every boundary point x of K.

Proof: Let x be an interior point of K. Let y be a boundary point of K. Denote $\mathsf{N} = (x-y)/\|x-y\|$. We claim that $\langle \mathsf{N}, \mathsf{M}\rangle < 0$ for all $\mathsf{M} \in \mathcal{N}(y)$. Indeed, choose $\delta > 0$ such that $\overline{B}_\delta(x) \subseteq K$. choose $\mathsf{M} \in \mathcal{N}(y)$. Observe that $x + \delta\mathsf{M} \in K$. Bearing in mind the definition of supporting normals:

$$\langle x - y, \mathsf{M}\rangle = \langle (x+\delta\mathsf{M}) - y, \mathsf{M}\rangle - \delta\langle \mathsf{M}, \mathsf{M}\rangle \leqslant -\delta < 0.$$

Since $\mathsf{M} \in \mathcal{N}(y)$ is arbitrary, the assertion follows and we conclude that $\mathsf{N}(y)$ is strictly contained in a hemisphere as desired. □

Lemma 4.28

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let x be a boundary point of K. If $\mathcal{N}(x)$ is strictly contained in a hemisphere, then there exists a supporting normal N to K at x such that $\langle \mathsf{N}, \mathsf{M}\rangle > 0$ for all $\mathsf{M} \in \mathcal{N}(x)$.

Proof: Upon applying a linear isometry, we may suppose that $\langle \mathsf{N}, e_{n+1}\rangle < 0$ for all $\mathsf{N} \in \mathcal{N}(x)$. Observe that $\mathcal{N}(x)$ is closed. If $-e_{n+1} \in \mathcal{N}(x)$, then we are done. Otherwise, suppose that $-e_{n+1} \notin \mathcal{N}(x)$. We define $\mathrm{Cone}(\mathcal{N}(x))$ by:

$$\mathrm{Cone}(\mathcal{N}(x)) = \{t\mathsf{N} \mid t \in [0,\infty[\ \&\ \mathsf{N} \in X\}.$$

Observe that $\mathrm{Cone}(\mathcal{N}(x))$ is closed and convex. Moreover, $-e_{n+1} \notin \mathrm{Cone}(\mathcal{N}(x))$. By Lemma 4.11, there exists a point $y \in \mathrm{Cone}(\mathcal{N}(x))$ minimising distance in $\mathrm{Cone}(\mathcal{N}(x))$ to $-e_{n+1}$. Moreover, there exists a supporting normal N to $\mathrm{Cone}(\mathcal{N}(x))$ at y such that $-e_{n+1} = y + \|e_{n+1} + y\|\mathsf{N}$. For all $t \in [0,\infty[$, since $ty \in \mathrm{Cone}(\mathcal{N}(x))$ for all $t \in [0,\infty[$, by definition of supporting normals:

$$\langle ty - y, \mathsf{N}\rangle \leqslant 0.$$

Differentiating this relation at $t = 1$ yields $\langle y, \mathsf{N}\rangle = 0$. Thus, by definition of supporting normals, for all $\mathsf{M} \in X \subseteq \mathrm{Cone}(\mathcal{N}(x))$, bearing in mind that $\langle \mathsf{M}, -e_{n+1}\rangle > 0$:

$$\begin{aligned} 0 &\geqslant \langle \mathsf{M} - y, \mathsf{N}\rangle \\ &= \langle \mathsf{M}, \mathsf{N}\rangle \\ &= \|e_n + y\|\langle \mathsf{M}, -e_{n+1} - y\rangle \\ &> \|e_n + y\|\|y\|\langle \mathsf{M}, -y/\|y\|\rangle. \end{aligned}$$

We denote $\mathsf{N} = y/\|y\|$. By definition of $\mathrm{Cone}(\mathcal{N}(x))$, $\mathsf{N} \in \mathcal{N}(x)$, and by the above reasoning, $\langle \mathsf{M}, \mathsf{N}\rangle > 0$ for all $\mathsf{M} \in \mathcal{N}(x)$ as desired. □

We now describe the singularities that may arise:

Theorem 4.29

Let $(K_m)_{m\in\mathbb{N}}$ and K_∞ be compact, convex subsets of $\mathbb{R}^{n+1}$. Suppose that K_∞ has non-trivial interior and $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense. Let $k>0$ be a real number, let U be an open subset of $\mathbb{R}^{n+1}$ and suppose that for all m, $(\partial K_m)\cap U$ is smooth with constant gaussian curvature equal to k. If $y\in(\partial K_\infty)\cap U$ then:

(1) either there exists $r>0$ such that $(\partial K_\infty)\cap B_r(y)$ is smooth with constant gaussian curvature equal to k; or

(2) K_∞ satisfies the local geodesic property at y.

Proof: Choose $y\in(\partial K_\infty)\cap U$ and suppose that K_∞ does not satisfy the local geodesic property at y. Let $(y_m)_{m\in\mathbb{N}}$ be a sequence converging to y such that $y_m\in\partial K_m$ for all m. Upon applying a sequence of affine isometries converging to the identity mapping, we may suppose that $y_m=0$ for all $m\in\mathbb{N}\cup\{\infty\}$. Since K_∞ has non-trivial interior, by Lemma 4.27, $\mathcal{N}(0;K_\infty)$ is strictly contained in a hemisphere. By Lemma 4.28, there exists $\mathsf{N}\in\mathcal{N}(0;K_\infty)$ such that for all $\langle\mathsf{N},\mathsf{M}\rangle>0$ for all $\mathsf{M}\in\mathcal{N}(0;K_\infty)$. By compactness of $\mathcal{N}(0;K_\infty)$, there exists $\theta\in[0,\pi/2[$ such that $\langle\mathsf{N},\mathsf{M}\rangle>3\cos(\theta)$ for all $\mathsf{M}\in\mathcal{N}(0;K_\infty)$. We denote $C=\tan(\theta)$.

By Lemma 4.3, there exists $r>0$ and $M\in\mathbb{N}$ such that $B_r(0)\subseteq U$ and, for all $m\geqslant M$, for all $x\in(\partial K_m)\cap B_r(0)$ and for all $\mathsf{M}\in\mathcal{N}(x;K_m)$, $\langle\mathsf{N},\mathsf{M}\rangle>2\cos(\theta)$. Upon extracting a subsequence, we may suppose that $M=1$. We denote $\rho=r/\sqrt{1+4C^2}$. By Lemma 4.23, there exists N', which we may choose as close to N as we wish such that for all $x\in K_\infty\setminus B_{\rho/2}(0)$, $\langle x,\mathsf{N}'\rangle<0$. Moreover, we may assume that for all $m\in\mathbb{N}\cup\{\infty\}$, for all $x\in(\partial K_m)\cap B_r(0)$ and for all $\mathsf{M}\in\mathcal{N}(x;K_m)$, $\langle\mathsf{N}',\mathsf{M}\rangle>\cos(\theta)$.

Upon applying a rotation, we may suppose that $\mathsf{N}'=-e_{n+1}$. By Theorem 4.13, for all $m\in\mathbb{N}\cup\{\infty\}$, there exists a convex, C-Lipschitz function $f_m:B'_\rho(0)\to]-C\rho,C\rho[$ such that $f_m=0$ and $(\partial K_m)\cap(B'_\rho(0)\times]-2C\rho,2C\rho[)$ coincides with the graph of f_m over $B'_\rho(0)$.

Since $f_m(0)=0$ and f_m is C-Lipschitz for all m, by the Arzela-Ascoli theorem, every subsequence of $(f_m)_{m\in\mathbb{N}}$ has a subsubsequence converging in the local uniform sense over $B'_\rho(0)$ to some limit f'_∞ say. Since $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense, it follows that $f'_\infty=f_\infty$, and we conclude that $(f_m)_{m\in\mathbb{N}}$ converges in the local uniform sense over $B'_\rho(0)$ to f_∞.

By construction, $f_\infty(x')>0$ for all $x'\in\partial B'_{\rho/2}(0)$. By compactness, there exists $\delta>0$ such that $f_\infty(x')>4\delta$ for all $x'\in\partial B'_{\rho/2}(0)$. Since $(f_m)_{m\in\mathbb{N}}$ converges locally uniformly to f_∞ over $B'_\rho(0)$, there exists $M\in\mathbb{N}$ such that for all $m\geqslant M$, and for all $x'\in B'_{\rho/2}(0)$, $f_m(x')>2\delta$. Upon extracting a subsequence, we may suppose that $M=1$.

For all m, since $f_m(0)=0$, and f_m is C-Lipschitz:

$$\|f_m|_{\overline{B}'_{\rho/2}(0)}\|_1\leqslant C(1+\rho/2).$$

Thus, by Lemma 4.26, there exists $C_2>0$ such that for all m and for all $x\in B'_{\rho/2}(0)$:

$$|2\delta-f_m(x)|^2\,\|D^2f_m(x)\|\leqslant C_2.$$

Since $f_\infty(0) = 0$, by continuity, there exists $s \in]0, \rho/2[$ such that $f_\infty(x') \leqslant \delta/2$ for all $x' \in \overline{B}'_s(0)$. Since $(f_m)_{m\in\mathbb{N}}$ converges locally uniformly to f_∞, there exists $M \in \mathbb{N}$ such that for all $m \geqslant M$, and for all $x \in \overline{B}'_s(0)$, $f_m(x') \leqslant \delta$. Upon extracting a subsequence, we may suppose that $M = 1$. Then, for all $m \in \mathbb{N}$ and for all $x \in \overline{B}'_s(0)$:

$$\|D^2 f_m(x)\| \leqslant C_2/\delta^2.$$

By the Krylov estimates (c.f Theorem 2.29), there exists $\alpha > 0$ and $C_{2,\alpha} > 0$ such that for all $m \in \mathbb{N}$:

$$\|f_m|_{\overline{B}'_{s/2}(0)}\|_{2+\alpha} \leqslant C_{2,\alpha}.$$

By the Schauder estimates (c.f. Theorem 2.28), for all $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all $m \in \mathbb{N}$:

$$\|f_m|_{\overline{B}'_{s/4}(0)}\|_k \leqslant C_k.$$

By the Arzela-Ascoli theorem, every subsequence of $(f_m)_{m\in\mathbb{N}}$ has a subsubsequence which converges in the C^∞ sense over $\overline{B}'_{s/4}(0)$ to some limit f'_∞ say. Since $(f_m)_{m\in\mathbb{N}}$ converges uniformly to f_∞, it follows that $f'_\infty = f_\infty$. We conclude that $(f_m)_{m\in\mathbb{N}}$ converges to f_∞ in the C^∞ sense over $\overline{B}'_{s/4}(0)$. In particular, f_∞ is smooth over $B'_{s/4}(0)$ and its graph has constant gaussian curvature equal to k. In other words, $(\partial K_\infty) \cap (B'_{s/4}(0) \times] - 2C\rho, 2C\rho[)$ is smooth and has constant gaussian curvature equal to k as desired. $\square$

5

Duality of Convex Sets

We continue our study of the elementary geometry of convex sets. In particular, we investigate the concept of duality for subsets of the sphere, showing how it is closely related to the concept of the convex hull, which we introduced in the preceeding chapter. We introduce the infinitesimal link of a given boundary point of a given compact, convex set with non-trivial interior. We define this to be an open subset of the sphere, and we show that it coincides with the dual of the set of supporting normal vectors to the convex set at that point. This consequently allows us to prove the most important result of this chapter, being Theorem 5.23 which determines the supporting normal set of the intersection of two given convex sets at any point on the boundary of this intersection.

While the results of this chapter are of use in the sequel, it is only tangential to the main flow of this text. In particular, Theorem 5.23, although interesting, may be substituted by ad-hoc arguments in the relatively straightforward cases where it will be applied.

5.1 Open Half Spaces and Convex Hulls.

Let N be a unit vector and let $t > 0$ be a positive real number (possibly $+\infty$). We define the subset $H(\mathsf{N}, t)$ of $\mathbb{R}^{n+1}$ by:

$$H(\mathsf{N}, t) = \{x \mid \langle x, \mathsf{N}\rangle < t\}.$$

We refer to $H(\mathsf{N}, t)$ as the **open half-space** normal to N of height t. Observe that, for all N, $H(\mathsf{N}, \infty) = \mathbb{R}^{n+1}$.

Lemma 5.1

If K is an open, convex subset of $\mathbb{R}^{n+1}$, then $K = (\overline{K})^o$.

Proof: Since K is open, $K \subseteq (\overline{K})^o$. We now show that $(\overline{K})^o \subseteq K$. By Lemma 4.14, $\overline{K}$ is convex. Choose $x \in (\overline{K})^o$. Without loss of generality, we may suppose that $x = 0$. Choose $\delta > 0$ such that $B_\delta(0) \subseteq \overline{K}$. Then $K \cap B_\delta(0)$ is a dense subset of $B_\delta(0)$. Upon applying a homothety, we may suppose that $\delta = 2$. Choose $x_1, ..., x_k \in B_1(0)$ such that:

$$\partial B_1(0) \subseteq \bigcup_{i=1}^{k} B_1(x_i).$$

Since $K \cap B_\delta(0)$ is dense, upon perturbing $x_1, ..., x_k$ if necessary, we may suppose that $x_i \in K$ for all i. Define L by:

$$L = \left\{ \sum_{i=1}^{k} t_i x_i \mid \sum_{i=1}^{k} t_i = 1 \right\}.$$

Observe that L is compact and convex. Moreover, by convexity of K, L is contained in K. We claim that 0 is an element of L. Indeed, suppose the contrary. Let $y \in L$ be the point minimising distance to 0. Denote $\mathsf{N} = -y/\|y\|$. By Lemma 4.5, N is a supporting normal to L at y. By definition of supporting normals, for all $z \in L$, $\langle z - y, \mathsf{N}\rangle \leqslant 0$. In particular, for all $1 \leqslant i \leqslant k$, since $x_i \in L$:

$$\langle x_i, \mathsf{N}\rangle = \langle x_i - y, \mathsf{N}\rangle + \langle y, \mathsf{N}\rangle \leqslant -\|y\| < 0.$$

Thus, for all i:

$$\|x_i - \mathsf{N}\|^2 = \|x_i\|^2 - 2\langle x_i, \mathsf{N}\rangle + \|\mathsf{N}\|^2 > 1.$$

However, by definition of $x_1, ..., x_k$, there exists $1 \leqslant i \leqslant k$ such that $\|x_i - \mathsf{N}\|^2 < 1$. This is absurd, and it follows that $0 \in L \subseteq K$ as asserted. Since $x \in (\overline{K})^o$ is arbitrary, it follows that $(\overline{K})^o \subseteq K$ and so the two sets coincide as desired. □

Theorem 5.2

For any subset X of $\mathbb{R}^{n+1}$, the convex hull of X coincides with the intersection of all open half-spaces containing X.

Proof: We denote by $\hat{X}$ the intersection of all open half-spaces containing X. Since every open half-space is also convex, it follows from the definition of the convex hull that $\mathrm{Conv}(X) \subseteq \hat{X}$. We now aim to show that $\hat{X} \subseteq \mathrm{Conv}(X)$. Indeed, suppose the contrary. Choose $x \in \hat{X} \setminus \mathrm{Conv}(X)$. By definition, there exists an open, convex set K such that $X \subseteq K$ and $x \notin K$. By Lemma 4.14, $\overline{K}$ is also convex. For all $r > 0$, denote $\overline{K}_r = \overline{K} \cap \overline{B}_r(x)$. Observe that for all $r > 0$, K_r is compact and convex. We have two cases to consider. Suppose first that $x \in \mathbb{R}^{n+1} \setminus \overline{K}$. Choose $r > 0$ such that $\overline{K}_r$ is non-empty. Let y be a point in $\overline{K}_{2r}$ minimising distance to x. Denote $\mathsf{N} = (x-y)/\|x-y\|$. By Lemma 4.5, N is a supporting normal to $\overline{K}_{2r}$ at y. Observe that $y \in \overline{K}_r \subseteq B_{2r}(x)$. Thus, by Lemma 4.9, for all $s \geqslant 2r$, N is also a supporting normal to $\overline{K}_s$ at y. In particular, for all

$z \in K$, there exists $s > 0$ such that $z \in \overline{K}_s$ and so, by definition of supporting normals $\langle z - y, \mathsf{N}\rangle \leqslant 0$. Thus:

$$\langle z - x, \mathsf{N}\rangle = \langle (z - y) + (y - x), \mathsf{N}\rangle \leqslant -\|x - y\| < 0.$$

Now suppose that $x \in \overline{K} \setminus K$. By Lemma 5.1, $K = (\overline{K})^o$, and so $x \in \overline{K} \setminus K = \overline{K} \setminus (\overline{K})^o = \partial\overline{K}$. Choose $r > 0$. Then $x \in \partial\overline{K}_r$. Let N be a supporting normal to $\overline{K}_r$ at r. By Lemma 4.9, for all $s \geqslant r$, N is also a supporting normal to $\overline{K}_s$ at y. In particular, for all $z \in K$, there exists $s > 0$ such that $z \in \overline{K}_s$ and so, by definition of supporting normals $\langle z - x, \mathsf{N}\rangle \leqslant 0$. Moreover, since K is open, for all $z \in K$, $\langle z - x, \mathsf{N}\rangle < 0$. In both cases, we conclude that $K \subseteq H(\mathsf{N}, \langle \mathsf{N}, x\rangle)$ and so, by definition, $\hat{X} \subseteq H(\mathsf{N}, \langle \mathsf{N}, x\rangle)$. However, $x \notin H(\mathsf{N}, \langle \mathsf{N}, x\rangle)$ and so $x \notin \hat{X}$. Since $x \notin \mathrm{Conv}(X)$ was arbitrary, it follows that $\mathbb{R}^{n+1} \setminus \mathrm{Conv}(X) \subseteq \mathbb{R}^{n+1} \setminus \hat{X}$. Taking complements yields $\hat{X} \subseteq \mathrm{Conv}(X)$, and the two sets therefore coincide as desired. □

We also have the following complement of Lemma 5.1:

Lemma 5.3

If K is a compact, convex subset of $\mathbb{R}^{n+1}$ with non-trivial interior, then $K = \overline{K^o}$.

Proof: Since $K^o \subseteq K$ and since K is closed, $\overline{K^o} \subseteq K$. Conversely, choose $x \in K$. Since K has non-trivial interior, there exists $y \in K^o$. Choose $\delta > 0$ such that $B_\delta(y) \subseteq K$. Bearing in mind that K is convex, for all $z \in B_{t\delta}(0)$ and for all $t \in]0, 1]$:

$$(1 - t)x + ty + z = (1 - t)x + t(y + t^{-1}z) \in K.$$

It follows that $B_{t\delta}((1 - t)x + ty) \subseteq K$ and so $(1 - t)x + ty \in K^o$. Taking limits as t tends to 0, we deduce that $x \in \overline{K^o}$. Since $x \in K$ is arbitrary, it follows that $K \subseteq \overline{K^o}$, and so the two sets coincide as desired. □

5.2 Convex Subsets of the Sphere.

Let $\mathsf{N}_0, \mathsf{N}_1 \in \Sigma^n$ be points in the sphere. We say that N_0 and N_1 are **non-antipodal** whenever $\mathsf{N}_0 + \mathsf{N}_1 \neq 0$. In this case, we define $\mathsf{N} : [0, 1] \to \Sigma^n$ by:

$$\mathsf{N}(s) = \frac{(1 - s)\mathsf{N}_0 + s\mathsf{N}_1}{\|(1 - s)\mathsf{N}_0 + s\mathsf{N}_1\|}.$$

Observe that since $\mathsf{N}_0 + \mathsf{N}_1 \neq 0$, N_s is defined for all s. We refer to N as the **great-circular arc** joining N_0 to N_1. This terminology is justified by the following result:

Lemma 5.4

If $\mathsf{N}_0, \mathsf{N}_1 \in \Sigma^n$ are distinct non-antipodal points of the sphere, then there exists a unique great circle C passing through N_0 and N_1. Moreover, if N is the great-circular arc joining N_0 to N_1 then, for all $s \in [0, 1]$, $\mathsf{N}(s)$ is an element of C.

Proof: Observe that every great circle in Σ^n coincides with the intersection of Σ^n with a plane in $\mathbb{R}^{n+1}$ passing through the origin. Conversely, the intersection of any such plane with Σ^n is a great circle. Since N_0 and N_1 are distinct and non-antipodal, they are linearly independant. There therefore exists a unique plane $E \subseteq \mathbb{R}^{n+1}$ passing through the origin such that $\mathsf{N}_0, \mathsf{N}_1 \in E$. $C = E \cap \Sigma^n$ is therefore a great circle passing through N_0 and N_1 and uniqueness follows by the uniqueness of E. Finally, for all s, $\mathsf{N}(s) \in E$ and $\mathsf{N}(s) \in \Sigma^n$, and it follows that $\mathsf{N}(s)$ is an element of $C = E \cap \Sigma^n$ as desired. □

Lemma 5.5

If X is a subset of Σ^n which is strictly contained in a hemisphere, then no two points of X are antipodal.

Proof: Suppose the contrary. Let $\mathsf{N} \in \Sigma^n$ be such that $\langle \mathsf{N}, \mathsf{M}' \rangle < 0$ for all $\mathsf{M}' \in X$. Let M'_1 and M'_2 be two antipodal points of X. In particular, $\mathsf{M}'_1 = -\mathsf{M}'_2$. Then $\langle \mathsf{N}, \mathsf{M}'_1 \rangle < 0$. However, $\langle \mathsf{N}, \mathsf{M}'_1 \rangle = -\langle \mathsf{N}, \mathsf{M}'_2 \rangle > 0$. This is absurd, and the result follows. □

Let X be a subset of Σ^n which is strictly contained in a hemisphere. By Lemma 5.5, no two points of X are antipodal and so there is a well defined great-circular arc joining any two points of X. We say that X is **convex** whenever it has in addition the property that for all $\mathsf{N}_0, \mathsf{N}_1 \in X$, the great-circular arc joining N_0 and N_1 is also contained in K.

For all $\mathsf{N} \in \Sigma^n$, we define the subset $\Sigma^n_-(\mathsf{N})$ of Σ^n by:

$$\Sigma^n_-(\mathsf{N}) = \{x \mid \langle x, \mathsf{N} \rangle < 0\}.$$

We refer to $\Sigma^n_-(\mathsf{N})$ is the **open hemisphere** defined by N. In particular, when $\mathsf{N} = e_n$, we define the **southern hemisphere** Σ^n_- of Σ^n by $\Sigma^n_- = \Sigma_-(e_n)$. We now identify $\mathbb{R}^{n+1}$ with the product $\mathbb{R}^n \times \mathbb{R}$. Observe that Σ^n_- then coincides with the intersection of Σ with $\mathbb{R}^n \times]-\infty, 0[$. We define the mapping $P : \Sigma^n_- \to \mathbb{R}^n$ by:

$$P(x', t) = -x'/t.$$

We refer to P as the **affine projection** of Σ^n_- onto $\mathbb{R}^n$.

Lemma 5.6

P defines a smooth diffeomorphism from Σ^n_- onto $\mathbb{R}^n$.

Proof: We define $\hat{P} : \mathbb{R}^n \times]-\infty, 0[\to \mathbb{R}^n$ by $\hat{P}(x', t) = -x'/t$. Observe that $\hat{P}$ is smooth and since P coincides with the restriction of $\hat{P}$ to Σ^n_-, it is therefore also smooth. We define $Q : \mathbb{R}^n \to \Sigma^n_-$ by $Q(x') = (x', -1)/\sqrt{1 + \|x'\|^2}$. Observe that Q is smooth. Moreover, for all $(x', t) \in \Sigma^n_-$, bearing in mind that $\|x'\|^2 + t^2 = 1$:

$$(Q \circ P)(x', t) = Q(-x'/t) = (-x'/t, -1)/\sqrt{1 + x^2/t^2} = (x', t).$$

Conversely, for all $x' \in \mathbb{R}^n$:

$$(P \circ Q)(x') = P((x', -1)/\sqrt{1 + \|x'\|^2}) = x'.$$

We conclude that P is a smooth diffeomorphism with inverse Q as desired. □

If X is a subset of Σ^n, we define $\mathrm{Cone}(X)$, the **cone** of X by:

$$\mathrm{Cone}(X) = \{tx \mid t \in]0, \infty] \;\&\; x \in X\}.$$

For any unit vector N in $\mathbb{R}^{n+1}$, we define the **linear half-space** $H(\mathsf{N})$ by $H(\mathsf{N}) = H(\mathsf{N}, 0)$.

Lemma 5.7

X is a convex subset of Σ^{n+1} if and only if $\mathrm{Cone}(X)$ *is convex and is contained in a linear half-space.*

Proof: Suppose first that X is convex. In particular, X is strictly contained in a hemisphere. Let $\mathsf{N} \in \Sigma^n$ be such that for all $\mathsf{M} \in X$, $\langle \mathsf{N}, \mathsf{M} \rangle < 0$. Choose $x \in \mathrm{Cone}(X)$. There exists $\mathsf{M} \in X$ and $t > 0$ such that $x = t\mathsf{M}$. Thus:

$$\langle \mathsf{N}, x \rangle = \langle \mathsf{N}, t\mathsf{M} \rangle < 0.$$

Since $x \in \mathrm{Cone}(X)$ is arbitrary, it follows that $\mathrm{Cone}(X)$ is contained in the linear half-space $H(\mathsf{N})$. Now choose $x_0, x_1 \in \mathrm{Cone}(X)$. Choose $\mathsf{N}_0, \mathsf{N}_1 \in X$ and $t_0, t_1 \in]0, \infty[$ such that $x_0 = t_0\mathsf{N}_0$ and $x_1 = t_1\mathsf{N}_1$. Let N be the great-circular arc joining N_0 to N_1. Since X is convex, $\mathsf{N}(s) \in X$ for all $s \in [0,1]$. Thus, for all s, $(1-s)\mathsf{N}_0 + s\mathsf{N}_1 = \|(1-s)\mathsf{N}_0 + s\mathsf{N}_1\|\mathsf{N}(s)$ is an element of $\mathrm{Cone}(X)$. In other words, the straight-line segment joining N_0 to N_1 is contained in $\mathrm{Cone}(X)$. By homogeneity, the straight-line segment joining $t_0\mathsf{N}_0$ to $t_1\mathsf{N}_1$ is also contained in $\mathrm{Cone}(X)$ and it follows that $\mathrm{Cone}(X)$ is convex as desired.

Now suppose that $\mathrm{Cone}(X)$ is convex and is contained in a linear half-space. Choose $\mathsf{N} \in \Sigma^n$ such that $\mathrm{Conv}(X) \subseteq H(\mathsf{N})$. Observe that $X \subseteq \mathrm{Cone}(X)$. Thus, for all $\mathsf{M} \in X$, $\langle \mathsf{N}, \mathsf{M} \rangle < 0$, and so X is strictly contained in a hemisphere. Now choose $\mathsf{N}_0, \mathsf{N}_1 \in X$. Let N be the great circular arc joining N_0 to N_1. Choose $s \in [0,1]$. Since $\mathrm{Cone}(X)$ is convex, $(1-s)\mathsf{N}_0 + s\mathsf{N}_1 \in \mathrm{Cone}(X)$. Thus, by homogeneity, $\mathsf{N}(s) = ((1-s)\mathsf{N}_0 + s\mathsf{N}_1)/\|(1-s)\mathsf{N}_0 + s\mathsf{N}_1\|$ is also an element of $\mathrm{Cone}(X)$. However, $\mathsf{N}(s) \in \Sigma^n$, and so $\mathsf{N}(s) \in \mathrm{Cone}(X) \cap \Sigma^n = X$. Since N_0, N_1 in X and $s \in [0,1]$ are arbitrary, it follows that X is convex as desired. □

Lemma 5.8

If X is a subset of Σ^n_-, then X is convex if and only if $P(X)$ is convex.

Remark: *Using this result, we are able to apply the theory of convex subsets of $\mathbb{R}^{n+1}$ as developed in Section 4 to convex subsets of the sphere.*

Proof: We identify $\mathbb{R}^n$ with the affine hyperplane $R = \mathbb{R}^n \times \{-1\}$ in $\mathbb{R}^{n+1}$. We observe that if X is a subset of Σ^n_-, then $P(X) = \mathrm{Cone}(X) \cap R$. Now suppose that X is convex. Then, by Lemma 5.7, $\mathrm{Cone}(X)$ is convex, and so $P(X) = \mathrm{Cone}(X) \cap R$ is also convex as desired. Conversely, suppose that $P(X) = \mathrm{Cone}(X) \cap R$ is convex. Then $\mathrm{Cone}(X)$ is convex, and so, by Lemma 5.7, X is convex, as desired. □

The mapping P sends the set of open hemispheres bijectively onto the set of open half-spaces:

Lemma 5.9

For every open hemisphere S^n_-, $P(\Sigma^n_- \cap S^n_-)$ is an open half-space. Conversely, for every open half-space H, there exists a unique open hemisphere S^n_- such that $P(\Sigma^n_- \cap S^n_-) = H$.

Proof: As in the proof of Lemma 5.8, we identify $\mathbb{R}^n$ with the affine hyperplane $R = \mathbb{R}^n \times \{-1\}$ in $\mathbb{R}^{n+1}$. Observe that every open hemisphere in Σ^n coincides with the intersection of Σ^n with a unique, open, linear half-space in $\mathbb{R}^{n+1}$. Conversely, the intersection of any open, linear half-space in $\mathbb{R}^{n+1}$ with Σ^n as an open hemisphere. Likewise, every open half-space in R coincides with the intersection of R with a unique, open, linear half-space in $\mathbb{R}^{n+1}$. Conversely, the intersection of any open, linear half-space in $\mathbb{R}^{n+1}$ with R is an open half-space in R. Now let S be an open hemisphere in Σ^n. Let H be the open, linear half-space in $\mathbb{R}^{n+1}$ whose intersection with Σ^n is S. Then $P(S) = H \cap R$ is an open half-space, as desired. Conversely, let H be an open half-space in R. Let H' be the unique open, linear half-space in $\mathbb{R}^{n+1}$ whose intersection with R is H. Denote $S = \Sigma^n \cap H'$. Then $P(S \cap S^n_-) = H$. Moreover, since H' is unique, so is S, and this completes the proof. □

Let X be a subset of Σ^n. If X is strictly contained in a hemisphere, then we define $\mathrm{Conv}(X)$ to be the intersection of all open convex subsets of Σ^n containing X. We call $\mathrm{Conv}(X)$ the **convex hull** of X. Upon applying a rotation, we may suppose that X is strictly contained in Σ^-. The following results are thus completely general:

Lemma 5.10

If X is strictly contained in Σ^n_-, then:

$$P(\mathrm{Conv}(X)) = \mathrm{Conv}(P(X)).$$

Proof: Choose $x \in \mathrm{Conv}(P(X))$. Let K be an open, convex subset of Σ^n containing X. Observe that Σ^n_- is also open and convex, and therefore so too is $K \cap \Sigma^n_-$. Observe that $P(X) \subseteq P(K \cap \Sigma^n_-)$. By Lemma 5.6, $P(K \cap \Sigma^n_-)$ is open. By Lemma 5.8, $P(K \cap \Sigma^n_-)$ is convex. By definition of $\mathrm{Conv}(P(X))$, $x \in P(K)$. By Lemma 5.6, $P^{-1}(x) \in K \cap \Sigma^n_-$. Since K is an arbitrary, open, convex subset of Σ^n_- containing X, it follows that $P^{-1}(x) \in \mathrm{Conv}(X)$ and so $x \in P(\mathrm{Conv}(X))$. Conversely, choose $x \in P(\mathrm{Conv}(X))$. By Lemma 5.6, $P^{-1}(x) \in \mathrm{Conv}(X)$. Let K be an open, convex subset of $\mathbb{R}^n$ such that $P(X) \subseteq K$. Observe that $X \subseteq P^{-1}(K)$. Moreover, since P is continuous, $P^{-1}(K)$ is open. By Lemma 5.8, $P^{-1}(K)$ is convex. By definition of $\mathrm{Conv}(X)$, $P^{-1}(x) \in P^{-1}(K)$. It follows that $x \in K$. Since K is an arbitrary, open, convex subset of $\mathbb{R}^n$ containing $P(X)$, it follows that $x \in \mathrm{Conv}(P(X))$, and we conclude that $P(\mathrm{Conv}(X))$ coincides with $\mathrm{Conv}(P(X))$ as desired. □

Lemma 5.11

Let X be a subset of Σ^n which is strictly contained in a hemisphere. If X is open, then $\mathrm{Conv}(X)$ *is open. If X is closed, then* $\mathrm{Conv}(X)$ *is closed.*

Proof: Suppose that X is open (resp. closed). By Lemma 5.6, $P(X)$ is open (resp. compact). By Lemmas 4.15 and 4.16, $\mathrm{Conv}(P(X))$ is open (resp. compact). By Lemma 5.6, $P^{-1}(\mathrm{Conv}(P(X))$ is open (resp. closed). Bearing in mind Lemmas 5.6 and 5.10:

$$P^{-1}(\mathrm{Conv}(P(X)) = P^{-1}(P(\mathrm{Conv}(X)) = \mathrm{Conv}(X),$$

and it follows that $\mathrm{Conv}(X)$ is open (resp. closed) as desired. □

Lemma 5.12

If K is a convex subset of Σ^n, and if K is either open or closed, then $\mathrm{Conv}(K) = K$.

Proof: By definition, $K \subseteq \mathrm{Conv}(K)$. If K is open, then by definition, $\mathrm{Conv}(K) \subseteq K$ and so the two sets coincide as desired. Suppose that K is closed. In particular, K is compact. Thus $P(K)$ is compact. By Lemma 5.8, $P(K)$ is convex. By Lemma 4.17, $\mathrm{Conv}(P(K)) = P(K)$. By Lemma 5.6, $P^{-1}(\mathrm{Conv}(P(K))) = K$. Thus, bearing in mind Lemmas 5.6 and 5.10:

$$K = P^{-1}(\mathrm{Conv}(P(K))) = P^{-1}(P(\mathrm{Conv}(K))) = \mathrm{Conv}(K),$$

as desired. □

Lemma 5.13

Let X be a subset of Σ^n. If X is strictly contained in a hemisphere, then $\mathrm{Conv}(X)$ *coincides with the intersection of all open hemispheres containing X.*

Proof: We denote by $\hat{X}$ the intersection of all open hemispheres containing X. We claim that $P(\hat{X}) = \mathrm{Conv}(P(X))$. Recall that, by Theorem 5.2, $\mathrm{Conv}(P(X))$ is the intersection of all open half-spaces containing $P(X)$. Choose $x \in \hat{X}$. Let H be an open half-space in $\mathbb{R}^n$ containing $P(X)$. Observe that $X \subseteq P^{-1}(H)$. By Lemma 5.9, there exists an open hemisphere S such that $P^{-1}(H) = S \cap \Sigma^n_-$. By definition of $\hat{X}$, $x \in S, \Sigma^n_-$. In particular, $x \in S \cap \Sigma^n_- = P^{-1}(H)$ and so $P(x) \in H$. Since H is an arbitrary open half-space containing $P(X)$, it follows by Theorem 5.2 that $P(x) \in \mathrm{Conv}(P(X))$. Conversely, choose $x \in \mathrm{Conv}(P(X))$. Let S be an open hemisphere containing X. Observe that $P(X) \subseteq P(S \cap \Sigma^n_-)$. By Lemma 5.9, $P(S \cap \Sigma^n_-)$ is an open half-space. By definition of $\mathrm{Conv}(P(X))$, $x \in P(S \cap \Sigma^n_-)$. By Lemma 5.6, $P^{-1}(x) \in S \cap \Sigma^n_- \subseteq S$. Since S is an arbitrary open hemisphere containing X it follows by definition of $\hat{X}$ that $P^{-1}(x) \in \hat{X}$ and so $x \in P(\hat{X})$. We conclude that $P(\hat{X}) = \mathrm{Conv}(P(X))$. Thus, by Lemma 5.10, $P(\hat{X}) = \mathrm{Conv}(P(X)) = P(\mathrm{Conv}(X))$ and so by Lemma 5.6, $\hat{X} = \mathrm{Conv}(X)$ as desired. □

5.3 Duality.

Let X be a subset of Σ^n. We define the **dual** subset X^* to X by:

$$X^* = \{\mathsf{M} \in \Sigma^n \mid \langle \mathsf{N}, \mathsf{M} \rangle < 0 \ \forall \ \mathsf{N} \in X\}.$$

Lemma 5.14

If X is non-empty, then X^* is strictly contained in a hemisphere. Conversely, if X is strictly contained in a hemisphere, then X^* is non-empty.

Proof: Suppose X is non-empty. Choose $\mathsf{N} \in X$. By definition, for all $\mathsf{M} \in X^*$, $\langle \mathsf{M}, \mathsf{N} \rangle < 0$ and so X^* is strictly contained in a hemisphere as desired. Now suppose that X is strictly contained in a hemisphere. Let $\mathsf{M} \in \Sigma^n$ be such that $\langle \mathsf{N}, \mathsf{M} \rangle < 0$ for all $\mathsf{N} \in X$. By definition, $M \in X^*$ and so X^* is non-empty, as desired. □

Lemma 5.15

If X is closed, then X^* is open. If X is open, then X^* is closed.

Proof: Suppose X is closed. Choose $\mathsf{M} \in X^*$. By compactness of X, there exists $\epsilon > 0$ such that $\langle \mathsf{N}, \mathsf{M} \rangle \leqslant -\epsilon$ for all N in X. If $\mathsf{M}' \in B_\epsilon(\mathsf{M}) \cap \Sigma^n$, then, for all $\mathsf{N} \in X$, bearing in mind the Cauchy-Schwarz inequality:

$$\begin{aligned}\langle \mathsf{N}, \mathsf{M}' \rangle &= \langle \mathsf{N}, \mathsf{M}' - \mathsf{M} \rangle + \langle \mathsf{N}, \mathsf{M} \rangle \\ &\leqslant \|\mathsf{N}\| \|\mathsf{M}' - \mathsf{M}\| + \langle \mathsf{N}, \mathsf{M} \rangle \\ &< \epsilon - \epsilon \\ &= 0.\end{aligned}$$

It follows that $B_\epsilon(\mathsf{M}) \cap \Sigma^n \subseteq X^*$ and so X^* is open as desired.

Suppose X is open. Choose $\mathsf{M} \in \Sigma^n \setminus X^*$. There exists $\mathsf{N} \in X$ such that $\langle \mathsf{N}, \mathsf{M} \rangle \geqslant 0$. For all $s > 0$, we denote $\mathsf{N}_s = (\mathsf{N} + s\mathsf{M})/\|\mathsf{N} + s\mathsf{M}\|$. For all $s > 0$:

$$\begin{aligned}\langle \mathsf{N}_s, \mathsf{M} \rangle &= \tfrac{1}{\|\mathsf{N}+s\mathsf{N}\|} \langle \mathsf{N} + s\mathsf{M}, \mathsf{M} \rangle \\ &\geqslant \tfrac{s}{\|\mathsf{N}+s\mathsf{N}\|} \\ &> 0.\end{aligned}$$

Since X is open, for sufficiently small s, $\mathsf{N}_s \in X$. Upon replacing N with N_s, we may therefore suppose that $\langle \mathsf{N}, \mathsf{M} \rangle > 0$. Denote $\epsilon = \langle \mathsf{N}, \mathsf{M} \rangle$. If $\mathsf{M}' \in B_\epsilon(\mathsf{M}) \cap \Sigma^n$, then, bearing in mind the Cauchy-Schwarz inequality:

$$\begin{aligned}\langle \mathsf{N}, \mathsf{M}' \rangle &= \langle \mathsf{N}, \mathsf{M}' - \mathsf{M} \rangle + \langle \mathsf{N}, \mathsf{M} \rangle \\ &\geqslant -\|\mathsf{N}\| \|\mathsf{M}' - \mathsf{M}\| + \langle \mathsf{N}, \mathsf{M} \rangle \\ &\geqslant -\epsilon + \epsilon \\ &= 0.\end{aligned}$$

It follows that $B_\epsilon(\mathsf{M}) \cap \Sigma^n \subseteq \Sigma^n \setminus X^*$. $\Sigma^n \setminus X^*$ is thus open and it follows that X^* is closed as desired. □

Lemma 5.16

If X is non-empty, then X^* is convex.

Proof: By Lemma 5.14, X^* is strictly contained in a hemisphere. Choose $\mathsf{M}_0, \mathsf{M}_1 \in X^*$. By Lemma 5.5, M_0 and M_1 are not antipodal. Let M be the great-circular arc joining M_0 to M_1. For all $\mathsf{N} \in X$ and for all $s \in [0, 1]$:

$$\begin{aligned}\langle \mathsf{N}, \mathsf{M}(s) \rangle &= \tfrac{1}{\|(1-s)\mathsf{M}_0+s\mathsf{M}_1\|} \langle \mathsf{N}, (1-s)\mathsf{M}_0 + s\mathsf{M}_1 \rangle \\ &= \tfrac{(1-s)}{\|(1-s)\mathsf{M}_0+s\mathsf{M}_1\|} \langle \mathsf{N}, \mathsf{M}_0 \rangle + \tfrac{s}{\|(1-s)\mathsf{M}_0+s\mathsf{M}_1\|} \langle \mathsf{N}, \mathsf{M}_1 \rangle \\ &> 0.\end{aligned}$$

It follows that $\mathsf{M}(s)$ is an element of X^* for all $s \in [0, 1]$. Since $\mathsf{M}_0, \mathsf{M}_1 \in X^*$ are arbitrary, we deduce that X^* is convex as desired. □

Lemma 5.17

If X is strictly contained in a hemisphere, then $X^{**} = \text{Conv}(X)$.

Proof: By Lemma 5.13, $\text{Conv}(X)$ is the intersection of all open hemispheres containing X. By definition, for $\mathsf{M} \in \Sigma^n$, X is contained in $\Sigma_-(\mathsf{M})$ if and only if M is an element of X^*. Hence:

$$X^{**} = \bigcap_{\mathsf{M} \in X^*} \Sigma_-(\mathsf{M}) = \bigcap_{X \subseteq \Sigma_-(\mathsf{M})} \Sigma_-(\mathsf{M}) = \text{Conv}(X),$$

as desired. □

Lemma 5.18

If X_1 and X_2 are both strictly contained in the same hemisphere, then $(X_1 \cup X_2)^* = X_1^* \cap X_2^*$.

Proof: Suppose $\mathsf{M} \in (X_1 \cup X_2)^*$. Then $\langle \mathsf{N}, \mathsf{M} \rangle < 0$ for all $\mathsf{N} \in X_1 \cup X_2$ and so $\mathsf{M} \in X_1^* \cap X_2^*$. Conversely, if $\mathsf{M} \in X_1^* \cap X_2^*$, then $\langle \mathsf{N}, \mathsf{M} \rangle < 0$ for all $\mathsf{N} \in X_1 \cup X_2$ and so $\mathsf{M} \in (X_1 \cup X_2)^*$. It follows that these two sets coincide as desired. □

Lemma 5.19

Let X_1 and X_2 be convex subsets of Σ^n contained in the same hemisphere. If X_1 and X_2 are either both open or both closed, then $(X_1 \cap X_2)^* = \text{Conv}(X_1^* \cup X_2^*)$.

Proof: Suppose that X_1 and X_2 are open (resp. closed). Since they are both convex, by Lemma 5.12, $X_1 = \text{Conv}(X_1)$ and $X_2 = \text{Conv}(X_2)$. By Lemma 5.17, $X_1 = \text{Conv}(X_1) = X_1^{**}$ and $X_2 = \text{Conv}(X_2) = X_2^{**}$. We denote $Y_1 = X_1^*$ and $Y_2 = X_2^*$. Bearing in mind Lemma 5.18:

$$(X_1^* \cup X_2^*)^* = (Y_1 \cup Y_2)^* = Y_1^* \cap Y_2^* = X_1 \cap X_2.$$

Thus, by Lemma 5.17:

$$\text{Conv}(X_1^* \cup X_2^*) = (X_1^* \cup X_2^*)^{**} = (X_1 \cap X_2)^*,$$

as desired. □

5.4 Links.

Let K be a compact, convex set with non-trivial interior. Let x be a boundary point of K. For $r > 0$, we define $\mathcal{L}_r(x; K) \subseteq \Sigma^n$, the **link** of K of radius r about x by:

$$\mathcal{L}_r(x; K) = \{\mathsf{N} \mid x + r\mathsf{N} \in K^o\}.$$

When there is no ambiguity, we denote $\mathcal{L}_r(x) = \mathcal{L}_r(x; K)$.

Lemma 5.20

For every boundary point x of K and for all $r < s$, $\mathcal{L}_s(x) \subseteq \mathcal{L}_r(x)$.

Proof: Indeed, choose $\mathsf{N} \in \mathcal{L}_s(x)$. Then $x + s\mathsf{N} \in K^o$. Viewing N as an element of $\mathbb{R}^{n+1}$, there exists $\delta > 0$ such that for all $V \in B_\delta(0)$, $x + s\mathsf{N} + V \in K$. By convexity, for all $V \in B_{r\delta/s}(0)$:

$$x + r\mathsf{N} + V = (1 - \frac{r}{s})x + \frac{r}{s}(x + s\mathsf{N} + \frac{s}{r}V) \in K.$$

It follows that $x + r\mathsf{N} \in K^o$, and so $\mathsf{N} \in \mathcal{L}_r(x)$. Since $\mathsf{N} \in \mathcal{L}_s(x)$ is arbitrary, it follows that $\mathcal{L}_s(x) \subseteq \mathcal{L}_r(x)$ as desired. □

$(\mathcal{L}_r(x))_{r>0}$ therefore constitutes an increasing, nested family of open sets. We define $\mathcal{L}(x; K) \subseteq \Sigma^n$, the **link** of K at x by:

$$\mathcal{L}(x; K) = \mathop{\cup}_{r>0} \mathcal{L}_r(x; K).$$

When there is no ambiguity, we denote $\mathcal{L}(x) = \mathcal{L}(x; K)$. Since it is the union of a family of open sets, $\mathcal{L}(x)$ is also open.

We recall from Section 4 that for all $x \in \partial K$, $\mathcal{N}(x; K)$ is the set of supporting normal vectors to K at x.

Lemma 5.21

Let K be a compact, convex set with non-trivial interior. Then for every boundary point x of K, $\mathcal{N}(x; K) = \mathcal{L}(x; K)^*$.

Proof: Suppose that $\mathsf{N} \in \mathcal{N}(x; K)$. By definition of supporting normals, for all $z \in K$, $\langle z - x, \mathsf{N}\rangle \leqslant 0$. In particular, for all $z \in K^o$, $\langle z - x, \mathsf{N}\rangle < 0$. Choose $\mathsf{M} \in \mathcal{L}(x; K)$. Choose $r > 0$ such that $\mathsf{M} \in \mathcal{L}_r(x; K)$. Then $x + r\mathsf{M} \in K^o$ and so:

$$r\langle \mathsf{M}, \mathsf{N}\rangle = \langle (x + r\mathsf{M}) - x, \mathsf{N}\rangle < 0.$$

In particular $\langle \mathsf{M}, \mathsf{N}\rangle < 0$ and so, since $\mathsf{M} \in \mathcal{L}(x; K)$ is arbitrary, it follows that $\mathsf{N} \in \mathcal{L}(x; K)^*$. Conversely, choose $\mathsf{N} \in \mathcal{L}(x; K)^*$. Choose $y \in K^o$. Denote $r = \|y - x\|$. Then $(y - x)/r \in \mathcal{L}_r(x; K)$. Thus:

$$\langle \mathsf{N}, y - x\rangle = r\langle \mathsf{N}, (y - x)/r\rangle < 0,$$

Thus, since $x \in K^o$ is arbitrary, it follows by continuity that for all $x \in \overline{K^o}$:

$$\langle \mathsf{N}, y - x\rangle \leqslant 0.$$

However, since K has non-trivial interior, by Lemma 5.3, $K = \overline{K^o}$, and so $\langle \mathsf{N}, y - x\rangle \leqslant 0$ for all $y \in K$. It follows that $\mathsf{N} \in \mathcal{N}(x; K)$ and the two sets therefore coincide as desired. □

Lemma 5.22

For every compact, convex set K with non-trivial interior, and for every boundary point x of K, $\mathcal{N}(x;K)$ is closed, convex and strictly contained in a hemisphere.

Proof: By Lemma 5.21, $\mathcal{N}(x;K) = \mathcal{L}(x;K)^*$. Since K has non-trivial interior, $\mathcal{L}(x;K)$ is non-empty, and so, by Lemma 5.14, $\mathcal{N}(x;K)$ is strictly contained in a hemisphere. Since $\mathcal{L}(x;K)$ is open, by Lemma 5.11, $\mathcal{N}(x;K)$ is closed. By Lemma 5.16, $\mathcal{N}(x;K)$ is convex, and this completes the proof. □

Theorem 5.23

Let K_1 and K_2 be two compact, convex sets whose intersection has non-trivial interior. Choose $x \in \partial(K_1 \cap K_2)$. Then:

(1) if $x \in (\partial K_1) \cap K_2^o$, then $\mathcal{N}(x;K_1 \cap K_2) = \mathcal{N}(x;K_1)$;

(2) if $x \in K_1^o \cap (\partial K_2)$, then $\mathcal{N}(x;K_1 \cap K_2) = \mathcal{N}(x;K_2)$; and

(3) if $x \in (\partial K_1) \cap (\partial K_2)$, then $\mathcal{N}(x;K_1 \cap K_2) = \mathrm{Conv}(\mathcal{N}(x;K_1) \cup \mathcal{N}(x;K_2))$.

Proof: Cases (1) and (2) follow from Lemma 4.9. Hence suppose that $x \in (\partial K_1) \cap (\partial K_2)$. Observe that:

$$\mathcal{L}(x;K_1 \cap K_2) = \mathcal{L}(x;K_1) \cap \mathcal{L}(x;K_2).$$

Thus, bearing in mind Lemmas 5.19 and 5.21:

$$\begin{aligned} \mathcal{N}(x;K_1 \cap K_2) &= \mathcal{L}(x;K_1 \cap K_2)^* \\ &= (\mathcal{L}(x;K_1) \cap \mathcal{L}(x;K_2))^* \\ &= \mathrm{Conv}(\mathcal{L}(x;K_1)^* \cup \mathcal{L}(x;K_2)^*) \\ &= \mathrm{Conv}(\mathcal{N}(x;K_1) \cup \mathcal{N}(x;K_2)), \end{aligned}$$

as desired. □

6

Weak Barriers

We use an approximation technique to solve a general Plateau problem in Euclidean Space. Let X be a compact set whose convex hull has non-trivial interior. This set replaces the prescribed boundary. Let k be a positive real number. We define a weak barrier of gaussian curvature at least k to be any compact, convex set containing X which is, roughly speaking, a limit in the Hausdorff sense of a sequence of compact, convex sets with smooth boundary and gaussian curvature at least k. We show that the set of weak barriers is closed under finite intersections and passage to the limit in the Hausdorff topology. This allows us to show that within the family of those weak barriers of gaussian curvature at least k which contain X, there exists a unique element K_0 minimising volume. Refining the proof that the family of weak barriers is closed under finite intersection, we define an excision operation which, bearing in mind Theorem 4.29, allows us to show that $(\partial K_0) \setminus X$ is only singular at points where it possesses the local geodesic property. Finally, an ad-hoc argument allows to exclude such points, from which we deduce that $(\partial K_0) \setminus X$ is smooth and has constant gaussian curvature equal to k. This construction allows to achieve our stated objective, being the proof Theorem 1.2, which solves a general Plateau problem for hypersurfaces of constant gaussian curvature Euclidean Space. We leave the enthusiastic reader to investigate how these techniques readily extend to the case where the curvature is prescribed by position in the ambient space as well the cases where the ambient space is either Hyperbolic Space or the sphere.

Proving the closure of the family of weak barriers under finite intersections is rather technical. Sections 6.1 to 6.3 inclusive are devoted to developing the necessary preliminary material for reaching this conclusion, which follows from a detailed study of the second derivatives of distance functions to convex sets (which are defined almost everywhere) and their behaviour under the application of smoothing functions. The most important result of these sections is Theorem 6.34 which shows that if two convex sets have smooth boundary of gaussian curvature at least k, and if their intersection has non-trivial interior, then this intersection can be approximated by convex sets with smooth boundary of curvature at least $k - \epsilon$, where $\epsilon > 0$ may be chosen as small as we wish. This result is useful for constructing convex barriers in more general settings, and we leave the enthusiastic reader to verify that it continues to hold in any riemannian manifold.

6.1 Distance Functions.

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let $d_K : \mathbb{R}^{n+1} \to [0, \infty[$ be the distance in $\mathbb{R}^{n+1}$ to K:

$$d_K(x) = \inf_{y \in K} \|x - y\|$$

We first aim to show that d_K is convex.

Lemma 6.1

The function $x \mapsto \|x\|$ is convex. The function $x \mapsto \|x\|^2$ is strictly convex.

Proof: Choose $x, y \in \mathbb{R}^{n+1}$. Then, bearing in mind the Cauchy-Schwarz inequality, for all t in the open interval $]0, 1[$:

$$\begin{aligned} \|(1-t)x + ty\|^2 &= (1-t)\|x\|^2 + 2(1-t)t\langle x, y\rangle + t^2\|y\|^2 \\ &\leqslant (1-t)\|x\|^2 + 2(1-t)t\|x\|\|y\| + t^2\|y\|^2 \\ &= ((1-t)\|x\| + t\|y\|)^2. \end{aligned} \quad (C)$$

Taking square roots, we conclude that $\|(1-t)x + ty\| \leqslant (1-t)\|x\| + t\|y\|$ and it follows that the function $x \mapsto \|x\|$ is convex as desired. Observe that since $t \neq 0, 1$, equality holds in (C) if and only if x and y are colinear and point in the same direction. Moreover, by strict convexity of the function $t \mapsto t^2$, for all $t \in]0, 1[$:

$$((1-t)\|x\| + t\|y\|)^2 \leqslant (1-t)\|x\|^2 + t\|y\|^2,$$

with equality if and only if $\|x\| = \|y\|$. It follows that for all $t \in]0, 1[$:

$$\|(1-t)x + ty\|^2 \leqslant (1-t)\|x\|^2 + t\|y\|^2,$$

with equality if and only if x and y are colinear, point in the same direction and have the same norm. That is, equality holds if and only if $x = y$. Thus, for all $x \neq y$ and for all $t \in]0, 1[$:

$$\|(1-t)x + ty\|^2 < (1-t)\|x\|^2 + t\|y\|^2,$$

and so the mapping $x \mapsto \|x\|^2$ is strictly convex, as desired. $\square$

Lemma 6.2

If K is a compact, convex subset of $\mathbb{R}^{n+1}$, then the function d_K is a convex function over $\mathbb{R}^{n+1}$.

Proof: Indeed, choose $x_0, x_1 \in \mathbb{R}^{n+1}$. For all $t \in [0,1]$, define $x_t = (1-t)x_0 + tx_1$. By compactness, there exists $y_0, y_1 \in K$ minimising the distance to x_0 and x_1 respectively. For all $t \in [0,1]$, denote $y_t = (1-t)y_0 + ty_1$. By convexity, $y_t \in K$ for all t, and so $d_K(x_t) \leqslant \|x_t - y_t\|$. However, for all $t \in [0,1]$:

$$\|x_t - y_t\| = \|(1-t)(x_0 - y_0) + t(x_1 - y_1)\|.$$

It follows from Lemma 6.1 that $t \mapsto \|x_t - y_t\|$ is a convex function of t. In particular, for all $t \in [0,1]$:

$$d_K(x_t) \leqslant \|x_t - y_t\| \leqslant (1-t)\|x_0 - y_0\| + t\|x_1 - y_1\| = (1-t)d_K(x_0) + td_K(x_1).$$

Since $x_0, x_1 \in \mathbb{R}^{n+1}$ and $t \in [0,1]$ are arbitrary, it follows that d_K is convex as desired. □

We now aim to show that there is a well-defined closest point projection from $\mathbb{R}^{n+1} \setminus K$ onto K.

Lemma 6.3

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Choose $x \in \mathbb{R}^{n+1} \setminus K$. There is at most one point y in the boundary of K with the property that $x = y + t\mathsf{N}$ for some $t > 0$ and for some supporting normal N to K at y.

Proof: Suppose the contrary. Let y and y' be two such boundary points of K. Let N and N' be supporting normals to K at y and y' respectively and let $t, t' > 0$ be such that $x = y + t\mathsf{N} = y' + t'\mathsf{N}'$. By definition of the supporting normal:

$$\langle y' - y, \mathsf{N}\rangle, \qquad \langle y - y', \mathsf{N}'\rangle \leqslant 0.$$

In particular:

$$\langle y' - y, x - y\rangle, \qquad \langle y - y', x - y'\rangle \leqslant 0.$$

Summing these two relations yields $\|y' - y\|^2 \leqslant 0$. Thus $\|y' - y\| = 0$ and so $y' = y$, and uniqueness follows. □

Lemma 6.4

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. For all $x \in \mathbb{R}^{n+1}$, there exists a unique point $y \in K$ minimising distance to x.

Proof: Choose $x \in \mathbb{R}^{n+1}$. Since K is compact, there exists $y \in K$ minimising distance to x. If $x \in K$, then $y = x$ and so y is unique. If $x \in \mathbb{R}^{n+1} \setminus K$, then we denote $\mathsf{N} = (x - y)/\|x - y\|$. By Lemma 4.5, y is a boundary point of K and N is a supporting normal to K at y. In particular $x = y + \|x - y\|\mathsf{N}$. By Lemma 6.3, there can only exist one such boundary point, and uniqueness follows. □

We define $\Pi_K : \mathbb{R}^{n+1} \to K$ to be the nearest point projection. We show that Π_K is related to the derivative of d_K. We first prove a preliminary result:

Lemma 6.5

Let $\varphi_1 \leqslant \varphi_2 \leqslant \varphi_3 : B_1(0) \to \mathbb{R}$ be continuous functions. Suppose that φ_1 and φ_3 are differentiable at 0. If $\varphi_1(0) = \varphi_3(0)$ and $D\varphi_1(0) = D\varphi_3(0)$, then φ_2 is also differentiable at 0 and $D\varphi_2(0) = D\varphi_1(0) = D\varphi_3(0)$.

Proof: We denote $A = D\varphi_1(0) = D\varphi_3(0)$. Choose $\epsilon > 0$, by definition of differentiability, there exists $\delta > 0$ such that if V is any vector with $\|V\| \leqslant \delta$:

$$\varphi_1(V) - \varphi_1(0) - A(V) \geqslant -\epsilon\|V\|.$$

Thus, by hypothesis:

$$\varphi_2(V) - \varphi_2(0) - A(V) \geqslant -\epsilon\|V\|.$$

Likewise, reducing δ if necessary:

$$\varphi_2(V) - \varphi_2(0) - A(V) \leqslant \epsilon\|V\|.$$

and so:

$$|\varphi_2(V) - \varphi_2(0) - A(V)| \leqslant \epsilon\|V\|.$$

Since $\epsilon > 0$ and $\|V\| \leqslant \delta$ are arbitrary, if follows that φ_2 is differentiable at 0 with derivative equal to A as desired. □

Lemma 6.6

If K be a compact, convex subset of $\mathbb{R}^{n+1}$, then d_K is differentiable at every point of $\mathbb{R}^{n+1} \setminus K$ and, for all $x \in \mathbb{R}^{n+1} \setminus K$:

$$\Pi_K(x) = x - d_K(x)Dd_K(x).$$

Remark: *This formula allows us in particular to deduce higher regularity of d_K from the regularity of Π_K.*

Proof: Choose $x \in \mathbb{R}^{n+1} \setminus K$. Denote $\mathsf{N} = (x - \Pi_K(x))/\|x - \Pi_K(x)\|$. By Lemma 4.5, N is a supporting normal to K at $\Pi_K(x)$. By definition of Π_K, for all $y \in \mathbb{R}^{n+1}$, $\Pi_K(y) \in K$. Thus, by definition of supporting normals, for all $y \in \mathbb{R}^{n+1}$, $\langle \Pi_K(y) - \Pi_K(x), \mathsf{N}\rangle \leqslant 0$. Thus, bearing in mind the Cauchy-Schwarz inequality and the fact that N has unit length, for all y:

$$\begin{aligned} d_K(y) &= \|y - \Pi_K(y)\| \\ &\geqslant \langle y - \Pi_K(y), \mathsf{N}\rangle \\ &= \langle y - \Pi_K(x), \mathsf{N}\rangle + \langle \Pi_K(x) - \Pi_K(y), \mathsf{N}\rangle \\ &\geqslant \langle y - \Pi_K(x), \mathsf{N}\rangle. \end{aligned}$$

On the other hand, $d_K(y) \leqslant d(y, \Pi_K(x))$. Combining these relations yields:

$$\langle y - \Pi_K(x), \mathsf{N}\rangle \leqslant d_K(y) \leqslant d(y, \Pi_K(x)).$$

Observe that the first and the last functions in this inequality are smooth at x. Moreover, they coincide at x and have derivative equal to N at that point. It follows by Lemma 6.5 that d_K is differentiable at x and $Dd_K(x) = \mathsf{N}$. In particular, bearing in mind the definition of N:

$$\Pi_K(x) = x - \|x - \Pi_K(x)\|\mathsf{N} = x - d_K(x)Dd_K(x),$$

as desired. □

Lemma 6.7

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. If x is a point of $\mathbb{R}^{n+1} \setminus K$, then d_K is twice differentiable at x if and only Π_K is differentiable at x. Moreover, at such a point, for all vectors V and W:

$$\langle D\Pi_K(x)(V), W\rangle = \langle \pi(V), \pi(W)\rangle - d_K(x)D^2 d_K(x)(V, W),$$

where π is the orthogonal projection from $\mathbb{R}^{n+1}$ onto $\langle Dd_K(x)\rangle^\perp$.

Proof: By Lemma 6.6, for all $x \in \mathbb{R}^{n+1} \setminus K$, d_K is differentiable at x and $\Pi_K(x) = x - d_K(x)Dd_K(x)$. Thus, by the product rule, if d_K is twice differentiable at x, then Π_K is differentiable at x. Conversely, $Dd_K(x) = (x - \Pi_K(x))/d_K(x)$. Thus, since $d_K(x)$ does not vanish over $\mathbb{R}^{n+1} \setminus K$, by the quotient rule, if Π_K is differentiable at x, then so is Dd_K. Finally, if Dd_K and Π_K are differentiable at x, then differentiating the relation in Lemma 6.6 yields:

$$\begin{aligned} \langle D\Pi_K(x)(V), W\rangle &= \langle V, W\rangle - \langle V, Dd_K(x)\rangle\langle W, Dd_K(x)\rangle - d_K(x)D^2 d_K(x)(V, W) \\ &= \langle \pi(V), \pi(W)\rangle - d_K(x)D^2 d_K(x)(V, W), \end{aligned}$$

as desired. □

Having established the relationship between Dd_K and Π_K, we determine the regularity of Π_K:

Lemma 6.8

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Then Π_K is 1-Lipschitz.

Proof: If $x, x' \in K$, then $\Pi_K(x) = x$ and $\Pi_K(x') = x'$. In particular, $\|\Pi_K(x) - \Pi_K(x')\| = \|x - x'\|$ as desired. If $x \in K$ and if $x' \in \mathbb{R}^{n+1} \setminus K$, then $\pi_K(x) = x$. Define $y' = \Pi_K(x')$. By Lemma 4.5, $(x' - y')/\|x' - y'\|$ is a supporting normal to K at y'. In particular, by definition of supporting normals $\langle x - y', x' - y'\rangle \leqslant 0$. Thus:

$$\begin{aligned} \|x - x'\|^2 &= \|(x - y') - (x' - y')\|^2 \\ &= \|x - y'\|^2 - 2\langle x - y', x' - y'\rangle + \|x' - y'\|^2 \\ &\geqslant \|x - y'\|^2 + \|x' - y'\|^2 \\ &\geqslant \|x - y'\|^2. \end{aligned}$$

Taking square roots yields $\|\Pi_K(x) - \Pi_K(x')\| \leqslant \|x - x'\|$ as desired. Finally, choose $x, x' \in \mathbb{R}^{n+1} \setminus K$. Denote $y = \Pi_K(x)$, $y' = \Pi_K(x')$. By Lemma 4.5, $(x - y)/\|x - y\|$ and $(x' - y')/\|x' - y'\|$ are the supporting normals to K at y and y' respectively. In particular, by definition of supporting normals:

$$\langle y' - y, x - y\rangle, \langle y - y', x' - y'\rangle \leqslant 0.$$

Consequently:

$$\begin{aligned} \langle x - x', y - y'\rangle &= \langle x - y, y - y'\rangle + \langle y - y', y - y'\rangle + \langle y' - x', y - y'\rangle \\ &\geqslant \|y - y'\|^2. \end{aligned}$$

Thus, bearing in mind the Cauchy-Schwarz inequality:

$$\|y - y'\|^2 \leqslant \langle x - x', y - y'\rangle \leqslant \|x - x'\|\|y - y'\|,$$

and so

$$\|y - y'\|(\|x - x'\| - \|y - y'\|) \geqslant 0.$$

It follows that $\|y - y'\| \leqslant \|x - x'\|$, as desired. □

Lemma 6.9

If K is a compact, convex subset of $\mathbb{R}^{n+1}$, then Π_K is differentiable almost everywhere. Moreover, the pointwise derivative of Π_K coincides with its distributional derivative, and $\|D\Pi_K(x)\|_{L^\infty} \leqslant 1$.

Proof: Since Π_K is Lipschitz, it follows from Rademacher's Theorem (c.f. Theorem 5.2 of [20]) that Π_K is differentiable almost everywhere and, moreover, that the pointwise derivative of Π_K coincides with its distributional derivative. Now choose $x \in \mathbb{R}^{n+1}$ such that Π_K is differentiable at x. For any vector V:

$$D\Pi_K(x)(V) = \operatorname*{Lim}_{t\rightarrow 0, t\neq 0} \frac{1}{t}(\Pi_K(x+tV) - \Pi_K(x))$$

Thus, bearing in mind Lemma 6.8 and the fact that the function $x \mapsto \|x\|$ is continuous:

$$\begin{aligned}\|D\Pi_K(x)(V)\| &= \| \operatorname{Lim}_{t\rightarrow 0,t\neq 0} \tfrac{1}{t}(\Pi_K(x+tV) - \Pi_K(x))\| \\ &= \operatorname{Lim}_{t\rightarrow 0,t\neq 0} \tfrac{1}{t}\|\Pi_K(x+tV) - \Pi_K(x)\| \\ &\leqslant \|V\|.\end{aligned}$$

Since V is arbitrary, it follows that $\|D\Pi_K(x)\| \leqslant 1$ and since x is arbitrary, we conclude that $\|D\Pi_K\|_{L^\infty} \leqslant 1$ as desired. □

Lemma 6.10

If K is a compact, convex subset of $\mathbb{R}^{n+1}$, then d_K is twice differentiable almost everywhere in $\mathbb{R}^{n+1} \setminus K$. Moreover, the pointwise second derivative of d_K coincides with its second-order distributional derivative, and if d_K is twice differentiable at $x \in \mathbb{R}^{n+1} \setminus K$, then $\|D^2 d_K(x)\| \leqslant 2/d_K(x)$.

Proof: By Lemmas 6.7 and 6.9, d_K is almost everywhere twice differentiable in $\mathbb{R}^{n+1} \setminus K$. By Lemma 6.6, for all $x \in \mathbb{R}^{n+1} \setminus K$, $Dd_K(x) = (x - \Pi_K(x))/d_K(x)$. Thus, since $d_K(x)$ never vanishes over this set, it follows from the quotient rules for pointwise derivatives and for distributional derivatives that the pointwise derivative of Dd_K coincides with its distributional derivative, and so the pointwise second-order derivative of d_K coincides with its second-order distributional derivative as desired. Finally, if Dd_K and $D\Pi_K$ are differentiable at x, for all vectors V and W:

$$D^2 d_K(V,W) = \frac{1}{d_K(x)}(\langle \pi(V), \pi(W)\rangle - \langle D\Pi_K(x)(V), W\rangle),$$

where π is the orthogonal projection from $\mathbb{R}^{n+1}$ onto $\langle Dd_K(x)\rangle^\perp$. Since both π and $D\Pi_K(x)$ have norm 1, we have:

$$\left|D^2 d_K(x)(V,W)\right| \leqslant \frac{2}{d_K(x)}\|V\|\|W\|,$$

and so $\|D^2 d_K(x)\| \leqslant 2/d_K(x)$ as desired. □

We also show that the second derivatives of d_K are almost everywhere symmetric:

Lemma 6.11

For almost all $x \in \mathbb{R}^{n+1} \setminus K$, d_K is twice differentiable at x and its second derivative is symmetric at that point.

Proof: By Lemma 6.10, d_K has L^∞_{loc} second-order, distributional derivatives over $\mathbb{R}^{n+1} \setminus K$. We denote the second-order distributional derivative of d_K by A. Then, for any $\varphi \in C^\infty_{\text{loc}}(\mathbb{R}^{n+1} \setminus K)$, and for all $1 \leqslant i, j \leqslant n$, by definition of distributional derivatives:

$$\begin{aligned}\int_{\mathbb{R}^{n+1}\setminus K} A(x)(\partial_i, \partial_j)\varphi(x)\mathrm{dVol}_x &= \int_{\mathbb{R}^{n+1}\setminus K} d_K(x)D^2\varphi(x)(\partial_j, \partial_i)\mathrm{dVol}_x \\ &= \int_{\mathbb{R}^{n+1}\setminus K} d_K(x)D^2\varphi(x)(\partial_i, \partial_j)\mathrm{dVol}_x \\ &= \int_{\mathbb{R}^{n+1}\setminus K} A(x)(\partial_j, \partial_i)\varphi(x)\mathrm{dVol}_x.\end{aligned}$$

Since $\varphi \in C^\infty_{\text{loc}}(\mathbb{R}^{n+1} \setminus K)$ is arbitrary, it follows that $A(x)(\partial_i, \partial_j) = A(x)(\partial_j, \partial_i)$ for almost all $x \in \mathbb{R}^{n+1} \setminus K$. Since $1 \leqslant i, j \leqslant n$ is arbitrary, it follows that $A(x)$ is symmetric for almost all $x \in \mathbb{R}^{n+1} \setminus K$. Thus, by Lemma 6.10, for almost all $x \in \mathbb{R}^{n+1} \setminus K$, d_K is twice differentiable at x, and for all $1 \leqslant i, j \leqslant n$:

$$D^2 d_K(x)(\partial_i, \partial_j) = A(x)(\partial_i, \partial_j) = A(x)(\partial_j, \partial_i) = D^2 d_K(x)(\partial_j, \partial_i).$$

$D^2 d_K(x)$ is therefore symmetric for almost all $x \in \mathbb{R}^{n+1} \setminus K$ where it is defined as desired. □

6.2 Convex Sets with Smooth Boundary.

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let U be an open subset of $\mathbb{R}^{n+1}$. We denote $U(K) = U \cap (\partial K)$, and we suppose that $U(K)$ is smooth. We henceforth use the terminology of riemannian geometry (c.f. [7]). Let $\mathsf{N} : U(K) \to \Sigma^n$ be the outward-pointing, unit, **normal** vector field over $U(K)$. Let A be the **shape operator** of $U(K)$ associated to this normal. That is, for all $x \in U(K)$ and for any vector V tangent to $U(K)$ at x, $A(x)(V) = D\mathsf{N}(x)(V)$.

If $M \in \mathrm{Symm}(2, \mathbb{R}^{n+1})$ is a symmetric matrix over $\mathbb{R}^{n+1}$, and if E is any subspace of $\mathbb{R}^{n+1}$, we denote by $\mathrm{Det}(M; E)$ the determinant of the restriction of M to E. We are interested in estimating $\mathrm{Det}(D^2 d_K; \langle Dd \rangle^\perp)$ near $U(K)$. This quantity will be used in the sequel to estimate the gaussian curvature of smooth hypersurfaces approximating K.

In this section, we study the regularity of d_K and Π_K over $\Pi_K^{-1}(U(K))$. We define $\Phi : U(K) \times [0, \infty[\to \mathbb{R}^{n+1}$ by $\Phi(x, t) = x + t\mathsf{N}(x)$.

Lemma 6.12

Φ is a smooth diffeomorphism from $U(K) \times [0, \infty[$ onto $\Pi_K^{-1}(U(K))$.

Proof: We first show that $\mathrm{Im}(\Phi) = \Pi_K^{-1}(U(K))$. Indeed, choose $(x, t) \in U(K) \times [0, \infty[$. Then $\Phi(x, t) = x + t\mathsf{N}(x)$. By Lemma 4.5, x minimises the distance to $\Phi(x, t)$ in K and so $x = (\Pi_K \circ \Phi)(x, t)$. In particular, $\Phi(x, t) \in \Pi_K^{-1}(U(K))$. Since $(x, t) \in U(K) \times [0, \infty[$ was arbitrary, it follows that $\mathrm{Im}(\Phi) \subseteq \Pi_K^{-1}(U(K))$. Conversely, choose $y \in \Pi_K^{-1}(U(K))$. Denote $x = \Pi_K(y) \in U(K)$. By definition, x minimises the distance in K to y. If $y \in K$, then $y = x = \Phi(x, 0)$ so that $y \in \mathrm{Im}(\Phi)$ as desired. Otherwise, $y \in \mathbb{R}^{n+1} \setminus K$. By Lemma

4.5, there exists $t \geqslant 0$ such that $y = x + t\mathsf{N}(x) = \Phi(x,t)$ and so $y \in \mathrm{Im}(\Phi)$ as desired. Since $y \in \Pi_K^{-1}(U(K))$ was arbitrary, it follows that $\Pi_K^{-1}(U(K)) \subseteq \mathrm{Im}(\Phi)$, and the two sets therefore coincide as desired.

We now claim that Φ is injective. Indeed, suppose the contrary. There exists $x, x' \in U(K)$ and $t, t' \in [0,\infty[$ such that $x + t\mathsf{N}(x) = x' + t'\mathsf{N}(x')$. It follows from Lemma 6.3 that $x = x'$ and so $t = t'$ as desired. It remains to show that Φ is smooth with smooth inverse. Choose $(x,t) \in U(K) \times [0,\infty[$. We denote by ∂_t the unit vector in the t direction. Observe that:

$$\begin{array}{rll} & D\Phi(x,t)(0,\partial_t) & = \mathsf{N}(x) \\ \Rightarrow & \|D\Phi(x,t)(0,\partial_t)\|^2 & = 1. \end{array}$$

Let X be a tangent vector to ∂K at x. Then:

$$\begin{array}{rll} & D\Phi(x)(X,0) & = X + tD\mathsf{N}(x)(X) \\ & & = X + tA(x)(X) \\ \Rightarrow & \|D\Phi(x)(X,0)\|^2 & = \|(\mathrm{Id} + tA(x))(X)\|^2. \end{array}$$

By convexity, $A(x)$ is non-negative definite, and so:

$$\|D\Phi(x)(X,0)\|^2 = \|(\mathrm{Id} + tA(x))(X)\|^2 \geqslant \|X\|^2.$$

Finally, bearing in mind that $\langle A(x)(X), \mathsf{N}(x)\rangle = 0$:

$$\langle D\Phi(x,t)(X,0), D\Phi(x,t)(0,\partial_t)\rangle = \langle X + tA(x)(X), \mathsf{N}(x)\rangle = 0.$$

If follows that $\|D\Phi(x,t)(V)\|^2 > 0$ for all non-zero V and so $D\Phi(x,t)$ is invertible. Since $(x,t) \in U \times [0,\infty[$ is arbitrary, it follows from the inverse function theorem that Φ is everywhere locally a smooth diffeomorphism. Thus, since Φ is injective, it is a smooth diffeomorphism as desired. □

Lemma 6.13

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let U be an open subset of $\mathbb{R}^{n+1}$. If $U(K)$ is smooth, then Π_K and d_K are smooth functions over $\Pi_K^{-1}(U(K)) \setminus K$. Moreover, for all vectors V and W:

$$\begin{array}{ll} Dd_K(x)(V) & = \langle \mathsf{N}(\Pi_K(x)), V\rangle, \\ D^2 d_K(x)(V,W) & = \langle A(\Pi_K(x))(D\Pi_K(x)(V)), W\rangle. \end{array}$$

Proof: Choose $(x,t) \in U(K) \times [0,\infty[$. Since $\Phi(x,t) = x + t\mathsf{N}(x)$, by Lemma 4.5, x minimises the distance in K to $\Phi(x,t)$. It follows that $(d_K \circ \Phi)(x,t) = t$ and, by definition, $(\Pi_K \circ \Phi)(x,t) = x$. In particular, $\Pi_K \circ \Phi$ and $d_K \circ \Phi$ are smooth functions. Composing with Φ^{-1}, it follows that d_K and Π_K are also smooth functions as desired. Now choose $x \in \Pi_K^{-1}(U(K)) \setminus K$. Observe that $\mathsf{N}(\Pi_K(x))$ is the unique supporting normal to K at x. Thus, by Lemma 6.6:

$$d_K(x) Dd_K(x)(V) = \langle x - \Pi_K(x), V\rangle = d_K(x)\langle \mathsf{N}(\Pi_K(x)), V\rangle.$$

Since $d_K(x) \neq 0$, it follows that $Dd_K(x)(V) = \langle \mathsf{N}(\Pi_K(x)), V\rangle$, as desired. Differentiating this relation yields $D^2 d_K(x)(V,W) = \langle A(\Pi_K(x))(D\Pi_K(x)(V)), W\rangle$, as desired. □

Lemma 6.14

For every compact subset X of U, there exists $B > 0$ such that for all $x \in \Pi_K^{-1}(X \cap U(K)) \setminus K$:

$$\|D\Pi_K(x) - \pi\| \leqslant Bd_K(x),$$

where π is the orthogonal projection onto $\langle Dd(x)\rangle^{\perp}$.

Proof: Let B be such that $\|A(y)\| \leqslant B$ for all $y \in X \cap U(K)$. By Lemma 6.9, $\|D\Pi_K(x)\| \leqslant 1$. By Lemma 6.13, for all vectors V and W:

$$\begin{aligned} \left|D^2 d_K(x)(V,W)\right| &= |\langle A(\Pi_K(x))(D\Pi_K(x)(V)), W\rangle| \\ &\leqslant \|A(\Pi_K(x))\|\|V\|\|W\| \\ &\leqslant B\|V\|\|W\|. \end{aligned}$$

By Lemma 6.7, for all vectors V and W:

$$\langle D\Pi_K(x)(V), W\rangle = \langle \pi(V), \pi(W)\rangle - d_K(x)D^2 d_K(x)(V,W),$$

where π is the orthogonal projection from $\mathbb{R}^{n+1}$ onto $\langle Dd_K(x)\rangle^{\perp}$. Since $\langle \pi(V), W - \pi(W)\rangle = 0$, it follows that:

$$\langle D\Pi_K(x)(V) - \pi(V), W\rangle = -d_K(x)D^2 d_K(x)(V,W).$$

Thus:

$$|\langle D\Pi_K(x)(V) - \pi(V), W\rangle| \leqslant d(x)\left|D^2 d_K(x)(V,W)\right| \leqslant Bd_K(x)\|V\|\|W\|,$$

and it follows that $\|D\Pi_K(x) - \pi\| \leqslant Bd_K(x)$ as desired. □

Lemma 6.15

Choose $k > 0$ and suppose that $U(K)$ has gaussian curvature everywhere at least k. For every compact subset X of U and for all $\epsilon > 0$, there exists $r > 0$ such that for all $x \in \Pi_K^{-1}(X \cap U(K)) \setminus K$, if $d_K(x) < r$, then $\mathrm{Det}(D^2 d_K(x); \langle Dd_K(x)\rangle^{\perp}) \geqslant (k-\epsilon)^n$.

Proof: By compactness, there exists $\delta > 0$ such that for all $x \in X \cap U(K)$ and for all $M \in B_\delta(A(x))$, $\mathrm{Det}(M; \langle Dd(x)\rangle^{\perp}) \geqslant (k-\epsilon)^n$. Let C_1 be such that for all $y \in X \cap U(K)$, $\|A(y)\| \leqslant C_1$. Let C_2 be as in Lemma 6.14. If $x \in \Pi_K^{-1}(X \cap U(K))$ is such that $d(x) < \delta/C_1C_2$, then, for all vectors V and W in $\langle Dd_K(x)\rangle^{\perp}$:

$$\begin{aligned} \left|\langle D^2 d(x)(V), W\rangle - \langle A(\Pi(x))(V), W\rangle\right| &= |\langle A(\Pi(x))(D\Pi(x)(V) - V), W\rangle| \\ &< \delta\|V\|\|W\|. \end{aligned}$$

and so $\mathrm{Det}(D^2 d(x); \langle Dd(x)\rangle^{\perp}) \geqslant (k-\epsilon)^n$ as desired. □

6.3 Intersecting Convex Sets.

Let K_1 and K_2 be compact, convex subsets of $\mathbb{R}^{n+1}$ whose intersection has non-trivial interior. Let U be an open subset of $\mathbb{R}^{n+1}$ and suppose that both $U(K_1)$ and $U(K_2)$ are smooth and have gaussian curvature at least k. Throughout the rest of this section we denote $K = K_1 \cap K_2$. We denote by N_1 and N_2 the outward-pointing, unit, normal vector fields over $K_1(U)$ and $K_2(U)$ and we denote by A_1 and A_2 their respective shape operators. Moreover, we denote $d = d_{K_1 \cap K_2}$, $d_1 = d_{K_1}$ and $d_2 = d_{K_2}$, and $\Pi = \Pi_{K_1 \cap K_2}$, $\Pi_1 = \Pi_{K_1}$ and $\Pi_2 = \Pi_{K_2}$. We recall by Lemmas 6.10 and 6.11 that d is almost everywhere twice differentiable with symmetric second derivative. We are interested in estimating lower bounds for $\mathrm{Det}(D^2 d; \langle Dd\rangle^\perp)$. There are four different cases to consider:

Lemma 6.16, Case 1

If $x \in \Pi^{-1}(U(K) \cap (\partial K_1) \cap K_2^o) \setminus K$ then $d = d_1$ and $\Pi = \Pi_1$.

Remark: *Observe that this set is open, and so $Dd = Dd_1$ and $D^2 d = D^2 d_1$ over this set.*

Proof: Denote $y = \Pi(x)$. Denote $\mathsf{N} = (x - y)/\|x - y\|$. By Lemma 4.5, N is a supporting normal to K at y. By Lemma 4.9, N is also a supporting normal to K_1 at y. By Lemma 4.5, y minimises distance in K_1 to x. In particular, $d_1(x) = \|x - y\| = d(x)$, and by definition of Π_1, $\Pi_1(x) = y = \Pi(x)$, as desired. □

Lemma 6.17, Case 2

If $x \in \Pi^{-1}(U(K) \cap (\partial K_2) \cap K_1^o) \setminus K$ then $d = d_2$ and $\Pi = \Pi_2$.

Proof: Denote $y = \Pi(x)$. Denote $\mathsf{N} = (x - y)/\|x - y\|$. By Lemma 4.5, N is a supporting normal to K at y. By Lemma 4.9, N is also a supporting normal to K_2 at y. By Lemma 4.5, y minimises distance in K_2 to x. In particular, $d_2(x) = \|x - y\| = d(x)$, and, by definition of Π_2, $\Pi_2(x) = y = \Pi(x)$, as desired. □

Lemma 6.18, Case 3

If $x \in \Pi^{-1}(U(K) \cap (\partial K_1) \cap (\partial K_2)) \setminus K$, if $(\mathsf{N}_1 \circ \Pi)(x) = (\mathsf{N}_2 \circ \Pi)(x)$, and if d is twice differentiable at x, then, for every vector V:

$$D^2 d(x)(V, V) \geqslant \mathrm{Max}(D^2 d_1(x)(V, V), D^2 d_2(x)(V, V)).$$

Proof: Denote $y = \Pi(x)$. By Lemma 4.5, $Dd(x)$ is a supporting normal to K at y. By Theorem 5.23, the set of supporting normals to K at y is the convex hull of $\{\mathsf{N}_1(y), \mathsf{N}_2(y)\}$. Since these two points coincide, this convex hull consists of a single point, and so $Dd(x) = \mathsf{N}_1(y) = \mathsf{N}_2(y)$. In particular, $Dd(x)$ is also a supporting normal to both K_1 and K_2 at y. It follows from Lemma 4.5 that y minimises the distance in K_1 to x. In particular, $d(x) = d_1(x)$. However, since $K \subseteq K_1$, for all y:

$$d(y) = \operatorname*{Inf}_{z \in K} \|y - z\| \geqslant \operatorname*{Inf}_{z \in K_1} \|y - z\| = d_1(y).$$

It follows upon differentiating that for all vectors V:

$$D^2 d(x)(V, V) \geqslant D^2 d_1(x)(V, V).$$

In like manner, we show that, $D^2d(x)(V,V) \geqslant D^2d_2(x)(V,V)$, and it follows that:

$$D^2d(x)(V,V) \geqslant \mathrm{Max}(D^2d_1(V,V), D^2d_2(V,V)),$$

as desired. □

Before treating the fourth case, we require the following preliminary result:

Lemma 6.19

If $x \in \Pi^{-1}(U(K)\cap(\partial K_1)\cap(\partial K_2)) \setminus K$ and if $(\mathsf{N}_1 \circ \Pi)(x) \neq (\mathsf{N}_2 \circ \Pi)(x)$, then there exists a unique element $s \in [0,1]$ such that:

$$Dd(x) = \frac{1-s}{l}(\mathsf{N}_1 \circ \Pi)(x) + \frac{s}{l}(\mathsf{N}_2 \circ \Pi)(x),$$

where $l = \|(1-s)(\mathsf{N}_1 \circ \Pi)(x) + s(\mathsf{N}_2 \circ \Pi)(x)\|$. In particular:

$$l \geqslant \|\mathsf{N}_1(y) + \mathsf{N}_2(y)\|/2.$$

Proof: Denote $y = \Pi(x)$. By Lemma 4.5, $Dd(x)$ is a supporting normal to K at $\Pi(x)$. By Theorem 5.23, the set of supporting normals to K at $\Pi(x)$ is the convex hull of $\{\mathsf{N}_1(y), \mathsf{N}_2(y)\}$. This coincides with the great-circular arc joining $\mathsf{N}_1(y)$ to $\mathsf{N}_2(y)$ (c.f. Section 5.2). The first assertion now follows by definition of this great-circular arc. Next observe that the vectors $\mathsf{N}_1(y) + \mathsf{N}_2(y)$ and $\mathsf{N}_1(y) - \mathsf{N}_2(y)$ are orthogonal. Thus:

$$\begin{aligned} l &= \|(1-s)\mathsf{N}_1(y) + s\mathsf{N}_2(y)\|^2 \\ &= \|(\mathsf{N}_1(y) + \mathsf{N}_2(y))/2 + (1/2 - s)(\mathsf{N}_1(y) - \mathsf{N}_2(y))\|^2 \\ &= \|\mathsf{N}_1(y) + \mathsf{N}_2(y)\|^2/4 + (1/2 - s)^2\|\mathsf{N}_1(y) - \mathsf{N}_2(y)\|^2 \\ &\geqslant \|\mathsf{N}_1(y) + \mathsf{N}_2(y)\|^2/4. \end{aligned}$$

Upon taking square roots, we obtain $l \geqslant \|\mathsf{N}_1(y) + \mathsf{N}_2(y)\|/2$, as desired. □

Lemma 6.20, Case 4a

If $x \in \Pi^{-1}(U(K)\cap(\partial K_1)\cap(\partial K_2)) \setminus K$, if $(\mathsf{N}_1 \circ \Pi)(x) \neq (\mathsf{N}_2 \circ \Pi)(x)$, and if d is twice differentiable at x, then for every vector V and for every vector W orthogonal to both $(\mathsf{N}_1 \circ \Pi)(x)$ and $(\mathsf{N}_2 \circ \Pi)(x)$:

$$D^2d(x)(V,W) = \frac{1-s}{l}\langle A_1(\Pi(x))(D\Pi(x)(V)), W\rangle + \frac{s}{l}\langle A_2(\Pi(x))(D\Pi(x)(V)), W\rangle,$$

where s and l are as in Lemma 6.19.

Remark: *Upon applying an isometry, we may suppose that $\langle e_n, e_{n+1}\rangle = \langle(\mathsf{N}_1 \circ \Pi)(x), (\mathsf{N}_2 \circ \Pi)(x)\rangle$. Consequently, when $D^2d(x)$ is symmetric, this result determines every component of $D^2d(x)$ except $D^2d(x)(e_i, e_j)$ for $(i,j) \in \{n, n+1\}^2$.*

Proof: Denote $y = \Pi(x)$. Let V be a vector in $\mathbb{R}^{n+1}$. Define $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$ by $\gamma(t) = x + tV$. Let $(t_m)_{m\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}$ converging to 0. For all m, we denote

$x_m = \gamma(t_m)$ and $y_m = (\Pi\circ\gamma)(t_m)$. Upon extracting a subsequence, we may suppose that one of the following holds:

1: $x_m \in \Pi^{-1}(U(K)\cap(\partial K_1)\cap K_2^o)\setminus K$ for all $m\in\mathbb{N}$. By Lemma 6.16, for all $m\in\mathbb{N}$, $Dd(x_m) = Dd_1(x_m)$. Taking limits and bearing in mind Lemma 6.13, it follows that:

$$Dd(x) = Dd_1(x) = (\mathsf{N}_1\circ\Pi)(x) = \mathsf{N}_1(y).$$

In particular, $s = 0$ and $l = 1$. Moreover, for all vectors W and for all m:

$$\frac{1}{t_m}\langle Dd(x_m) - Dd(x), W\rangle = \frac{1}{t_m}\langle \mathsf{N}_1(y_m) - \mathsf{N}_1(y), W\rangle.$$

By the chain rule, upon taking limits, we obtain:

$$D^2d(x)(V,W) = \langle A_1(y)(D\Pi(x)(V)), W\rangle,$$

as desired.

2: $x_m \in \Pi^{-1}(U(K)\cap(\partial K_2)\cap K_1^o)\setminus K$ for all $m\in\mathbb{N}$. As in Step (1), we show that $s = 1$, $l = 1$ and :

$$D^2d(x)(V,W) = \langle A_2(y)(D\Pi(x)(V)), W\rangle,$$

as desired.

3: $x_m \in \Pi^{-1}(U(K)\cap(\partial K_1)\cap(\partial K_2))\setminus K$. For all $m\in\mathbb{N}$, we denote $\mathsf{N}_{1,m} = \mathsf{N}_1(y_m)$ and $\mathsf{N}_{2,m} = \mathsf{N}_2(y_m)$. Observe that for sufficiently large m, $\mathsf{N}_{1,m} \neq \mathsf{N}_{2,m}$. Thus by Lemma 6.19, for all $m\in\mathbb{N}$ there exists a unique element $s_m\in[0,1]$ such that:

$$Dd(x_m) = \frac{1-s_m}{l_m}\mathsf{N}_{1,m} + \frac{s_m}{l_m}\mathsf{N}_{2,m},$$

where $l_m = \|(1-s_m)\mathsf{N}_{1,m} + s_m\mathsf{N}_{2,m}\|$. Since N_1, N_2, Dd and Π are continuous, $(s_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$ converge to the limits s_∞ and l_∞ respectively. Let W be a vector normal to both $\mathsf{N}_1(y)$ and $\mathsf{N}_2(y)$. In particular W is normal to $Dd(x)$. For all m:

$$\begin{aligned}\tfrac{1}{t_m}\langle Dd(x_m) - Dd(x), W\rangle &= \tfrac{1}{t_m}\langle Dd(x_m), W\rangle\\ &= \tfrac{1-s_m}{l_m t_m}\langle \mathsf{N}_{1,m}, W\rangle + \tfrac{1-s_m}{l_m t_m}\langle \mathsf{N}_{2,m}, W\rangle\\ &= \tfrac{1-s_m}{l_m t_m}\langle \mathsf{N}_{1,m} - \mathsf{N}_1(y), W\rangle + \tfrac{1-s_m}{l_m t_m}\langle \mathsf{N}_{2,m} - \mathsf{N}_2(y), W\rangle.\end{aligned}$$

By the chain rule, upon taking limits, we obtain:

$$D^2d(x)(V,W) = \frac{1-s}{l}\langle A_1(y)(D\Pi(x)(V)), W\rangle + \frac{s}{l}\langle A_2(y)(D\Pi(x)(V)), W\rangle,$$

as desired. □

Lemma 6.21, Case 4b

If $x \in \Pi^{-1}(U(K)\cap(\partial K_1)\cap(\partial K_2)) \setminus K$, if $(\mathsf{N}_1 \circ \Pi)(x) \neq (\mathsf{N}_2 \circ \Pi)(x)$, and if d is twice differentiable at x, then for any vector W and for any vector V tangent to $\langle(\mathsf{N}_1\circ\Pi)(x), (\mathsf{N}_2\circ\Pi)(x)\rangle$ and normal to $Dd(x)$:

$$D^2d(x)(V,W) = \frac{1}{d(x)}\langle V, W\rangle.$$

Remark: *Upon applying an isometry, we may suppose that $\langle e_n, e_{n+1}\rangle = \langle(\mathsf{N}_1\circ\Pi)(x), (\mathsf{N}_2\circ\Pi)(x)\rangle$ and $e_{n+1} = Dd(x)$. Consequently when $D^2d(x)$ is symmetric, this result along with Lemma 6.20 determines every component of $D^2d(x)$ except $D^2d(x)(e_{n+1}, e_{n+1})$. In fact, we may show that $D^2d(x)(e_{n+1}, e_{n+1}) = 0$. Since this this is not necessary for out purposes, we leave this result as an exercise for the interested reader.*

Proof: Denote $y = \Pi(x)$. By Theorem 5.23, the set of supporting normals to K at y coincides with the convex hull of $\{\mathsf{N}_1(y), \mathsf{N}_2(y)\}$. This in turn coincides with the image of the great-circular arc joining $\mathsf{N}_1(y)$ to $\mathsf{N}_2(y)$ (c.f. Section 5.2). We denote this great-circular arc by N. In particular, for all r, $\mathsf{N}(r)$ is a supporting normal to K at y. We define $\gamma : [0,1] \to \mathbb{R}^{n+1}$ by $\gamma(r) = y + d(x)\mathsf{N}(r)$. By Lemma 4.5, for all t, y is the point in K minimising distance to the point $\gamma(r)$. In particular, by Lemma 6.6, for all r, $(Dd\circ\gamma)(r) = \mathsf{N}(r)$. Let s be as in Lemma 6.19. Since V is in the plane spanned by $\mathsf{N}_1(y)$ and $\mathsf{N}_2(y)$ but is normal to $Dd(x)$, it follows that V is colinear with $(\partial_r\gamma)(s)$. Thus, upon multiplying by a scalar factor, we may suppose that $V = (\partial_r\gamma)(s)$. Thus:

$$\begin{aligned}\langle D^2d(x)(V), W\rangle &= \langle\partial_r(Dd\circ\gamma)(s), W\rangle\\ &= \langle(\partial_r\mathsf{N})(s), W\rangle\\ &= \tfrac{1}{d(x)}\langle(\partial_r\gamma)(s), W\rangle\\ &= \tfrac{1}{d(x)}\langle V, W\rangle,\end{aligned}$$

as desired. □

Lemma 6.22

For every compact subset X of U there exists $B > 0$ with the property that if $x \in \Pi^{-1}(X\cap U(K)\cap(\partial K_1)\cap(\partial K_2))\setminus K$, if $(\mathsf{N}_1\circ\Pi)(x) \neq (\mathsf{N}_2\circ\Pi)(x)$, if d is twice differentiable at x, and if $D^2d(x)$ is symmetric, then:

$$\|D\Pi(x) - \pi^{1,2}\| \leqslant Bd(x),$$

where $\pi^{1,2}$ is the orthonogonal projection from $\mathbb{R}^{n+1}$ onto $\langle(\mathsf{N}_1\circ\Pi)(x), (\mathsf{N}_2\circ\Pi)(x)\rangle^{\perp}$.

Proof: Denote $y = \Pi(x)$. Let $B > 0$ be such that $\|A_1(x)\| \leqslant B$ and $\|A_2(x)\| \leqslant B$ for all $x \in X\cap U(K_1)$ and for all $x \in X\cap U(K_2)$ respectively. By Lemma 6.7, Π is differentiable at x and, for all vectors V and W:

$$\langle D\Pi(x)(V), W\rangle = \langle\pi(V), \pi(W)\rangle - d(x)D^2d(x)(V,W),$$

where π is the orthogonal projection from $\mathbb{R}^{n+1}$ onto $\langle Dd(x)\rangle^\perp$. Observe, in particular, that since D^2d is symmetric, for all vectors V and W:

$$\langle D\Pi(x)(V), W\rangle = \langle D\Pi(x)(W), V\rangle.$$

Let V be any vector in $\mathbb{R}^{n+1}$. Define $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$ by $\gamma(t) = x + tV$. Since $\gamma(t) \in K$ for all t, it follows by definition of supporting normals that for each $\mathsf{N} \in \{\mathsf{N}_1(y), \mathsf{N}_2(y)\}$ and for all t:

$$\langle (\Pi \circ \gamma)(t) - y, \mathsf{N}\rangle \leqslant 0.$$

By the chain rule, differentiating this relation yields:

$$\langle D\Pi(x)(V), \mathsf{N}\rangle = 0.$$

Thus, by linearity and symmetry, for any vector W in $\langle \mathsf{N}_1(y), \mathsf{N}_2(y)\rangle$:

$$\langle D\Pi(x)(W), V\rangle = \langle D\Pi(x)(V), W\rangle = 0.$$

Now let V and W be both orthogonal to $\langle \mathsf{N}_1(y), \mathsf{N}_2(y)\rangle$. In particular, V and W are both orthogonal to $Dd(x)$. Thus, by Lemma 6.7:

$$\langle D\Pi(x)(V), W\rangle = \langle V, W\rangle - d(x)D^2d(x)(V, W).$$

Let s and l be as in Lemma 6.19. Then, by Lemma 6.20 and bearing in mind Lemma 6.9 and the definition of B:

$$\begin{aligned} \left|D^2d(x)(V,W)\right| &= \left|\tfrac{1-s}{l}\langle A_1(y)(D\Pi(x)(V)), W\rangle + \tfrac{s}{l}\langle A_2(y)(D\Pi(x)(V), W)\rangle\right| \\ &\leqslant \tfrac{1-s}{l}\|B\|\|V\|\|W\| + \tfrac{s}{l}\|B\|\|V\|\|W\| \\ &\leqslant \|B\|\|V\|\|W\|. \end{aligned}$$

Thus:

$$|\langle D\Pi(x)(V), W\rangle - \langle V, W\rangle| \leqslant Bd(x)\|V\|\|W\|.$$

Combining these relations, we conclude that $\|D\Pi(x) - \pi^{1,2}\| \leqslant Bd(x)$ as desired. □

Lemma 6.23

For every compact subset X of U there exists $L > 0$ such that for all $x \in (\partial K_1) \cap (\partial K_2) \cap X$:

$$\|\mathsf{N}_1(x) + \mathsf{N}_2(x)\|/2 \geqslant L.$$

Proof: Suppose the contrary. By compactness, there exists $x \in (\partial K_1) \cap (\partial K_2) \cap X$ such that $\mathsf{N}_1(x) + \mathsf{N}_2(x) = 0$. By definition of supporting normals, for all $y \in K_1 \cap K_2$, $\langle y - x, \mathsf{N}_1\rangle \leqslant 0$ and $\langle y - x, \mathsf{N}_2\rangle \leqslant 0$. Since $\mathsf{N}_1(x) = -\mathsf{N}_2(x)$, if follows that, for all $y \in K_1 \cap K_2$, $\langle y - x, \mathsf{N}_1\rangle = \langle y - x, \mathsf{N}_2\rangle = 0$. In other words, $K_1 \cap K_2$ is contained in the hyperplane normal to N_1 passing through x. In particular, $K_1 \cap K_2$ has trivial interior. This is absurd, and the assertion follows. □

Lemma 6.24

For every compact subset X of U and for all $\epsilon > 0$, there exists $r > 0$ with the property that if $x \in \Pi^{-1}(X \cap U(K)) \setminus K$, if $d(x) < r$ and if $D^2 d(x)$ is defined and is symmetric, then:

$$\mathrm{Det}(D^2 d(x); \langle Dd(x)\rangle^{\perp}) \geqslant (k-\epsilon)^n.$$

Proof: We consider the following cases:

1: Suppose that $x \in \Pi^{-1}(X \cap U(K) \cap (\partial K_1) \cap K_2^o) \setminus K$. By Lemma 6.16, $D^2 d(x) = D^2 d_1(x)$, and the result follows by Lemma 6.15.

2: Suppose that $x \in \Pi^{-1}(X \cap U(K) \cap (\partial K_2) \cap K_1^o) \setminus K$. By Lemma 6.17, $D^2 d(x) = D^2 d_2(x)$, and the result follows by Lemma 6.15.

3: Suppose that $x \in \Pi^{-1}(X \cap U(K) \cap (\partial K_1) \cap (\partial K_2)) \setminus K$ and $\mathsf{N}_1(\Pi(x)) = \mathsf{N}_2(\Pi(x))$. By Lemma 6.18, for all vectors $V \in \mathbb{R}^{n+1}$:

$$D^2 d(x)(V,V) \geqslant \mathrm{Max}(D^2 d_1(x)(V,V), D^2 d_2(x)(V,V)).$$

In particular, bearing in mind that $Dd(x) = Dd_1(x) = Dd_2(x)$:

$$\mathrm{Det}(D^2 d(x); \langle Dd(x)\rangle^{\perp}) \geqslant \mathrm{Det}(D^2 d_1(x); \langle Dd_1(x)\rangle^{\perp}), \mathrm{Det}(D^2 d_2(x); \langle Dd_2(x)\rangle^{\perp}),$$

and the result now follows by Lemma 6.15.

4: Denote $y = \Pi(x)$. Suppose that $x \in \Pi^{-1}(X \cap U(K) \cap (\partial K_1) \cap (\partial K_2)) \setminus K$. Suppose moreover that $\mathsf{N}_1(\Pi(x)) \neq \mathsf{N}_2(\Pi(x))$. Let s and l be as in Lemma 6.19. Let $B_1 > 0$ be such that $(1/B_1)\mathrm{Id} \leqslant A_1(y) \leqslant B_1\mathrm{Id}$ and $(1/B_1)\mathrm{Id} \leqslant A_2(y) \leqslant B_1\mathrm{Id}$ for all $y \in X \cap U(K_1)$ and for all $y \in X \cap U(K_2)$ respectively. Let $B_2 \geqslant 0$ be as in Lemma 6.22. Define $r > 0$ by $r = 1/(2B_1^2 B_2)$. Then if $d(x) < r$, for all vectors V normal to $\mathsf{N}_1(y)$ and $\mathsf{N}_2(y)$:

$$\begin{aligned}\langle A_1(y)(D\Pi(x)(V)), V\rangle &= \langle A_1(y)(V), V\rangle + \langle A_1(y)((D\Pi(x) - \pi^{1,2})(V)), V\rangle \\ &\geqslant \tfrac{1}{B_1}\|V\|^2 - \tfrac{1}{2B_1}\|V\|^2 \\ &= \tfrac{1}{2B_1}\|V\|^2\end{aligned}$$

Likewise, for all such x and V:

$$\langle A_2(y)(D\Pi(x)(V)), V\rangle \geqslant \frac{1}{2B_1}\|V\|^2.$$

Thus, if s and l are as in Lemma 6.19, by Lemma 6.20, for all such x and V:

$$\begin{aligned}D^2 d(x)(V,V) &= \tfrac{1-s}{l}\langle A_1(y)(D\Pi(x)(V)), V\rangle + \tfrac{s}{l}\langle A_2(y)(D\Pi(x)(V)), V\rangle \\ &\geqslant \tfrac{1-s}{2B_1 l}\|V\|^2 + \tfrac{s}{2B_1 l}\|V\|^2 \\ &\geqslant \tfrac{1}{2B_1 L}\|V\|^2,\end{aligned}$$

where L is as in Lemma 6.23. Upon applying an isometry, we may suppose that the plane spanned by e_n and e_{n+1} coincides with the plane spanned by $\mathsf{N}_1(y)$ and $\mathsf{N}_2(y)$. In addition, upon applying a further isometry, we may suppose that $e_{n+1} = Dd(x)$. We denote by M the restriction of $D^2d(x)$ to $\langle e_1, ..., e_{n-1}\rangle$. By the preceeding discussion, $M \geqslant (1/2B_1L)\mathrm{Id}$. By Lemma 6.21, for all i:

$$D^2d(x)(e_i, e_n) = D^2d(x)(e_n, e_i) = \frac{1}{d(x)}\delta_{in}.$$

Reducing r if necessary, we may suppose that $r < (2B_1)^{1-n}(k-\epsilon)^{-n}$. Then, if $d(x) < r$:

$$\mathrm{Det}(D^2d(x), \langle Dd(x)\rangle^{\perp}) \geqslant (k-\epsilon)^n,$$

as desired. □

6.4 Smoothing Functions.

Let $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ be a smooth function such that $\chi \geqslant 0$, $\chi = 0$ outside the unit ball $B_1(0)$ and:

$$\int_{\mathbb{R}^{n+1}} \chi(x)\mathrm{dVol}_x = 1$$

For all $s > 0$, we define $\chi_s \in C_0^\infty(\mathbb{R}^{n+1})$ by:

$$\chi_s(x) = s^{-(n+1)}\chi(x/s).$$

Let E be a finite dimensional vector space. For any function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}, E)$, and for all $s > 0$, we define the function $f_s : \mathbb{R}^{n+1} \to E$ by:

$$f_s(x) = \int_{\mathbb{R}^{n+1}} f(x-y)\chi_s(y)\mathrm{dVol}_y.$$

We recall the following properties of smoothing functions:

Lemma 6.25

For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and for all $s > 0$, f_s is continuous.

Remark: *In fact, as is well known, f_s is smooth.*

Proof: Choose $x \in \mathbb{R}^{n+1}$ and $s > 0$. By local uniform continuity, there exists $\delta > 0$ such that if $\|y\| < s$ and $\|z - y\| < \delta$, then $|\chi_s(y) - \chi_s(z)| < \epsilon$. Thus, if $\|z - x\| < \delta$, using a change of variable, we obtain:

$$\begin{aligned}\|f_s(z) - f_s(x)\| &= \|\textstyle\int_{\mathbb{R}^{n+1}} f(z-y)\chi_s(y) - f(x-y)\chi_s(y)\mathrm{dVol}_y\| \\ &= \|\textstyle\int_{\mathbb{R}^{n+1}} f(x-y)(\chi_s(y+(z-x)) - \chi_s(y))\mathrm{dVol}_y\| \\ &\leqslant \textstyle\int_{\mathbb{R}^{n+1}} \|f(x-y)\|\,|\chi_s(y+(z-x)) - \chi_s(y)|\,\mathrm{dVol}_y \\ &\leqslant \epsilon\textstyle\int_{B_{R+\delta}(0)} \|f(x-y)\|\mathrm{dVol}_y.\end{aligned}$$

Since ϵ may be chosen arbitrarily small, continuity of f_s at x follows. Since $x \in \mathbb{R}^{n+1}$ is arbitrary, it follows that f_s is continuous as desired. □

Lemma 6.26

Choose $f \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$. If f is continuous, then $(f_s)_{s>0}$ converges to f locally uniformly as r tends to 0.

Proof: Choose $\epsilon > 0$ and $R > 0$. By uniform continuity, there exists $\delta > 0$ such that if $\|x\| < R$ and if $\|x - y\| < \delta$, then $\|f(x) - f(y)\| < \delta$. Then, for $s < \delta$ and for $\|x\| < R$, bearing in mind that χ_s is non-negative and has integral equal to 1:

$$\begin{aligned}\|f(x) - f_s(x)\| &= \|f(x) - \textstyle\int_{\mathbb{R}^{n+1}} f(x-y)\chi_s(y)\mathrm{dVol}_y\| \\ &= \|\textstyle\int_{\mathbb{R}^{n+1}} (f(x) - f(x-y))\chi_s(y)\mathrm{dVol}_y\| \\ &\leqslant \textstyle\int_{\mathbb{R}^{n+1}} \|f(x) - f(x-y)\|\chi_s(y)\mathrm{dVol}_y \\ &\leqslant \epsilon \textstyle\int_{\mathbb{R}^{n+1}} \chi_r(y)\mathrm{dVol}_y \\ &= \epsilon.\end{aligned}$$

Since $R, \epsilon > 0$ are arbitrary, it follows that $(f_s)_{s>0}$ converges locally uniformly to f over $\mathbb{R}^{n+1}$ as s tends to 0 as desired. □

Lemma 6.27

Choose $f \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$. If f has L^1_{loc} distributional derivatives, then for all $s > 0$, f_s is differentiable and $D(f_r) = (Df)_r$.

Proof: Choose $x \in \mathbb{R}^{n+1}$ and $r > 0$. Choose $\epsilon > 0$. Since χ_s is smooth, there exists $\eta > 0$ with the property that for all y and for all vectors V such that $\|V\| \leqslant \eta$:

$$|\chi_s(y+V) - \chi_s(y) - (D\chi_s)(y)(V)| \leqslant \eta\|V\|.$$

Thus, bearing in mind the definition of the distributional derivative and using a change of variable formula, for all V such that $\|V\| \leqslant \eta$:

$$\begin{aligned}\|f_s(x+V) - f_s(x) - (Df)_s(x)(V)\| &= \|\textstyle\int_{\mathbb{R}^{n+1}} f(x+V-y)\chi_s(y) - f(x-y)\chi_s(y) \\ &\qquad -(Df)(x-y)(V)\chi_s(y)\mathrm{dVol}_y\| \\ &= \|\textstyle\int_{\mathbb{R}^{n+1}} f(x-y)(\chi_s(y+V) \\ &\qquad -\chi_s(y) - (D\chi_s)(y)(V))\mathrm{dVol}_y\| \\ &\leqslant \epsilon\|V\| \textstyle\int_{B_s(x)} \|f(x-y)\|\mathrm{dVol}_y.\end{aligned}$$

Since ϵ may be chosen arbitrarily small and since V is arbitrary, it follows that f_s is differentiable at x with derivative equal to $(Df)_s(x)$, as desired. □

Combining these results yields:

Theorem 6.28

If $f \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$ is C^k, then for all $s > 0$, $J^k(f_s) = (J^k f)_s$. and $(f_s)_{s>0}$ converges to f in the C^k_{loc} sense as s tends to 0.

Proof: We work by induction on k. By Lemmas 6.25 and 6.26, the result holds when $k = 0$. Suppose that the result holds for $k = l$. Choose $f \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$ such that f is

C^{l+1}. In particular, Df is C^l. Bearing in mind the induction hypothesis, for all s, we obtain:

$$J^{l+1}(f_s) = J^l(D(f_s)) = J^l((Df)_s) = (J^l(Df))_s = (J^{l+1}f)_s.$$

Moreover, $(f_s)_{s>0}$ and $(J^l(Df)_s)_{s>0}$ converge locally uniformly to f and $J^l(Df)$ respectively as s tends to 0. Thus $(J^{l+1}(f_s))_{s>0}$ converges locally uniformly to $J^{l+1}f$, and the result follows by induction. □

Of particular use to us is the following:

Lemma 6.29

Let E be a finite dimensional vector space. Let K be a compact, convex subset of E. Let U be an open subset of $\mathbb{R}^{n+1}$. Let $f \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$ be such that for almost all $x \in U$, $f(x) \in K$. Then for all $s > 0$ and for all $x \in U$ with the property that $B_{1/s}(x) \in U$, we have $f_s(x) \in K$.

Remark: *The same result holds under the hypothesis that K is merely closed. We prove the result in the form given above in order to be consistent with the formalism developed in Section 4. Lemma 6.29 as stated is, in any case, sufficient for our purposes.*

Proof: We use the terminology of Section 5. Let $H(\mathsf{N}, t)$ be an open half-space of E containing K. By compactness, there exists $\epsilon > 0$ such that $\langle z, \mathsf{N}\rangle \leqslant t - \epsilon$ for all $z \in K$. Choose $s > 0$ and $x \in \mathbb{R}^{n+1}$ with the property that $B_{1/s}(x) \subseteq U$. Then, bearing in mind that χ is positive:

$$\begin{aligned}\langle f_s(x), \mathsf{N}\rangle &= \langle \textstyle\int_{B_s(0)} f(x-y)\chi_s(y)\mathrm{dVol}_y, \mathsf{N}\rangle \\ &= \textstyle\int_{B_s(0)} \langle f(x-y), \mathsf{N}\rangle \chi_s(y)\mathrm{dVol}_y \\ &\leqslant \textstyle\int_{B_s(0)} (t-\epsilon)\chi_s(y)\mathrm{dVol}_y \\ &= (t-\epsilon).\end{aligned}$$

Thus, by definition of open half-spaces, $f_s(x) \in H(\mathsf{N}, t)$. Recall by Lemmas 4.17 and 5.2 that K coincides with the intersection of all open half-spaces containing K. Since $H(\mathsf{N}, t)$ is an arbitrary open half-space containing K, it follows that $f_s(x) \in K$ as desired. □

6.5 Smoothing the Intersection.

We return to the situation discussed in Section 6.3. Thus, let K_1 and K_2 be compact, convex subsets of $\mathbb{R}^{n+1}$ whose intersection has non-trivial interior. Let U be an open subset of $\mathbb{R}^{n+1}$ and suppose that both $U(K_1)$ and $U(K_2)$ are smooth and have gaussian curvature at least k. As before, we denote $K = K_1 \cap K_2$ and we denote $d = d_{K_1 \cap K_2}$, $d_1 = d_{K_1}$ and $d_2 = d_{K_2}$. We recall the following version of the submersion theorem:

Lemma 6.30

Let $U \subseteq \mathbb{R}^{n+1}$ be an open set. Let $f : U \to \mathbb{R}$ be a smooth mapping and denote $\Sigma = f^{-1}(\{0\})$. If 0 is a regular value of f, then Σ is a smooth, embedded submanifold. Moreover, for all $x \in \Sigma$, $Df(x)/\|Df(x)\|$ is a unit, normal vector field over Σ, and if *we denote by A the shape operator of Σ with respect to this normal, for all $x \in \Sigma$ and for all X, Y tangent to Σ at x:*

$$A(x)(X,Y) = \frac{1}{\|Df(x)\|}D^2f(x)(X,Y).$$

Proof: If 0 is a regular value of f, then it follows by the submersion theorem (c.f. [13]) that Σ is a smooth, embedded submanifold of U. Choose $x \in U$ and let X be a tangent vector to Σ at x. Let $\gamma :]-\epsilon, \epsilon[\rightarrow \Sigma$ be a smooth curve such that $\gamma(0) = x$ and $\gamma'(0) = X$. Observe that $(f \circ \gamma)(t) = 0$ for all t. Thus, by the chain rule:

$$\langle Df(x), X\rangle = \langle Df(x), \gamma'(0)\rangle = (f \circ \gamma)'(0) = 0.$$

Since X is an arbitrary vector tangent to Σ at x, it follows that $Df(x)$ is normal to Σ at x. Observe that $\|Df(x)\| \neq 0$, and it follows that $Df(x)/\|Df(x)\|$ is a unit normal vector to Σ at x as desired. Now let X and Y be tangent vectors to Σ at x. We denote $\mathsf{N} = Df/\|Df\|$. By definition of A, and bearing in mind the chain rule and the product rule:

$$\begin{aligned}\langle A(x)(X), Y\rangle &= \langle D\mathsf{N}(x)(X), Y\rangle \\ &= \langle D(Df/\|Df\|)(x)(X), Y\rangle \\ &= \tfrac{1}{\|Df(x)\|}\langle D^2f(x)(X), Y\rangle - \tfrac{1}{\|Df(x)\|^3}\langle Df(x), Y\rangle\langle Df(x), X\rangle.\end{aligned}$$

However, by the previous discussion, $Df(x)$ is normal to Σ at x, and so:

$$\langle A(x)(X), Y\rangle = \frac{1}{\|Df(x)\|}D^2f(x)(X,Y),$$

as desired. □

For all $k, B > 0$, and for all $\mathsf{N} \in \Sigma^n$, we define the set $\kappa(k, B, \mathsf{N}) \subseteq \mathrm{Symm}(2, \mathbb{R}^{n+1})$ by:

$$\kappa(k, B, \mathsf{N}) = \left\{A \mid \|A\| \leqslant B,\ A \geqslant 0,\ \mathrm{Det}(A; \langle\mathsf{N}\rangle^\perp) \geqslant k^n\right\}.$$

Lemma 6.31

For all $k, B > 0$ and for all $\mathsf{N} \in \Sigma^n$, $\kappa(k, B, \mathsf{N})$ is compact and convex.

Proof: The set of all matrices of norm no greater than B is compact. Since $\kappa(k, B, \mathsf{N})$ is a closed subset of this set, it follows that it too is compact. Choose $M_0, M_1 \in \kappa(k, B, \mathsf{N})$ and $t \in [0, 1]$. By convexity of the norm, $\|(1-t)M_0 + tM_1\| \leqslant (1-t)\|M_0\| + t\|M_1\| \leqslant B$. Moreover, $(1-t)M_0 + tM_1 \geqslant 0$. Finally, by Lemma 2.8:

$$\mathrm{Det}((1-t)M_0 + tM_1; \langle\mathsf{N}\rangle^\perp)^{1/n} \geqslant (1-t)\mathrm{Det}(M_0; \langle\mathsf{N}\rangle^\perp)^{1/n} + t\mathrm{Det}(M_1; \langle\mathsf{N}\rangle^\perp)^{1/n} \geqslant k,$$

We conclude that $(1-t)M_0 + tM_1 \in \kappa(k, B, \mathsf{N})$. Since $M_0, M_1 \in \kappa(k, B, \mathsf{N})$ and $t \in [0, 1]$ are arbitrary, it follows that $\kappa(k, B, \mathsf{N})$ is convex as desired. □

Lemma 6.32

For every compact subset X of U and for all $\epsilon > 0$, there exists $\rho > 0$ with the property that if $x \in \Pi^{-1}(X \cap U(K)) \setminus K$, if $d(x) < \rho$ and if $D^2d(x)$ is defined and is symmetric, then:

$$D^2d(x) \in \kappa(k - \epsilon, 2/d(x), Dd(x)).$$

Proof: By Lemma 6.10, $\|D^2d(x)\| \leqslant 2/d(x)$. The result now follows by Lemma 6.24 and by definition of $\kappa(k - \epsilon, 2/d(x), Dd(x))$. $\square$

We now consider smoothings of d as described in Section 6.4:

Lemma 6.33

For every compact subset X of U and for all $\epsilon > 0$, there exists $\rho > 0$ with the property that for all $r \in]0, \rho[$, there exists $S > 0$ such that if $s > S$, if $x \in X$ and if $d_s(x) = s$, then $0 < \|Dd_s(x)\| \leqslant 1$ and:

$$D^2d_s(x) \in \kappa(k - \epsilon, 4/r, Dd_s(x)/\|Dd_s(x)\|).$$

Proof: Choose $S_1 > 0$ such that $\overline{B}_{1/S_1}(X) \subseteq U$. We denote $X_1 = \overline{B}_{1/S_1}(X)$. Observe that since X is compact, so too is X_1. We claim that there exists a compact subset X_2 of U and $\rho_1 > 0$ such that:

$$X_1 \cap d^{-1}(]0, \rho_1[) \subseteq \Pi^{-1}(X_2 \cap U(K)) \setminus K.$$

Indeed, suppose the contrary. There exists a sequence $(x_m)_{m\in\mathbb{N}}$ in X_1 with the properties that $d(x_m) > 0$ for all m, $(d(x_m))_{m\in\mathbb{N}}$ converges to 0 and $(\Pi(x_m))_{m\in\mathbb{N}}$ is not contained in any compact subset of U. For all m, we denote $y_m = \Pi(x_m)$ and $\mathsf{N}_m = Dd(x_m)$. Since K is compact, there exists $y_\infty \in K$ towards which $(y_m)_{m\in\mathbb{N}}$ subconverges. By hypothesis, y_∞ lies in the boundary of U. By Lemma 6.6, for all m, $x_m = y_m + d(x_m)\mathsf{N}_m$. In particular, since $(d(x_m))_{m\in\mathbb{N}}$ converges to 0 and since N_m has unit length for all m, it follows that $(x_m)_{m\in\mathbb{N}}$ also subconverges to y_∞. Since x_m is an element of X_1 for all m, and since X_1 is compact, it follows that y_∞ is also an element of X_1. This is absurd, since X_1 is contained in the interior of U, and the assertion follows.

By Lemma 6.32, there exists $\rho_2 < \rho_1$ with the property that if $x \in \Pi^{-1}(X_2 \cap U(K)) \setminus K$, if $d(x) < \rho_2$ and if $D^2d(x)$ is defined and is symmetric, then:

$$D^2d(x) \in \kappa(k - \epsilon/4, 2/d(x), Dd(x)).$$

Choose $r \in]0, \rho_2[$. Choose $S_2 > S_1$ and $\eta_1 > 0$ such that $\eta_1 + 1/S_2 < \mathrm{Min}(r/2, \rho_2 - r)$. Observe that there exists $\delta_1 \in]0, 1[$ such that if N is any vector in Σ^n and if $V \in \overline{B}_{\delta_1}(\mathsf{N})$, then:

$$\kappa(k - \epsilon/4, 4/r, V/\|V\|) \subseteq \kappa(k - \epsilon/2, 4/r, \mathsf{N}) \subseteq \kappa(k - \epsilon, 4/r, V/\|V\|). \qquad (D)$$

Since K is compact, so too is $d^{-1}([r - \eta_1, r + \eta_1])$. There therefore exists $S_3 > S_2$ such that if $x \in d^{-1}([r - \eta_1, r + \eta_1])$ and if $y \in B_{1/S_3}(x)$, then $Dd(y) \subseteq \overline{B}_{\delta_1}(Dd(x))$. Since

$\overline{B}_{\delta_1}(Dd(x))$ is compact and convex, by Lemma 6.29, if $s > S_3$, then $Dd_s(x) \in \overline{B}_{\delta_1}(Dd(x))$. In particular, since $\delta_1 < 1$, 0 is not an element of $\overline{B}_{\delta_1}(Dd(x))$ and so $Dd_s(x) \neq 0$. Moreover, since $\overline{B}_1(0)$ is compact and convex, by Lemma 6.29 again, if $s > S_3$, then $Dd_s(x) \in \overline{B}_1(0)$ and so $\|Dd_s(x)\| \leqslant 1$.

If $x \in d^{-1}([r - \eta_1, r + \eta_1]) \cap X$ and if $y \in B_{1/S_3}(x)$, then, in particular, y is an element of $X_1 \cap d^{-1}([r/2, \rho_2]) \cap (\Pi^{-1}(X_2 \cap U(K)) \setminus K)$. Thus, by definition of ρ_2, if $D^2d(y)$ is defined and is symmetric, then:

$$D^2d(y) \in \kappa(k - \epsilon/4, 2/d(y), Dd(y)) \subseteq \kappa(k - \epsilon/4, 4/r, Dd(y)).$$

However, by definition of S_3, $Dd(y) \in \overline{B}_{\delta_1}(Dd(x))$ and so, by (D):

$$D^2d(y) \in \kappa(k - \epsilon/2, 4/r, Dd(x)).$$

By Lemma 6.31, $\kappa(k - \epsilon/2, 4/r, Dd(x))$ is compact and convex. Thus, by Lemma 6.29, if $s > S_3$, then:

$$D^2d_s(x) \in \kappa(k - \epsilon/2, 4/r, Dd(x)).$$

Since $Dd_s(x) \in \overline{B}_{\delta_1}(Dd(x))$, it follows by (D) again that:

$$D^2d_s(x) \in \kappa(k - \epsilon, 4/r, Dd_s(x)).$$

Choose $S_4 > S_3$ such that $1/S_4 < \eta_1$. Then for all $x \in \mathbb{R}^{n+1}$, if $y \in B_{1/S_4}(x)$, then:

$$d(y) \in [d(x) - \eta_1, d(x) + \eta_1].$$

Since this set is convex and compact, by Lemma 6.29, if $s > S_4$, then:

$$d_s(x) \in [d(x) - \eta_1, d(x) + \eta_1].$$

In particular, if $x \in X$ and if $d_s(x) = r$, then $d(x) \in [r - \eta_1, r + \eta_1]$ and so:

$$D^2d_s(x) \in \kappa(k - \epsilon, 4/r, Dd(x)),$$

as desired. $\square$

Theorem 6.34

For every compact subset X of U and for all $\epsilon > 0$, there exists $\rho > 0$ with the property that for all $r < \rho$, there exists $S > 0$ such that if $s > S$, if $x \in X$ and if $d_s(x) = r$, then $d_s^{-1}(\{r\})$ is smooth near x and has gaussian curvature at least $k - \epsilon$ at x.

Proof: Let ρ be as in Lemma 6.33. Choose $r < \rho$. Let S be as in Lemma 6.33. Choose $s > S$. We denote $\Sigma_{r,s} = d_s^{-1}(\{r\})$. Choose $x \in X \cap \Sigma_{r,s}$. By Lemma 6.33, $Dd_s(x) \neq 0$ and $\|Dd_s(x)\| \leqslant 1$. Thus, by Lemma 6.30, $\Sigma_{r,s}$ is smooth near x and $Dd_s(x)/\|Dd_s(x)\|$ is the normal to $\Sigma_{r,s}$ at x. Moreover, if we denote by $A(x)$ the shape operator of $\Sigma_{r,s}$ at x with respect to this normal, then, for all vectors X and Y tangent to $\Sigma_{r,s}$ at x:

$$A(x)(X, Y) = \frac{1}{\|Dd_s(x)\|} D^2d_s(x)(X, Y).$$

Thus, bearing in mind that $\|Dd_s(x)\| \leqslant 1$, if we denote by $\kappa(x)$ the gaussian curvature of Σ at x, then:

$$\kappa(x) = \mathrm{Det}(A(x))^{1/n} \geqslant \mathrm{Det}(D^2d(x); \langle Dd_s(x)\rangle^{\perp})^{1/n}.$$

However, by Lemma 6.33:

$$D^2d(x) \in \kappa(k - \epsilon, 4/r, Dd_s(x)/\|Dd_s(x)\|),$$

and so $\kappa(x) \geqslant k - \epsilon$, as desired. $\square$

6.6 Weak Barriers.

Let $U \subseteq \mathbb{R}^{n+1}$ be an open set. Let $k > 0$ be a positive real number. Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. We say that K is a **strong barrier** of gaussian curvature at least k inside U whenever $(\partial K) \cap U$ is smooth and has gaussian curvature at least k at every point. We say that K is a **weak barrier** of gaussian curvature at least k inside U whenever there exists a sequence $(\epsilon_m)_{m\in\mathbb{N}} > 0$ converging to 0, an increasing sequence $(V_m)_{m\in\mathbb{N}}$ of open sets and a sequence $(K_m)_{m\in\mathbb{N}}$ of convex sets converging to K in the Hausdorff sense with the properties that $U = \cup_{m\in\mathbb{N}} V_m$ and, for all m, K_m is a strong barrier of gaussian curvature at least $k - \epsilon_m$ inside V_m.

We first show that the set of weak barriers is closed under taking limits in the Hausdorff topology:

Lemma 6.35

Let $U \subseteq \mathbb{R}^{n+1}$ be an open set. Let $k > 0$ be a positive real number. Let $(U_m)_{m\in\mathbb{N}}$ be an increasing sequence of open sets such that $U = \cup_{m\in\mathbb{N}} U_m$. Let $(k_m)_{m\in\mathbb{N}}$ be a sequence positive real numbers converging to k. Let $(K_m)_{m\in\mathbb{N}}, K_\infty$ be compact, convex subsets of $\mathbb{R}^{n+1}$ and suppose that $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense. If K_m is a weak barrier of gaussian curvature at least k inside U_m for all m, then K_∞ is a weak barrier of gaussian curvature at least k inside U.

Proof: Upon extracting a subsequence, we may suppose that for all m, $d_H(K_m, K_\infty) \leqslant 1/m$ and $k_m \geqslant k - 1/m$. For all m, let $(\epsilon_{m,p})_{p\in\mathbb{N}} > 0$ be a sequence converging to 0, let $(V_{m,p})_{p\in\mathbb{N}}$ be an increasing sequence of open subsets of U_m such that $U_m = \cup_{p\in\mathbb{N}} V_{m,p}$ and let $(K_{m,p})_{p\in\mathbb{N}}$ be a sequence of convex sets converging to K_m in the Hausdorff sense such that, for all m, $K_{m,p}$ is a strong barrier of gaussian curvature at least $k_m - \epsilon_{m,p}$ inside $V_{m,p}$. Upon extracting subsequences, we may suppose, in addition, that for all m and for all p, $\epsilon_{m,p} \leqslant 1/p$ and $d_H(K_{m,p}, K_m) \leqslant 1/p$. Let $(V_m)_{m\in\mathbb{N}}$ be an increasing sequence of relatively compact open subsets of U such that $U = \cup_{m\in\mathbb{N}} V_m$. Upon extracting subsequences, we may suppose moreover, that $V_p \subseteq V_{m,p}$ for all m and for all p. For all m, we now define $K'_m = K_{m,m}$. Then, for all m, $d_H(K'_m, K_\infty) \leqslant 2/m$ and K'_m is a strong barrier of gaussian curvature at least $k_m - 1/m \geqslant k - \frac{2}{m}$ over V_m. In particular, $(K'_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense and we conclude that K_∞ is a weak barrier of gaussian curvature at least k over U as desired. □

We now show that the set of weak barriers in closed under intersection:

Lemma 6.36

Let $U \subseteq \mathbb{R}^{n+1}$ be an open set. Let $k > 0$ be a positive real number. Let K_1 and K_2 be compact, convex subsets of $\mathbb{R}^{n+1}$. If K_1 and K_2 are both weak barriers of gaussian curvature at least k inside U, and if $K_1 \cap K_2$ has non-trivial interior, then $K_1 \cap K_2$ is also a weak barrier of gaussian curvature at least k inside U.

Proof: By definition, for each $i \in \{1, 2\}$, there exist an increasing sequence $(V_{i,m})_{m\in\mathbb{N}}$ of open subsets of U, a sequence $(\epsilon_{i,m})_{m\in\mathbb{N}}$ of positive real numbers converging to 0, and a sequence $(K_{i,m})_{m\in\mathbb{N}}$ of compact, convex subsets of $\mathbb{R}^{n+1}$ converging to K_i in the Hausdorff sense with the properties that $U = \cup_{m\in\mathbb{N}} V_{i,m}$ and, for all m and for all $x \in (\partial K_{i,m}) \cap V_{i,m}$, $(\partial K_{i,m})$ is smooth near x and has gaussian curvature at least $k - \epsilon_{i,m}$ at x. Let V_m be an increasing sequence of relatively compact, open subsets of U such that $U = \cup_{m\in\mathbb{N}} V_m$. By compactness, upon extracting subsequences, we may suppose that for all m, $V_m \subseteq V_{1,m}, V_{2,m}$. For all m, we denote $\epsilon_m = \mathrm{Max}(\epsilon_{1,m}, \epsilon_{2,m})$. Then $(\epsilon_m)_{m\in\mathbb{N}}$ also converges to 0, and, for all m, for each i, and for all $x \in (\partial K_{i,m}) \cap V_m$, $(\partial K_{i,m})$ is smooth near x and has gaussian curvature at least $k - \epsilon_m$ at x.

Let $(W_m)_{m\in\mathbb{N}}$ be a sequence of relatively compact open subsets of U such that $U = \cup_{m\in\mathbb{N}} W_m$ and, for all m, $\overline{W}_m \subseteq V_m$. For all m, we denote $K_m = K_{1,m} \cap K_{2,m}$. Observe that $(K_m)_{m\in\mathbb{N}}$ converges to $K_1 \cap K_2$ in the Hausdorff sense. Upon extracting a subsequence, we may therefore suppose that for all m, $d_H(K_m, K_1 \cap K_2) \leqslant 1/m$. Choose $m \in \mathbb{N}$. We denote by d_m the distance in $\mathbb{R}^{n+1}$ to K_m. By Theorem 6.34, there exists $r < 2/3m$ and $S > 2/r$ such that if $s > S$, if $x \in \overline{W}_m$ and if $d_{m,s}(x) = r$, then $d_{m,s}^{-1}(\{r\})$ is smooth near x and has gaussian curvature at least $k - 2\epsilon_m$ at x. In particular, if we denote $K'_m = d_{m,s}^{-1}(]-\infty, r])$ then for all m, K'_m is a strong barrier of gaussian curvature at least $k - 2\epsilon_m$ in W_m.

For all $x \in \mathbb{R}^{n+1}$, if $y \in \overline{B}_{1/S}(x)$, then $d_m(y) \in [d_m(x) - r/2, d_m(x) + r/2]$. Thus, by Lemma 6.29, if $s > S$, then for all $x \in \mathbb{R}^{n+1}$, $d_{m,s}(x) \in [d_m(x) - r/2, d_m(x) + r/2]$. Thus:

$$K_m = d_m^{-1}(]-\infty, 0]) \subseteq d_m^{-1}(]-\infty, r/2]) \subseteq K'_m \subseteq d_m^{-1}(-]\infty, 3r/2]).$$

In particular, $d_H(K_m, K'_m) \leqslant 3r/2 < 1/m$. It follows by the triangle inequality that $d_H(K'_m, K_1 \cap K_2) < 2/m$. Thus $(K'_m)_{m\in\mathbb{N}}$ converges to $K_1 \cap K_2$ in the Hausdorff sense, and we conclude that $K_1 \cap K_2$ is a weak barrier of gaussian curvature at least k in U as desired. □

We refine Lemma 6.36 in order to construct a local excision operation which allows us to obtain regularity for extremal weak barriers as we shall see presently.

Lemma 6.37

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let V be an open, convex subset of $\mathbb{R}^{n+1}$. Let L be a compact, convex subset of $\overline{V}$. If $K \cap (\partial V) \subseteq L$, then $(K \setminus \overline{V}) \cup (K \cap L)$ is compact and convex.

Proof: We denote $K' = (K \setminus \overline{V}) \cup (K \cap L)$. Observe that since $K \cap (\partial V) \subseteq L$, $K' \subseteq (K \setminus V) \cup (K \cap L)$. In particular, since both $K \setminus V$ and $K \cap L$ are compact, so too is K'. Choose $x, x' \in K'$. For all $t \in [0,1]$, we denote $x_t = (1-t)x + tx'$. We claim that $x_t \in K'$ for all t. Observe that x and x' are both elements of K. Thus, by convexity, $x_t \in K$ for all t. Let I be the set of all t such that $x_t \in \overline{V}$. Since $\overline{V}$ is compact and convex, so is I. In particular, I is a closed subinterval of $[0,1]$. Observe that if $t \in (\partial I) \cap]0,1[$, then x_t is an element of $K \cap (\partial V)$, and so, by hypothesis, x_t is an element of L. If $t \in (\partial I) \cap \{0,1\}$, then, $x_t \in \overline{V}$ and so, by definition of K', $x_t \in K \cap L \subseteq L$. Combining these cases, we conclude that $x_t \in L$ for each $t \in \partial I$. By convexity, it follows that $x_t \in L$ for all $t \in I$. In particular, $x_t \in K \cap L \subseteq K'$ for all $t \in I$. However, for all $t \in [0,1] \setminus I$, $x_t \in K \setminus \overline{V} \subseteq K'$, and we conclude that $x_t \in K'$ for all $t \in [0,1]$. Since $x, x' \in K'$ are arbitrary, it follows that K' is convex as desired. □

Lemma 6.38

Choose $k > 0$. Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let U be an open subset of $\mathbb{R}^{n+1}$ and suppose that K is a strong barrier of gaussian curvature at least k in U. Let V be an open, convex subset of $\mathbb{R}^{n+1}$ whose closure is contained in U, and let L be a compact, convex subset of $\overline{V}$. If L is a strong barrier of gaussian curvature at least k in V and if $K\cap(\partial V)$ is contained in the relative interior of L in $\overline{V}$, then $(K\setminus\overline{V})\cup(K\cap L)$ is a weak barrier of gaussian curvature at least k in U.

Proof: We denote $K' = (K\setminus\overline{V})\cup(K\cap L)$. Let d', d_K and d_L be the respective distances in $\mathbb{R}^{n+1}$ to K', K and L. Likewise, let Π', Π_K and Π_L be the respective closest point projections onto K', K and L. Since $K\setminus V$ is compact, and since $K\cap(\partial V)$ is contained in the relative interior of L in $\overline{V}$, there exists $\delta > 0$ such that for all $x\in K\setminus V$, $(\overline{B}_\delta(x)\cap\overline{V})\subseteq L$. Consequently, for all such x:

$$\overline{B}_\delta(x)\cap K\cap\overline{V}\subseteq\overline{B}_\delta(x)\cap K\cap L = \overline{B}_\delta(x)\cap K'\cap\overline{V}.$$

Thus, for all such x:

$$\begin{aligned}\overline{B}_\delta(x)\cap K &= (\overline{B}_\delta(x)\cap(K\setminus\overline{V}))\cup(\overline{B}_\delta(x)\cap(K\cap\overline{V}))\\ &\subseteq(\overline{B}_\delta(x)\cap(K'\setminus\overline{V}))\cup(\overline{B}_\delta(x)\cap(K'\cap\overline{V}))\\ &= \overline{B}_\delta(x)\cap K'.\end{aligned}$$

However, $K'\subseteq K$, and so $\overline{B}_\delta(x)\cap K'\subseteq\overline{B}_\delta(x)\cap K$. It follows that for all $x\in K\setminus V$:

$$B_\delta(x)\cap K' = B_\delta(x)\cap K. \qquad (E)$$

We now define X by:

$$X = K'\setminus\bigcup_{x\in K'\setminus V}B_\delta(x).$$

Observe that X is compact. Moreover, by definition, X is contained in V. Choose $\rho_1 > 0$ such that $\overline{B}_{2\rho_1}(X)\subseteq V$. Consider $y\in\mathbb{R}^{n+1}\setminus\overline{B}_{\rho_1}(X)$ and suppose that $d'(y) < \rho_1$. Let $z = \Pi'(y)$ be the point in K' minimising distance to y. Observe that $z\notin X$. It follows that there exists $z'\in K'\setminus V$ such that $z\in B_\delta(z')$. We denote $\mathsf{N} = (y-z)/\|y-z\|$. By Lemma 4.5, N is a supporting normal to K' at z. By Lemma 4.9, N is a supporting normal to $\overline{B}_\delta(z')\cap K'$ at z. However, by (E), $\overline{B}_\delta(z')\cap K' = \overline{B}_\delta(z')\cap K$, and so, by Lemma 4.9 again, N is a supporting normal to K at z. It follows by Lemma 4.5 that z is also the closest point in K to y. In particular:

$$d_K(y) = \|y-z\| = d'(y).$$

We denote $X_1 = \overline{B}_{2\rho_1}(X)$. Observe that X_1 is compact. We claim that for all $r < \rho_1$, there exists $S_1 := S_1(r) > 0$ such that if $s\geqslant S_1$, if $x\notin X_1$ and if $d'_s(x) = r$, then there exists a neighbourhood Ω of x such that $d'_s(y) = d_{K,s}(y)$ for all $y\in\Omega$. Indeed, choose $r < \rho_1$. Choose $\eta_1, S_0 > 0$ such that $\eta_1 + 1/S_0 < \mathrm{Min}(r/2, \rho_1 - r)$. Observe that if $x\notin X_1$, if $d'(x) \in [r - \eta_1, r + \eta_1]$, and if $y \in \overline{B}_{1/S_0}(x)$, then $d'(y) < \rho_1$ and $y \notin \overline{B}_{\rho_1}(X)$. Thus, by the preceeding discussion, $d'(y) = d_K(y)$. Consequently, for all $s \geqslant S_0$:

$$d'_s(x) = d_{K,s}(x).$$

Choose $S_1 := S_1(r) > \mathrm{Max}(2/\eta_1, S_0)$. Choose $s > S_1$. For all $x \in \mathbb{R}^{n+1}$ and for all $y \in B_{1/S_1}(x)$, $d'(y) \in [d'(x) - \eta_1/2, d'(x) + \eta_1/2]$. Thus, by Lemma 6.29, $d'_s(x) \in [d'(x) - \eta_1/2, d'(x) + \eta_1/2]$. Thus, if $x \notin X_1$ and $d'_s(x) \in [d'(x) - \eta_1/2, d'(x) + \eta_1/2]$, then $d'(x) \in [r - \eta_1, r + \eta_1]$ and so $d'_s(x) = d_{K,s}(x)$. Since X_1 is compact, if $x \notin X_1$, there exists a neighbourhood Ω of x such that $y \notin X_1$ for all $y \in \Omega$. Moreover, by continuity, if, in addition, $d'_s(x) = r$, we may suppose that $d'_s(y) \in [r - \eta_1/2, r + \eta_1/2]$ for all $y \in \Omega$. It follows that $d'_s(y) = d_{K,s}(y)$ for all $y \in \Omega$, as desired.

Let $(W_m)_{m\in\mathbb{N}}$ be an increasing family of relatively compact open subsets of U with closure contained in U such that $U = \cup_{m\in\mathbb{N}} W_m$. Choose $m \in \mathbb{N}$. By Theorem 6.34, there exists $\rho_2 < \rho_1$ with the property that for all $r < \rho_2$, there exists $S_2 := S_2(r) > 0$ such that if $s \geqslant S_2$, if $x \in W_m \cap X_1$ and if $d'_s(x) = r$, then $(d'_s)^{-1}(\{r\})$ is smooth near x and. has gaussian curvature at least $k - 1/m$ at x. Choose $R > 0$ such that $K \subseteq \overline{B}_R(0)$. By Theorem 6.34 again (with $K_2 = \overline{B}_R(0)$), there exists $\rho_3 < \rho_2$ with the property that for all $r < \rho_3$, there exists $S_3 := S_3(r) > 0$ such that if $s \geqslant S_3$, if $x \in W_m$ and if $d_{K,s}(x) = r$, then $(d_{K,s})^{-1}(\{r\})$ is smooth near x and has gaussian curvature at least $k - 1/m$ at x.

Now fix $r < \mathrm{Min}(\rho_3, 2/3m)$ and choose $S_4 := S_4(r) > \mathrm{Max}(S_1(r), S_2(r), S_3(r), 2/r)$. By definition of S_1, if $s \geqslant S_4$, if $x \in W_m \setminus X_1$, and if $d'_s(x) = r$, then there exists a neighbourhood Ω of x such that $d_s(y) = d_{K,s}(y)$ for all $y \in \Omega$. In particular, $(d'_s)^{-1}(\{r\}) \cap \Omega = (d_{K,s})^{-1}(\{r\}) \cap \Omega$. Thus, by definition of S_3, $(d'_s)^{-1}(\{r\})$ is smooth near x and has gaussian curvature at least $k - 1/m$ at x. On the other hand, by definition of S_2, if $s \geqslant S_4$, if $x \in X_1$, and if $d'_s(x) = r$, then $(d'_s)^{-1}(\{r\})$ is smooth near x and has gaussian curvature at least $k - 1/m$ at x. Thus, if we denote $K_m = (d'_s)^{-1}(]0, r])$, then K_m is a strong barrier of gaussian curvature at least $k - 1/m$ in W_m.

By definition of S_4, for all $x \in \mathbb{R}^{n+1}$ and for all $y \in \overline{B}_{1/S_4}(x)$, $d'(y) \in [d'(x) - r/2, d'(x) + r/2]$. Thus, for all $x \in \mathbb{R}^{n+1}$, by Lemma 6.29, $d'_s(x) \in [d'(x) - r/2, d'(x) + r/2]$. Thus:

$$K = (d')^{-1}(] - \infty, 0]) \subseteq (d')^{-1}(] - \infty, r/2]) \subseteq K_m,$$

and, since $r < 2/3m$:

$$K_m \subseteq (d')^{-1}(] - \infty, 3r/2]) \subseteq (d')^{-1}(] - \infty, 1/m]).$$

It follows that $d_H(K, K_m) \leqslant 1/m$. Since this holds for all m, it follows that $(K_m)_{m\in\mathbb{N}}$ converges to K in the Hausdorff sense, and so K is a weak barrier of gaussian curvature at least k over U as desired. □

6.7 The Plateau Problem.

The machinery developed in the preceeding sections allows us to solve a general version of the Plateau Problem. Let K be a compact, convex subset of $\mathbb{R}^{n+1}$ with smooth boundary and non-trivial interior. Let X be a closed subset of the boundary of K such that $\mathrm{Conv}(X)$ also has non-trivial interior. Choose $k > 0$, and suppose that ∂K has gaussian curvature at least k at every point of $(\partial K) \setminus X$. Observe that, using the terminology of the preceeding section, this means that K is a strong barrier of gaussian curvature at least k in $\mathbb{R}^{n+1} \setminus X$. We define the family $\mathcal{B}(k, K, X)$ to be the set of all compact, convex subsets K' of $\mathbb{R}^{n+1}$ with the properties that $X \subseteq K' \subseteq K$ and K' is a weak barrier of gaussian curvature at least k in $\mathbb{R}^{n+1} \setminus X$. Since strong barriers are also weak barriers, we see that K itself is an element of $\mathcal{B}(k, K, X)$ and so this family is non-empty.

Lemma 6.39

If L is an element of $\mathcal{B}(k, K, X)$, then L has non-trivial interior.

Proof: By definition, L is compact and convex. Thus, by Lemma 4.17, $L = \mathrm{Conv}(L)$. Since $X \subseteq L$, it follows that $\mathrm{Conv}(X) \subseteq \mathrm{Conv}(L) = L$. Since $\mathrm{Conv}(X)$ has non-trivial interior, it follows that L too has non-trivial interior, as desired. □

For any Borel measurable subset X of $\mathbb{R}^{n+1}$, we define the **volume** of X to be its $(n+1)$-dimensional Lebesgue measure. We denote the volume of X by $\mathrm{Vol}(X)$.

Lemma 6.40

There exists $V_0 > 0$ such that:

$$V_0 = \operatorname*{Inf}_{L \in \mathcal{B}(k,K,X)} \mathrm{Vol}(L).$$

Proof: Choose $L \in \mathcal{B}(k, K, X)$. By definition, L is compact and convex. Thus, by Lemma 4.17, $L = \mathrm{Conv}(L)$. Since $X \subseteq L$, it follows that $\mathrm{Conv}(X) \subseteq \mathrm{Conv}(L) = L$. Since $\mathrm{Conv}(X)$ has non-trivial interior, $\mathrm{Vol}(\mathrm{Conv}(X)) > 0$. Thus, by monotonicity of Lebesgue measure, $\mathrm{Vol}(L) \geqslant \mathrm{Vol}(\mathrm{Conv}(X))$. It follows that:

$$V_0 = \operatorname*{Inf}_{L \in \mathcal{B}(k,K,X)} \mathrm{Vol}(L) \geqslant \mathrm{Vol}(\mathrm{Conv}(X)) > 0,$$

as desired. □

Lemma 6.41

Let $K_0 \subseteq K_1$ be compact, convex subsets of $\mathbb{R}^{n+1}$. If $K_0 \neq K_1$, then $\mathrm{Vol}(K_0) < \mathrm{Vol}(K_1)$.

Proof: Choose $x \in K_1 \setminus K_0$. Since K_0 is compact, there exists $\delta_1 > 0$ such that $B_{\delta_1}(x) \cap K_0 = \emptyset$. Let y be an interior point of K_0. There exists $\delta_2 > 0$ such that $B_{\delta_2}(y) \subseteq K_0$. By convexity, for all $t \in]0,1]$, and for all $z \in B_{t\delta_2}(0)$:

$$(1-t)x + ty + z = (1-t)x + t(y + z/t) \in K_1.$$

Thus, for all $t \in]0,1]$, $B_{t\delta_2}((1-t)x+ty) \subseteq K_1$. Choose $t > 0$ such that $t(\|y-x\|+\delta_2) < \delta_1$. In particular $B_{t\delta_2}((1-t)x+ty) \cap K_0 = \emptyset$. Thus, by additivity and monotonicity of Lebesgue measure:

$$\mathrm{Vol}(K_1) \geqslant \mathrm{Vol}(K_0) + \mathrm{Vol}(B_{t\delta_2}((1-t)x+ty)) > \mathrm{Vol}(K_0),$$

as desired. □

We now show that the infimal volume is realised by a unique element of $\mathcal{B}(k, K, X)$:

Lemma 6.42

There exists a unique element $K_0 \in \mathcal{B}(k, K, X)$ such that for all $L \in \mathcal{B}(k, K, X)$:

$$\mathrm{Vol}(K_0) \leqslant \mathrm{Vol}(L).$$

Proof: We first show uniqueness. Indeed, suppose that there exists $K_0 \neq K_0' \in \mathcal{B}(k, K, X)$ such that for all $L \in \mathcal{B}(k, X, X)$:

$$\mathrm{Vol}(K_0), \mathrm{Vol}(K_0') \leqslant \mathrm{Vol}(L).$$

In particular, $\mathrm{Vol}(K_0) \leqslant \mathrm{Vol}(K_0')$ and $\mathrm{Vol}(K_0') \leqslant \mathrm{Vol}(K_0)$ and so $\mathrm{Vol}(K_0) = \mathrm{Vol}(K_0')$. Since $K_0 \neq K_0'$, without loss of generality, we may assume that $K_0 \cap K_0' \neq K_0$. Since X is contained in each of K_0 and K_0', it is also contained in $K_0 \cap K_0'$. Moreover, since both K_0 and K_0' are contained in K, so is $K_0 \cap K_0'$. Finally, by Lemma 6.36, $K_0 \cap K_0'$ is a weak barrier of gaussian curvature at least k over $\mathbb{R}^{n+1} \setminus X$ and it follows that $K_0 \cap K_0'$ is an element of $\mathcal{B}(k, K, X)$. However, by Lemma 6.41, $\mathrm{Vol}(K_0 \cap K_0') < \mathrm{Vol}(K_0)$. This contradicts minimality of K_0, and uniqueness follows.

We now prove existence. Define $V_0 > 0$ by:

$$V_0 = \operatorname*{Inf}_{L \in \mathcal{B}(k,K,X)} \mathrm{Vol}(L).$$

Let $(L_m)_{m\in\mathbb{N}} \in \mathcal{B}(k, K, X)$ be a sequence such that $(\mathrm{Vol}(L_m))_{m\in\mathbb{N}}$ converges to V_0. For all m, we define $K_m = L_1 \cap ... \cap L_m$. For all m, $X \subseteq K_m \subseteq K$, and, by Lemma 6.36, K_m is a weak barrier of gaussian curvature at least k over $\mathbb{R}^{n+1} \setminus X$, and so $K_m \in \mathcal{B}(k, K, X)$. Moreover, by definition of V_0, and by monotonicity of the Lebesgue measure, for all m, $V_0 \leqslant \mathrm{Vol}(K_m) \leqslant \mathrm{Vol}(L_m)$. In particular, $(\mathrm{Vol}(K_m))_{m\in\mathbb{N}}$ also converges to V_0. We define K_∞ by:

$$K_\infty = \bigcap_{m\in\mathbb{N}} K_m.$$

Observe that for all m, $\mathrm{Vol}(K_\infty) \leqslant \mathrm{Vol}(K_m)$. Taking limits yields $\mathrm{Vol}(K_\infty) \leqslant V_0$. We claim that $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense. Indeed, suppose the contrary. Since $K_\infty \subseteq K_m$ for all m, we may suppose that there exists $\epsilon > 0$ with the property that, upon extracting a subsequence, for all m, there exists $x_m \in K_m$ such that $d(x_m, K_\infty) \geqslant \epsilon$. Observe that $x_m \in K$ for all m. Thus, by compactness, upon extracting a subsequence, we may suppose that there exists $x_\infty \in K$ towards which $(x_m)_{m\in\mathbb{N}}$ converges. By continuity:

$$d(x_\infty, L_\infty) = \operatorname*{Lim}_{m\to\infty} d(x_m, K_\infty) \geqslant \epsilon.$$

Observe that $x_m \in K_l$ for all $l \leqslant m$. It follows upon taking limits that $x_\infty \in K_l$ for all l. Thus, taking the intersection over all l, we conclude that $x_\infty \in K_\infty$. In particular, $d(x_\infty, K_\infty) = 0$. This is absurd, and it follows that $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense as asserted. By Lemma 6.35, K_∞ is therefore also an element of $\mathcal{B}(k,K,X)$. In particular $V_0 \leqslant \mathrm{Vol}(K_\infty)$, and so $\mathrm{Vol}(K_\infty) = V_0$. K_∞ is therefore an element of $\mathcal{B}(k,K,X)$ minimising volume, as desired. □

We now show that the volume minimiser solves the Plateau Problem up to singularities of the type described in Section 4:

Lemma 6.43

Let $K_0 \in \mathcal{B}(k,K,X)$ by such that $\mathrm{Vol}(K_0) \leqslant \mathrm{Vol}(L)$ for all $L \in \mathcal{B}(k,K,X)$. Choose $x \in (\partial K_0) \setminus X$. Then either:

(1) (∂K_0) is smooth near x and has gaussian curvature equal to k at x; or

(2) K_0 satisfies the local geodesic property at x.

Proof: Choose $x \in (\partial K_0) \setminus X$ and suppose that K_0 does not satisfy the local geodesic property at x. Let U be a relatively compact neighbourhood of x whose closure is contained in $\mathbb{R}^{n+1} \setminus X$. By definition of weak barriers, there exists a sequence $(K_m)_{m\in\mathbb{N}}$ of compact, convex subsets of $\mathbb{R}^{n+1}$ with the properties that $(K_m)_{m\in\mathbb{N}}$ converges to K_0 in the Hausdorff sense and for all m, K_m is a strong barrier of gaussian curvature at least $k - 1/m$ in U. We denote $K_\infty = K_0$ and $x_\infty = x_0$.

Let $(x_m)_{m\in\mathbb{N}}$ be a sequence converging to x_∞ such that $x_m \in \partial K_m$ for all m. Upon applying a sequence of affine isometries converging to the identity mapping, we may suppose that $x_m = 0$ for all $m \in \mathbb{N} \cup \{\infty\}$. Since K_∞ has non-trivial interior, by Lemma 4.27, $\mathcal{N}(0;K_\infty)$ is strictly contained in a hemisphere. By Lemma 4.28, there exists $\mathsf{N} \in \mathcal{N}(0;K_\infty)$ such that $\langle \mathsf{N}, \mathsf{M}\rangle > 0$ for all $\mathsf{M} \in \mathcal{N}(0;K_\infty)$. By compactness of $\mathcal{N}(0;K_\infty)$, there exists $\theta \in [0,\pi/2[$ such that $\langle \mathsf{N}, \mathsf{M}\rangle > 3\cos(\theta)$ for all $\mathsf{M} \in \mathcal{N}(0;K_\infty)$. We denote $C = \tan(\theta)$.

By Lemma 4.3, there exists $r > 0$ and $M \in \mathbb{N}$ such that $\overline{B}_r(0) \subseteq U$ and, for all $m \geqslant M$, for all $x \in (\partial K_m) \cap B_r(0)$ and for all $\mathsf{M} \in \mathcal{N}(x;K_m)$, $\langle \mathsf{N}, \mathsf{M}\rangle > 2\cos(\theta)$. Upon extracting a subsequence, we may suppose that $M = 1$. We denote $\rho = r/\sqrt{1+4C^2}$. By Lemma 4.23, there exists N', which we may choose as close to N as we wish such that for all $x \in K_\infty \setminus B_{\rho/2}(0)$, $\langle x, \mathsf{N}'\rangle < 0$. Moreover, we may assume that for all $m \in \mathbb{N} \cup \{\infty\}$, for all $x \in (\partial K_m) \cap B_r(0)$ and for all $\mathsf{M} \in \mathcal{N}(x;K_m)$, $\langle \mathsf{N}', \mathsf{M}\rangle > \cos(\theta)$.

Upon applying a rotation, we may suppose that $\mathsf{N}' = -e_{n+1}$. By Theorem 4.13, for all $m \in \mathbb{N} \cup \{\infty\}$, there exists a convex, C-Lipschitz function $\hat{f}_m : B'_\rho(0) \to]-C\rho, C\rho[$ such that $\hat{f}_m = 0$ and $(\partial K_m) \cap (B'_\rho(0) \times]-2C\rho, 2C\rho[)$ coincides with the graph of $\hat{f}_m$ over $B'_\rho(0)$.

Since $\hat{f}_m(0) = 0$ and $\hat{f}_m$ is C-Lipschitz for all m, by the Arzela-Ascoli theorem, every subsequence of $(\hat{f}_m)_{m\in\mathbb{N}}$ has a subsubsequence converging in the local uniform sense over $B'_\rho(0)$ to some limit $\hat{f}'_\infty$ say. Since $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense, it follows that $\hat{f}'_\infty = \hat{f}_\infty$, and we conclude that $(\hat{f}_m)_{m\in\mathbb{N}}$ converges in the local uniform sense over $B'_\rho(0)$ to $\hat{f}_\infty$.

By construction, $\hat{f}_\infty(x') > 0$ for all $x' \in \partial B'_{\rho/2}(0)$. By compactness, there exists $\delta > 0$ such that $\hat{f}_\infty(x') > 2\delta$ for all $x' \in \partial B'_{\rho/2}(0)$. Since $(\hat{f}_m)_{m\in\mathbb{N}}$ converges locally uniformly to $\hat{f}_\infty$ over $B'_\rho(0)$, there exists $M \in \mathbb{N}$ such that for all $m \geqslant M$, and for all $x' \in \partial B'_{\rho/2}(0)$, $\hat{f}_m(x') > \delta$. Upon extracting a subsequence, we may suppose that $M = 1$.

Choose $m \in \mathbb{N}$. Observe that $\hat{f}_m$ is smooth. We denote $\overline{\Omega}_m = \hat{f}_m^{-1}(]-\infty,\delta])$. Observe that $\overline{\Omega}_m$ is a compact, convex subset of $B'_{\rho/2}(0)$. Since $B'_\rho(0)$ is a convex set, and since $\hat{f}_m$ is a convex function, $D\hat{f}_m$ only vanishes at the absolute minimum of $\hat{f}_m$ over $B'_{\rho/2}(0)$. Since $\hat{f}_m(0) = 0$, it follows that the absolute minimum of $\hat{f}_m$ in $B'_{\rho/2}(0)$ is contained in the interior of $\overline{\Omega}_m$. In particular, $D\hat{f}_m$ does not vanish at any boundary point of $\overline{\Omega}_m$, and so $\overline{\Omega}_m$ has smooth boundary. By Theorem 1.4, there exists a unique, smooth, strictly convex function $f_m : \overline{\Omega}_m \to]-\infty,\delta]$ such that $f_m(x') = \delta$ for all $x' \in \partial\Omega_m$ and the graph of f_m has constant gaussian curvature equal to k. By convexity, $f_m \leqslant \delta$. By Lemma 2.12, $f_m \geqslant \hat{f}_m$.

We define $V_m = \Omega_m \times]-(3/2)C\rho,(3/2)C\rho[$. Observe that V_m is open and convex. Moreover, $\overline{V}_m \subseteq \overline{B}_r(0) \subseteq U$. We define the subset L_m of $\overline{V}_m$ by:

$$L_m = \left\{(x',t) \mid x' \in \overline{\Omega}_m \ \&\ f_m(x') \leqslant t \leqslant (3/2)C\rho\right\}.$$

Observe that L_m is compact and convex. Moreover, since $f_m \geqslant \hat{f}_m$, it follows that $L_m \subseteq K_m \cap \overline{V}_m$. We define $K'_m = (K_m \setminus \overline{V}) \cup L_m = (K_m \setminus \overline{V}) \cup (K_m \cap L_m)$. We claim that K'_m is a weak barrier of gaussian curvature at least k in $\mathbb{R}^{n+1} \setminus X$. Indeed, for all $s \in [0,(1/2)C\rho[$, we define the subset $L_{m,s}$ of $\overline{V}_m$ by:

$$L_{m,s} = \left\{(x',t) \mid x' \in \overline{\Omega}_m \ \&\ f_m(x') - s \leqslant t \leqslant (3/2)C\rho\right\}.$$

We define $K'_{m,s} = (K_m \setminus \overline{V}) \cup (K_m \cap L_{m,s})$. For all $s > 0$, by definition of $\hat{f}_m$, $K_m \cap \partial V_m$ is contained in the relative interior of L_m in V_m. Thus, by Lemma 6.38, $K'_{m,s}$ is a weak barrier of gaussian curvature at least k over $\mathbb{R}^{n+1} \setminus X$. However, $(K'_{m,s})_{s\in[0,(1/2)C\rho[}$ converges to $K'_{m,0} = K'_m$ in the Hausdorff sense as s tends to 0, and so, by Lemma 6.35, $K'_{m,0} = K'_m$ is also a weak barrier of gaussian curvature at least k in $\mathbb{R}^{n+1}$ as desired.

By Lemma 4.1, there exists a compact subset K'_∞ of $\mathbb{R}^{n+1}$ towards which $(K'_m)_{m\in\mathbb{N}}$ subconverges. By Lemma 4.2, K'_∞ is also convex. We claim that $K'_\infty = K_\infty$. Indeed, suppose the contrary. By Lemma 6.35, K'_∞ is a weak barrier of gaussian curvature at least k in $\mathbb{R}^{n+1} \setminus X$. Moreover, for all m, $K_m \setminus B_r(0) \subseteq K_m \setminus V_m \subseteq K'_m$, and upon taking limits it follows that $K_\infty \setminus B_r(0) \subseteq K'_\infty$ and, in particular, $X \subseteq K_\infty \setminus B_r(0) \subseteq K'_\infty$. Finally, for all m, $K'_m \subseteq K_m$, and upon taking limits, it follows that $K'_\infty \subseteq K_\infty \subseteq K$. We conclude that K'_∞ is an element of $\mathcal{B}(k,K,X)$. However, since $K'_\infty \subseteq K_\infty$ and $K'_\infty \neq K_\infty$, it follows from Lemma 6.41 that $\mathrm{Vol}(K'_\infty) < \mathrm{Vol}(K_\infty)$. This is absurd by minimality of K_∞ and we conclude that $K'_\infty = K_\infty$ as asserted.

By continuity, there exists $\rho' < \rho$ such that for all $x' \in \overline{B}'_{\rho'}(0)$, $\hat{f}_\infty(x') < \delta/2$. Since $(\hat{f}_m)_{m\in\mathbb{N}}$ converges to $\hat{f}_\infty$ uniformly over $\overline{B}'_{\rho'}(0)$, there exists $M \in \mathbb{N}$ such that for $m \geqslant M$, and for all $x' \in \overline{B}'_{\rho'}(0)$, $\hat{f}_m(x') \leqslant \delta$. In particular, for all $m \geqslant M$, $\overline{B}'_{\rho'}(0) \subseteq \overline{\Omega}_m$. For all m, we therefore define $W = B'_{\rho'}(0)\times] - (3/2)C\rho, (3/2)C\rho[$. Then, for all m, $(\partial K_m) \cap W = (\partial L_m) \cap W$ is smooth with constant gaussian curvature equal to k. Since K_∞ does not satisfy the local geodesic property at x, by Theorem 4.29, $(\partial K_\infty) \cap W$ is smooth with constant gaussian curvature equal to k, as desired. □

Using an ad-hoc argument, we are also able to exclude the existence of boundary points of K_0 satisfying the Local Geodesic Property:

Lemma 6.44

Let $K_0 \in \mathcal{B}(k, K, X)$ by such that $\mathrm{Vol}(K_0) \leqslant \mathrm{Vol}(L)$ for all $L \in \mathcal{B}(k, K, X)$. If X has C^2 boundary, then K_0 does not satisfy the local geodesic property at any point of $(\partial K_0) \setminus X$.

Remark: *In order to prove this result, we make appeal to Theorem* 1.1 *of* [2]. *Thus result, which proves existence of solutions to the classical Monge-Ampère equation, may in fact be obtained by adapting the techniques of Sections 2 and 3. The main extra difficulty lies in proving the existence of a lower barrier for arbitrary boundary data. However, in the case of the classical Monge-Ampère equation, this follows from the strict convexity of the domain. We leave the details as an exercise for the motivated reader.*

Proof: Suppose the contrary. Let Y be the set of all points in $(\partial K_0) \setminus X$ here K_0 satifies the local geodesic property. We first claim that $X \cup Y$ is closed. Indeed, let $x \in (\partial K_0)$ lie in the complement of $X \cup Y$. In particular, K_0 does not satisfy the local geodesic property at x. Thus, by Lemma 6.43, there exists a nieghbourhood U of x such that $(\partial K_0) \cap U$ is smooth and strictly convex. In particular, K_0 does not satisfy the local geodesic property at any point of $(\partial K_0) \cap U$ and so $U \cap Y = \emptyset$. Since x lies in the complement of X and since X is closed, upon reducing U if necessary, we may suppose that $U \cap X = \emptyset$, and so $U \cap (X \cup Y) = \emptyset$. Since $x \in (\partial K_0) \setminus (X \cup Y)$ is arbitrary, it follows that $(\partial K_0) \setminus (X \cup Y)$ is open and so $(X \cup Y)$ is closed as asserted.

By Lemma 4.19, $Y \subseteq \mathrm{Conv}(X)$. Choose $y \in Y$. Let N be a supporting normal to K_0 at y. Let H be the affine hyperplane normal to N passing through Y. Observe that $(Y \cap H) \subseteq \mathrm{Conv}(X \cap H)$. In particular, since Y is non-empty, so is $\mathrm{Conv}(X \cap H)$, and so $X \cap H$ consists of at least two distinct points, x_1 and x_2 say. Let Γ be the closed straight-line segment joining x_1 to x_2. By convexity, $\Gamma \subseteq K_0$. Moreover, $\Gamma \subseteq H$ and so, in particular, Γ lies on the boundary of K_0.

Upon applying an affine isometry, we may suppose that $x_1 = -te_n$ and $x_2 = te_n$ for some $t > 0$, $\mathsf{N} = -e_{n+1}$, and $H = \{x \mid \langle x, e_{n+1}\rangle = 0\}$. In particular, $0 \in \partial K_0$ and $-e_{n+1}$ is a supporting normal to K_0 at 0. Observe that, since K is strictly convex, 0 is an interior point of K. By definition of supporting normals $\langle x, -e_{n+1}\rangle \leqslant 0$ for all $x \in X \subseteq K_0$, and since $\langle x_1, e_{n+1}\rangle = \langle x_2, e_{n+1}\rangle = 0$, it follows that x_1 and x_2 are both boundary points of X. Moreover, ∂X is tangent to H at these points, and since K is strictly convex, Γ is not tangent to ∂X at these points. Consequently, for each i:

$$H = T_{x_i}(\partial X) \oplus T_{x_i}\Gamma = T_{x_i}(\partial X) \oplus \langle e_n\rangle.$$

We identify $\langle e_1, ..., e_{n-1}\rangle$ with $\mathbb{R}^{n-1}$ and for all $r > 0$, we denote by $B''_r(0)$ the ball of radius r about 0 in $\mathbb{R}^{n-1}$. By transversality, and since ∂X is C^2, there exists $r > 0$, open subsets U_1 and U_2 of x_1 and x_2 respectively and C^2 functions $\xi_1, \xi_2 : B''_r(0) \to \mathbb{R}$ and $\eta_1, \eta_2 : B''_r(0) \to [0, \infty[$ such that, for each i, $(\partial X) \cap U_i$ coincides with the graph of (ξ_i, η_i) over $B''_r(0)$. We identify $\langle e_1, ..., e_n \rangle$ with $\mathbb{R}^n$ and we denote by $B'_r(0)$ the ball of radius r about 0 in $\mathbb{R}^n$. Since $\xi_1(0) = -t$ and $\xi_2(0) = t$, by continuity, upon reducing r if necessary, we may suppose that for each i, $\xi_i(x'') > r$ for all $x'' \in B''_r(0)$. We then define $f : B'_r(0) \to [0, \infty[$ by:

$$\hat{f}(x'', s) = \frac{s - \xi_1(x'')}{\xi_2(x'') - \xi_1(x'')}\eta_1(x'') + \frac{\xi_2(x'') - s}{\xi_2(x'') - \xi_1(x'')}\eta_2(x'').$$

Since $\eta_1(0) = \eta_2(0) = 0$, $\hat{f}(0, s) = 0$ for all s. Moreover, by convexity, $(x', \hat{f}(x')) \in K_0$ for all $x' \in B'_r(0)$. Finally, since ξ_1, ξ_2, η_1 and η_2 are C^2, it follows that $\hat{f}$ is also C^2. By Theorem 4.13, upon reducing r if necessary, there exists $C > 0$ and a convex, Lipschitz function $g : B'_r(0) \to]-Cr, Cr[$ such that the intersection of (∂K_0) with $B'_r(0) \times]-2Cr, 2Cr[$ coincides with the graph of g over $B'_r(0)$. Since $\hat{f}$ attains a local maximum of 0 at 0, and since $\hat{f}$ is C^1, $D\hat{f}(0) = 0$ and so, upon reducing r if necessary, we may suppose that $\hat{f}(x') \in]-2Cr, 2Cr[$ for all $x' \in B'_r(0)$. It then follows by definition of g that $\hat{f}(x') \geqslant g(x')$ for all $x' \in B'_r(0)$. Finally, since $\hat{f}$ is C^2, we may perturb it to a smooth function $\hat{f}' : B'_r(0) \to]-\infty, 0]$ such that $\hat{f}'(0, s) = 0$ for all s and $\hat{f}'(x') \geqslant g(x')$ for all $x' \in B'_r(0)$.

By Theorem 1.1 of [2], there exists a smooth, strictly convex function $f : B'_{r/2}(0) \to \mathbb{R}$ such that f coincides with $\hat{f}'$ along $\partial B'_{r/2}(0)$ and $\mathrm{Det}(\mathrm{Hess}(f)) = k/2$. In particular, by Lemma 1.3 the graph of f has gaussian curvature no greater than $k/2$. Observe that since K_0 is a weak barrier of gaussian curvature at least k in $\mathbb{R}^{n+1} \setminus X$, there exists a sequence of smooth, strictly convex functions $(g_m)_{m \in \mathbb{N}} : B'_r(0) \to \mathbb{R}$ converging uniformly to g such that for all m, the graph of g_m has gaussian curvature at least $k/2$. It then follows by Lemma 2.12 that for all $m \in \mathbb{N}$:

$$\operatorname*{Inf}_{x' \in B'_{r/2}(0)} f(x') - g_m(x') = \operatorname*{Inf}_{x' \in \partial B'_{r/2}(0)} f(x') - g_m(x'),$$

and so, upon taking limits, we obtain:

$$\operatorname*{Inf}_{x' \in B'_{r/2}(0)} f(x') - g(x') = \operatorname*{Inf}_{x' \in \partial B'_{r/2}(0)} f(x') - g(x') \geqslant 0.$$

In particular, $f(0) - g(0) \geqslant 0$, and so $g(0) \leqslant f(0)$. However, since $f(0)$ is smooth and strictly convex over $B'_{r/2}(0)$, and since $f(0, \pm r/2) = \hat{f}'(0, \pm r/2) = 0$, it follows that $f(0) < 0$ and so $g(0) < 0$. However, by definition of g, $(0, g(0))$ is an element of K_0. By definition of supporting normals $-g(0) = \langle (0, g(0)) - (0, 0), -e_{n+1} \rangle \leqslant 0$, and so $g(0) \geqslant 0$. This is absurd, and it follows that Y is empty, as desired. □

Combining these results, we obtain the general solution to the Plateau problem described in the introduction:

Theorem 1.2

Choose $k > 0$. Let K be a compact, convex subset of $\mathbb{R}^{n+1}$ with smooth boundary. Let X be a closed subset of ∂K with C^2 boundary $C = \partial X$. If ∂K has gaussian curvature bounded below by k at every point of $(\partial K) \setminus X$, then there exists a compact, strictly convex, $C^{0,1}$ embedded hypersurface $S \subseteq \mathbb{R}^{n+1}$ with the properties that:

(1) $S \subseteq K$;

(2) $\partial S = C$; and

(3) $S \setminus \partial S$ is smooth and has constant gaussian curvature equal to k.

Remark: *In fact, if one supposes that C is C^∞, then we may show that S also has smooth boundary. We shall not study this here.*

Proof: Since K is convex, ∂K is diffeomorphic to the unit sphere Σ^n in $\mathbb{R}^{n+1}$. Since C is embedded and diffeomorphic to the unit sphere $\Sigma^{n-1} \in \mathbb{R}^n$, it follows from the Jordan Sphere Theorem (c.f. [9]) that C divides ∂K into two connected components. Let Ω be one of these components. We denote $X = (\partial K) \setminus \Omega$. Observe that $(\partial X) = C$. Consider the family $\mathcal{B}(k, K, X)$. Observe that K is an element of $\mathcal{B}(k, K, X)$ and so this family is non-trivial. By Lemma 6.42, there exists a compact, convex set $K_0 \in \mathcal{B}(k, K, X)$ minimising volume over all elements of $\mathcal{B}(k, K, X)$. By Lemma 6.44, K_0 does not possess the local geodesic property at any point of $(\partial K_0) \setminus X$. By Lemma 6.43, $(\partial K_0) \setminus X$ is smooth and has constant gaussian curvature equal to k. We denote $S = (\partial K_0) \setminus X^o$.

By definition $S \subseteq K_0 \subseteq K$. Since S is a subset of the boundary of a convex set with non-trivial interior, it follows from Theorem 4.13 that S is a $C^{0,1}$ embedding. By definition $\partial S = \partial X = C$, and finally, $S \setminus (\partial S)$ is smooth and has constant gaussian curvature equal to k, as desired. □

Barcelona-Granada, May-June, 2012

A

Terminology

Derivatives: For any vector spaces E, F, let $\mathrm{Symm}(n, E) \otimes F$ denote the space of symmetric multilinear forms from E into F. When $F = \mathbb{R}$, we denote simply $\mathrm{Symm}(n, E) = \mathrm{Symm}(n, E) \otimes \mathbb{R}$. For any open subset $U \subseteq E$ and for any k-times differentiable function $f : U \to F$, we denote the k'th total derivative by $D^k f : U \to \mathrm{Symm}(k, E) \otimes F$. For any point $p \in U$ and for k vectors $V_1, ..., V_k \in E$, we denote $D^k f(p)(V_1, ..., V_k) \in F$ the image of the k-tuplet $(V_1, ..., V_k)$ under the action of $D^k f$ at the point P.

For any vector spaces E and F, for any open subset U of E, and for all $k \in \mathbb{N}$, we denote by $C^k(U, F)$ the space of k-times continuously differentiable functions from U into F. We denote by $C^\infty(U, F)$ the space of functions from U into F which have continuous derivatives of arbitrarily high order. When $F = \mathbb{R}$, we denote simply $C^k(U) = C^k(U, \mathbb{R})$ and $C^\infty(U) = C^\infty(U, \mathbb{R})$.

For any $k \in \mathbb{N}$ and for any $f \in C^k(U)$, we define $J^k(f) \in C^0(U, \oplus_{k=0}^m \mathrm{Symm}(n, E))$ by:

$$J^k(f)(x) = (f(x), Df(x), ..., D^k f(x)).$$

We refer to $J^k(f)$ as the k-jet of f.

Canonical Basis of Euclidean Space: For all n, we denote by $\mathbb{R}^n$, n-dimensional, real space and by $e_1, ..., e_n$ its canonical basis. We denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product and by $\| \cdot \|$ the Euclidean norm. For any open subset $U \subseteq \mathbb{R}^n$, for any k-times differentiable function $f : U \to \mathbb{R}$, and for any k-tuple of indices $1 \leqslant i_1, ..., i_k \leqslant n$, we define the function $(\partial_{i_1} ... \partial_{i_k} f)$ such that for all $x \in U$:

$$(\partial_{i_1} ... \partial_{i_k} f)(x) = D^k f(x)(e_{i_1}, ..., e_{i_k}).$$

We will also use the more concise notation:

$$f_{i_1...i_k} := (\partial_{i_1}...\partial_{i_k}f).$$

Distributional Derivatives: Let E be a vector space furnished with a volume form dVol. Let U be an open subset of E and let $f : U \to \mathbb{R}$ be a real valued function which is locally L^1. Let $g = (g_0, g_1, ..., g_k) : U \to \oplus_{k=0}^m \text{Symm}(n, E)$ be locally L^1. We say that g is the k'th order distributional derivative of f whenever it has the property that for any smooth function φ with compact support, for all $1 \leqslant k \leqslant n$, and for all vectors $V_1, ..., V_k$:

$$\int_E f(x)(D^k\varphi)(x)(V_1, ...V_k)\text{dVol} = (-1)^k \int_E g_k(x)(V_1, ..., V_k)\varphi(x)\text{dVol}.$$

Smooth Functions on Sets with Boundary: Ω will always represent a bounded, strictly convex, open subset of $\mathbb{R}^n$. Given any vector space E, a function $f : \overline{\Omega} \to E$ is said to be C^k whenever there exists an extension $\hat{f}$ of f to $\mathbb{R}^n$ which is k-times continuously diferentiable. By Whitney's Extension Theorem (c.f. [20]), the extension $\hat{f}$ can be chosen such that for all $k \leqslant l$:

$$\|D^k\hat{f}\|_{L^\infty} = \|D^k\hat{f}|_{\overline{\Omega}}\|_{L^\infty} = \|D^kf\|_{L^\infty}.$$

We say that f is smooth whenever it is C^k for all finite k. Given any open subset U of E, we denote by $C^\infty(\overline{\Omega}, U)$ the set of all smooth functions from $\overline{\Omega}$ into E taking values in U. In particular, when $U = E = \mathbb{R}$, we denote $C^\infty(\overline{\Omega}) = C^\infty(\overline{\Omega}, \mathbb{R})$.

Non-linear Operators: Given open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \text{Symm}(2, \mathbb{R}^n)$ and a smooth function $F : \mathbb{R} \times \text{Symm}(1, \mathbb{R}^n) \times V \to \mathbb{R}$, for any function $f : U \to \mathbb{R}$ with the property that $D^2f(x) \in V$ for all $x \in U$, we define the function $F(f, Df, D^2f)$ such that, for all $x \in U$:

$$F(f, Df, D^2f)(x) = F(f(x), Df(x), D^2f(x)).$$

F thus represents the most general second-order, non-linear partial differential operator acting on functions over U which is homogeneous in the spatial variables.

Decomposition of Euclidean Space: We often decompose $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \mathbb{R}$. For all $r > 0$ and for all $x \in \mathbb{R}^{n+1}$, we denote by $B_r(x)$ the open ball of radius r about x in $\mathbb{R}^{n+1}$. For all $r > 0$ and for all $x' \in \mathbb{R}^n$, we denote by $B'_r(x)$ the open ball of radius r about x in $\mathbb{R}^n$.

Metrics: Let X and Y be two compact subsets of $\mathbb{R}^{n+1}$. We define the Hausdorff distance between X and Y by:

$$d_H(X, Y) = \mathop{\text{Sup}}_{x\in X} \mathop{\text{Inf}}_{y \in Y} \|x - y\| + \mathop{\text{Sup}}_{y \in Y} \mathop{\text{Inf}}_{x\in X} \|x - y\|.$$

We denote by Σ^n the sphere of unit radius in $\mathbb{R}^{n+1}$. We define the spherical distance $d_\Sigma : \Sigma^n \times \Sigma^n \to \mathbb{R}$ by:

$$d_\Sigma(\mathsf{N}, \mathsf{M}) = \cos^{-1}\langle \mathsf{N}, \mathsf{M}\rangle.$$

The spherical distance thus measures the angle between two points in the sphere. Let X and Y be two compact subsets of Σ^n. We define the spherical-Hausdorff distance between X and Y by:

$$d_{H,\Sigma}(X,Y) = \operatorname*{Sup}_{x\in X}\operatorname*{Inf}_{y\in Y} d_\Sigma(x,y) + \operatorname*{Sup}_{y\in Y}\operatorname*{Inf}_{x\in X} d_\Sigma(x,y).$$

Miscellaneous: If X is any subset of $\mathbb{R}^n$, we denote its closure by $\overline{X}$, its interior by X^o and its boundary by ∂X.

Let E be a vector space furnished with an inner product. For vectors X and Y in E, we denote by $\langle X, Y\rangle$ the inner product of X with Y.

Let E be any vector space. For vectors $X_1, ..., X_n$, we denote by $\langle X_1, ..., X_n\rangle$ the linear subspace of E spanned by $X_1, ..., X_n$. This should not be confused with the inner product. It will be clear from the context which is meant.

B

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C

Bibliography

[1] Brezis H., *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, (2011)

[2] Caffarelli L., Nirenberg L., Spruck J., The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge Ampère equation, *Comm. Pure Appl. Math.* **37** (1984), no. 3, 369–402

[3] Caffarelli L., Kohn J. J., Nirenberg L., Spruck J., The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge Ampère, and uniformly elliptic, equations, *Comm. Pure Appl. Math.* **38** (1985), no. 2, 209–252

[4] Caffarelli L., Nirenberg L., Spruck J., Nonlinear second-order elliptic equations. V. The Dirichlet problem for Weingarten hypersurfaces, *Comm. Pure Appl. Math.* **41** (1988), no. 1, 47–70

[5] Calabi E., Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens, *Michigan Math. J.* **5** (1958), 105–126

[6] do Carmo M. P., *Differential Geometry of Curves and Surfaces*, Pearson, (1976)

[7] do Carmo M. P., *Riemannian Geometry*, Birkhaüser, Boston-Basel-Berlin, (1992)

[8] Crandall M. G., Ishii H., Lions P.-L., User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.* **27** (1992), no. 1, 1–67

[9] Dold A., *Lectures on Algebraic Topology*, Classics in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, (1995)

[10] Friedlander F. G., Joshi M., *Introduction to the Theory of Distributions*, Cambridge University Press, Cambridge, (1998)

[11] Gilbarg D., Trudinger N. S., *Elliptic partical differential equations of second order*, Die Grundlehren der mathemathischen Wissenschagten, **224**, Springer-Verlag, Berlin, New York (1977)

[12] Guan B., Spruck J., The existence of hypersurfaces of constant Gauss curvature with prescribed boundary, *J. Differential Geom.* **62** (2002), no. 2, 259–287

[13] Guillemin V., Pollack A., *Differential Topology*, Prentice-Hall, Englewood Cliffs, N.J., (1974)

[14] Gutierrez C. E., *The Monge-Ampere Equation*, Progress in Nonlinear Differential Equations and Their Applications, Birkhaüser, (2001)

[15] Pogorelov A. V., On the improper convex affine hyperspheres, *Geometriae Dedicata* **1** (1972), no. 1, 33–46

[16] Rudin W., *Principles of Mathematical Analysis*, International Series in Pure & Applied Mathematics, McGraw-Hill, (1976)

[17] Rudin W., *Real & Complex Analysis*, McGraw-Hill, (1987)

[18] Rudin W., *Functional Analysis*, International Series in Pure & Applied Mathematics, McGraw-Hill, (1990)

[19] Sheng W., Urbas J., Wang X., Interior curvature bounds for a class of curvature equations. (English summary), *Duke Math. J.* **123** (2004), no. 2, 235–264

[20] Simon L., *Lectures on geometric measure theory*, Centre for Mathematical Analysis, Australian National University (1984)

[21] Smale S., An infinite dimensional version of Sard's theorem, *Amer. J. Math.*, **87**, (1965), 861–866

[22] Trudinger N. S., Wang X., On locally locally convex hypersurfaces with boundary, *J. Reine Angew. Math.* **551** (2002), 11–32