PDE Techniques in Non-Linear Problems

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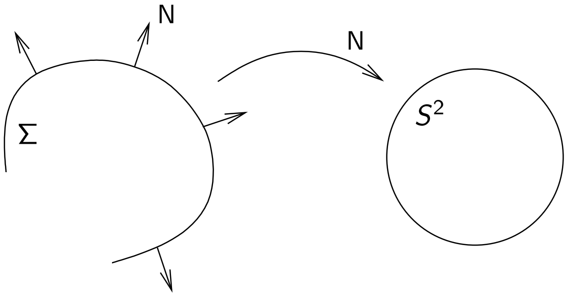
Introduction - Embeddings

Let Σ be an oriented, embedded surface in $\mathbb{R}^3$.

Let S^2 be the unit sphere in $\mathbb{R}^3$.

$\mathsf{N} : \Sigma \to S^2$ be the unit normal vector field over Σ compatible with the orientation.



Introduction - The Shape Operator and Curvature

We define the operator A, acting on tangent vectors of Σ, by:

$$A(X) = D_X \mathsf{N}.$$

We refer to A as the **shape operator** or **Weingarten operator** of Σ.

We define the function $K_\Sigma : \Sigma \to \mathbb{R}$ by:

$$K_\Sigma(x) = \mathrm{Det}(A(x)).$$

We call K the **gaussian curvature** or **extrinsic curvature** of Σ.

Introduction - Why Gaussian Curvature?

Gauss discovered that the gaussian curvature of a surface is an intrinsic property:

Theorem, Gauss, (1827)

If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.

What does this mean?

It means that if Σ_1 and Σ_2 are two embedded surfaces in $\mathbb{R}^3$, and $\Phi : \Sigma_1 \to \Sigma_2$ is an isometry, then:

$$K_{\Sigma_2} \circ \Phi = K_{\Sigma_1}.$$

As Gauss observed, this is remarkable, as the curvature, a-priori determined by **extrinsic** information, is, in fact, an **intrinsic** property.

Introduction - Constant Curvature - I

Having introduced the concept of curvature, it is natural to ask which objects have constant curvature.

The study of surfaces of constant curvature at first glance seems trivial:

Theorem, Hopf

A closed (i.e. compact, without boundary), embedded surface in $\mathbb{R}^3$ *of constant gaussian curvature is totally umbilic. In other words, it is a sphere.*

In fact, this is a special result concerning the **global rigidity** of surfaces of constant gaussian curvature.

Introduction - Constant Curvature - II

The non-closed case is very different:

Theorem

Choose $k \in \mathbb{R}$ *and let* $f :]a, b[\to \mathbb{R}$ *be a solution of the ordinary differential equation:*

$$f'' = -kf\sqrt{1 + (f')^2}^3 .$$

Then the surface generated by rotating f *about the* x*-axis has constant extrinsic curvature equal to* k*.*

Introduction - The Plateau Problem

The case of surfaces of rotation in $\mathbb{R}^3$ is relatively straightforward.

We therefore consider the following, more general problem:

Let $k \in \mathbb{R}$ be a real number. Let Γ be a smooth, closed curve in $\mathbb{R}$.

Does there exists a compact, embedded surface Σ whose boundary coincides with Γ and which has constant gaussian curvature equal to k?

Introduction - The Parametric Plateau Problem

We ask the same question for hypersurfaces in $\mathbb{R}^{n+1}$ for all n.

For various (good) reasons, we restrict attention to locally strictly convex hypersurfaces.

Theorem 1.2

Choose $k > 0$. Let K be a compact, convex subset of $\mathbb{R}^{n+1}$ with smooth boundary. Let Ω be an open subset of ∂K with C^2 boundary $C = \partial\Omega$. If ∂K has gaussian curvature bounded below by k at every point of Ω then there exists a compact, strictly convex, $C^{0,1}$ embedded hypersurface $S \subseteq \mathbb{R}^{n+1}$ with the properties that:

(1) $S \subseteq K$;

(2) $\partial S = C$; and

(3) $S \setminus \partial S$ is smooth and has constant gaussian curvature equal to k.

Part I - The Non-Parametric Problem

Let Ω be a bounded, strictly convex, open subset of $\mathbb{R}^n$ with smooth boundary.

As a first step towards proving Theorem 1.2, we obtain:

Theorem 1.4

Choose $k > 0$. If there exists a smooth, strictly convex function $\hat{f} : \overline{\Omega} \to]-\infty, 0]$ such that:

(1) $\hat{f}|_{\partial\Omega} = 0$; and

(2) the gaussian curvature of Graph$(\hat{f})$ is everywhere at least k;

then there exists a smooth, strictly convex function $f : \overline{\Omega} \to]-\infty, 0[$ such that:

(1) $f|_{\partial\Omega} = 0$; and

(2) the gaussian curvature of Graph(f) is everywhere equal to k.

Part I - The Gauss Curvature Equation

Lemma 1.3

Let $f : \overline{\Omega} \to \mathbb{R}$ *be a smooth function. If we define* $\kappa : \overline{\Omega} \to \mathbb{R}$ *such that for all* $x \in \overline{\Omega}$, $\kappa(x)$ *is the gaussian curvature of the graph of* f *at the point* $(x, f(x))$*, then* $\kappa(x)$ *is given by:*

$$\kappa(x) = Det(D^2 f(x))/(1 + \|Dd(x)\|^2)^{(n+2)/2}.$$

Moreover, $Graph(f)$ *is locally strictly convex if and only if* f *is strictly convex.*

We henceforth denote:

$$F(A) = \mathrm{Det}(A)^{1/n}, \qquad G(X) = (1 + \|X\|^2)^{(n+2)/2n}.$$

We are therefore looking for solutions of:

$$F(D^2 f) - kG(Df) = 0.$$

Part I - The Candidate and Data Spaces

We define Cand $\subseteq C_0^\infty(\overline{\Omega})$ by:

$$\text{Cand} = \left\{ f \;\middle|\; \begin{matrix} f \text{ strictly convex;} \\ f|_{\partial\Omega} = 0. \end{matrix} \right\}$$

We refer to Cand as the **candidate space**.

We define Data $\subseteq C^\infty(\overline{\Omega})$ by:

$$\text{Data} = \left\{ \kappa \mid 0 < \kappa < F(D^2\hat{f})/G(D\hat{f}) \right\}.$$

We refer to Data as the **data space**.

Observe that both Cand and Data are **open**.

Part I - The Functional Formulation

We define $\Phi : \mathrm{Cand} \times \mathrm{Data} \to C^\infty(\overline{\Omega})$ by:

$$\Phi(f,\kappa) = F(D^2 f) - \kappa G(Df).$$

We define $\mathrm{Soln} \subseteq \mathrm{Cand} \times \mathrm{Data}$ by:

$$\mathrm{Soln} = \Phi^{-1}(\{0\}).$$

We refer to Soln as the **solution space**.

Let $\Pi : \mathrm{Cand} \times \mathrm{Data} \to \mathrm{Data}$ be the projection onto the second factor.

Let $\Pi_{\mathrm{Soln}} : \mathrm{Soln} \to \mathrm{Data}$ be the restriction of Π to Soln.

Observe that there exists a solution for κ if and only if $\Pi_{\mathrm{Soln}}^{-1}(\{\kappa\})$ is non-trivial.

Part I - The Finite Dimensional Model - I

Let X, Y and Z be finite dimensional vector spaces.

Let C and D be open subsets of X and Y respectively.

Let $F : C \times D \to Z$ be a smooth function.

We define:

$$S = F^{-1}(\{0\}) = \{(x, y) \mid F(x, y) = 0\}.$$

We refer to S as the **solution space**.

Let $\Pi : C \times D \to D$ be the projection onto the second factor.

We denote by $\Pi_S : S \to D$ the restriction of Π to S.

Part I - The Finite Dimensional Model - II

We assume the following:

(A) $\mathrm{Dim}(X) = \mathrm{Dim}(Z)$;

(B) DF is surjective at every point of S;

(C) Π_S is a proper map from S onto D.

This allows us to conclude the following:

Theorem

There exists an open, dense subset D_0 of D such that:

(1) for all $d \in D_0$, $\Pi_S^{-1}(\{d\})$ is finite; and

(2) for all $d, d' \in D_0$:

$$\left|\Pi_S^{-1}(\{d\})\right| = \left|\Pi_S^{-1}(\{d'\})\right| \ \mathit{Mod}\ 2.$$

In other words, if there exists a unique, (non-degenerate) solution for one data point, then there exists a solution for almost every data point

Part I - The Finite Dimensional Model - III

How does this work?

Theorem, Submersion Theorem

Let U be an open subset of $\mathbb{R}^m$ and let $F : \Omega \to \mathbb{R}^n$ be a smooth mapping.

If DF is surjective at every point of $F^{-1}(\{0\})$, then $F^{-1}(\{0\})$ is a smooth, embedded submanifold of U of dimension equal to $m - n$.

It follows from (A) and (B) that S is a smooth, embedded submanifold of $C \times D$ of dimension equal to:

$$\mathrm{Dim}(X) + \mathrm{Dim}(Y) - \mathrm{Dim}(Z) = \mathrm{Dim}(Y) = \mathrm{Dim}(D).$$

Part I - The Finite Dimensional Model - IV

Theorem, Differential Topological Degree

Let M and N be smooth, finite-dimensional manifolds of the same dimension. Suppose, moreover, that N is connected.

Let $\pi : M \to N$ be a smooth, proper mapping.

There exists an open, dense subset $N_0 \subseteq N$ such that:

(1) for all $y \in N_0$, $\pi^{-1}(\{y\})$ is finite;

(2) for all $y, y' \in N_0$:

$$\left|\pi^{-1}(\{y\})\right| = \left|\pi^{-1}(\{y'\})\right| \ \mathit{Mod}\ 2.$$

We have established that S is a smooth, finite dimensional manifold and $\mathrm{Dim}(S) = \mathrm{Dim}(D)$. By (C), Π_S is proper, and the differential topological degree follows.

Part I - Infinite Dimensions - Differential Structure

Before we can perform analysis, we require a useful differential structure.

For all $(k,\alpha)\in\mathbb{N}\times]0,1[$, let $C^{k,\alpha}(\overline{\Omega})$ be the **Banach space** of $k+\alpha$-times Hölder differentiable functions.

For all (k,α), we define $\mathrm{Cand}^{k+2,\alpha}\subseteq C_0^{k+2,\alpha}(\overline{\Omega})$ by:

$$\mathrm{Cand}^{k+2,\alpha}=\left\{f\ \middle|\ \begin{matrix} f\text{ strictly convex;}\\ f|_{\partial\Omega}=0.\end{matrix}\right\}$$

For all (k,α), we define $\mathrm{Data}^{k,\alpha}\subseteq C^{k,\alpha}(\overline{\Omega})$ by:

$$\mathrm{Data}^{k,\alpha}=\left\{\kappa\mid 0<\kappa<F(D^2\hat{f})/G(D\hat{f})\right\}.$$

Observe that both $\mathrm{Cand}^{k+2,\alpha}$ and $\mathrm{Data}^{k,\alpha}$ are **open**.

Part I - Infinite Dimensions - A Banach Space Set-up

For all (k, α), we define $\Phi^{k,\alpha} : \mathrm{Cand}^{k+2,\alpha} \times \mathrm{Data}^{k,\alpha} \to C^{k,\alpha}(\overline{\Omega})$ by:

$$\Phi^{k,\alpha}(f, \kappa) = F(D^2 f) - \kappa G(Df).$$

We define $\mathrm{Soln}^{k,\alpha} \subseteq \mathrm{Cand}^{k+2,\alpha} \times \mathrm{Data}^{k,\alpha}$ by:

$$\mathrm{Soln}^{k,\alpha} = (\Phi^{k,\alpha})^{-1}(\{0\}).$$

Let $\Pi^{k,\alpha} : \mathrm{Cand}^{k+2,\alpha} \times \mathrm{Data}^{k,\alpha} \to \mathrm{Data}^{k,\alpha}$ be the canonical projection.

Let $\Pi^{k,\alpha}_{\mathrm{Soln}} : \mathrm{Soln}^{k,\alpha} \to \mathrm{Data}^{k,\alpha}$ be the restriction of $\Pi^{k,\alpha}$ to $\mathrm{Soln}^{k,\alpha}$

Part I - Infinite Dimensions - The Smooth Mapping

We henceforth fix $k + \alpha \gg 0$ and henceforth suppress it from the notation where possible.

Lemma 3.19**, Part I**

Φ *defines a smooth mapping from Cand* $\times$ *Data into* $C^{k,\alpha}(\overline{\Omega})$.

We will show:

(A) for all $(f, \kappa) \in$ Cand $\times$ Data, $D_1\Phi_{(f,\kappa)}$ is Fredholm of index 0;

(B) $D\Phi$ is surjective at every point of Soln;

(C) Π_{Soln} is a proper mapping from Soln onto Data.

Part I - Infinite Dimensions - Fredholm Mappings

Observe that for all $(f, \kappa) \in \text{Cand} \times \text{Data}$, $D\Phi$ defines a bounded linear map from $C_0^{k+2,\alpha}(\overline{\Omega}) \times C^{k,\alpha}(\overline{\Omega})$ into $C^{k,\alpha}(\overline{\Omega})$.

Property (A) follows from:

Lemma 3.19**, Part** 2

For all k *and for all* $(f, \kappa) \in \mathit{Cand} \times \mathit{Data}$*,* $D_1\Phi_{f,\kappa}$ *is Fredholm of index* 0*.*

Part I - Infinite Dimensions - Surjectivity

Property (B) follows from:

Lemma

For all k and for all $(f, \kappa) \in$ Soln, $D\Phi_{f,\kappa}$ is surjective.

Part I - Infinite Dimensions - Properness

Property (C) follows from:

Theorem 2.30

The mapping Π_{Soln} *defines a proper mapping from Soln into Data.*

This properness result will be discussed in Part II.

Part I - Finite Dimensional Reduction - Step 1

The properness of Π_{Soln} combined with surjectivity yields a finite dimensional reduction of the data space:

Lemma 3.21

For any compact subset K of Data, there exists a finite dimensional subspace E of $C^{k,\alpha}(\overline{\Omega})$ and $r > 0$ with the property that if (f, κ) is such that:

(1) $\Phi(f, \kappa) = 0$; and

(2) $\kappa - \kappa_0 \in B_r(0) \subseteq E$ for some $\kappa_0 \in K$;

then the restriction of $D\Phi_{(f,\kappa)}$ to $C_0^{k+2,\alpha}(\overline{\Omega}) \times E$ is a surjective mapping onto $C^{k,\alpha}(\overline{\Omega})$.

In particular, since $D_1\Phi_{(f,\kappa)}$ is Fredholm of index 0, it follows that this restriction is also Fredholm of index equal to $\mathrm{Dim}(E)$.

Part I - Finite Dimensional Reduction - Step 2

We denote $\mathrm{Data}(E) = K + B_r(0)$.

We define $\mathrm{Soln}(E)$ by:

$$\mathrm{Soln}(E) = \mathrm{Soln} \cap (\mathrm{Cand} \times \mathrm{Data}(E)).$$

Theorem, Infinite Dimensional Submersion Theorem

Let X and Y be Banach spaces. Let Ω be an open subset of X. Let $\mathcal{F} : \Omega \to Y$ be a smooth mapping such that $D\mathcal{F}$ is Fredholm at every point of Ω.

If $D\mathcal{F}$ is surjective at every point of $\mathcal{F}^{-1}(\{0\})$, then $\mathcal{F}^{-1}(\{0\})$ is a smooth, embedded (not necessarily separable) submanifold of Ω of dimension equal to $\mathrm{Ind}(D\mathcal{F})$.

$\mathrm{Soln}(E)$ is therefore a smooth (not necessarily separable) manifold of dimension equal to $\mathrm{Dim}(E)$.

Part I - Infinite Dimensions - The Degree

Since $\Pi_{\mathsf{Soln}} : \mathsf{Soln}(E) \rightarrow \mathsf{Data}(E)$ is proper, $\mathsf{Soln}(E)$ is **separable**.

Theorem, Differential Topological Degree

There exists an open, dense subset $Data(E)_0 \subseteq Data(E)$ such that:

(1) for all $d \in Data(E)_0$, $\pi^{-1}(\{d\})$ is finite;

(2) for all $d, d' \in Data(E)_0$:

$$\left|\pi^{-1}(\{d'\})\right| = \left|\pi^{-1}(\{d'\})\right| \ \mathit{Mod}\ 2.$$

Part I - Extending the Functional Formulation - I

Choose $\kappa_0 \in \text{Data}$.

For $\kappa \in \text{Data}$ and $t \in [0, 1]$, we define κ_t by:

$$\kappa_t = (1 - t)\kappa_0 + t\kappa.$$

For $t \in [0, 1]$, we define G_t by:

$$G_t = (1 - t) + tG.$$

Part I - Extending The Functional Formulation - II

We define $\Phi : \mathrm{Cand} \times \mathrm{Data} \times [0,1] \to C^\infty(\overline{\Omega})$ by:

$$\Phi(f, \kappa, t) = F(D^2 f) - \kappa_t G_t(Df).$$

We define $\mathrm{Soln} \subseteq \mathrm{Cand} \times \mathrm{Data} \times [0,1]$ by:

$$\mathrm{Soln} = \Phi^{-1}(\{0\})$$

Let $\Pi : \mathrm{Cand} \times \mathrm{Data} \times [0,1] \to \mathrm{Data} \times [0,1]$ be the canonical projection.

Let $\Pi_{\mathrm{Soln}} : \mathrm{Soln} \to \mathrm{Data} \times [0,1]$ be the restriction of Π to Soln.

Part I - Calculating The Degree - I

We calculate the degree by considering the case $t = 0$.

For $\alpha \in [0, 1]$, we denote:

$$f_\alpha = \alpha \hat{f}, \qquad \kappa_0 = F(D^2 f_0) = \mathrm{Det}(D^2 f_0)^{1/n}$$

Observe that, for α sufficiently small:

$$\kappa_\alpha = F(D^2 f_0) < \alpha^n F(D^2 \hat{f}) \leqslant F(D^2 \hat{f})/G(D\hat{f}).$$

In other words, $\kappa_\alpha \in \mathrm{Data}$

Part I - Calculating The Degree - II

By definition, $\Phi(f_\alpha, \kappa_\alpha, 0) = 0$.

Lemma

If $f \in$ Cand satisfies $\Phi(f, \kappa_\alpha, 0) = 0$, then $f = f_\alpha$.

Lemma

$D_1\Phi_{(f_\alpha, \kappa_\alpha, 0)}$ is invertible.

It follows that $\Pi_{\mathrm{Soln}}^{-1}(\{(\kappa_\alpha, 0)\}) = \{(f_\alpha, \kappa_\alpha, 0)\}$, and that $(f_\alpha, \kappa_\alpha, 0)$ is non-degenerate.

The degree is therefore equal to 1, and existence follows.

Part II - The Plateau Problem for Graphs

We study the following problem:

(1) Let Ω be an open, bounded, convex subset of $\mathbb{R}^n$ with C^∞ boundary;

(2) let $\kappa \in C^\infty(\overline{\Omega},]0, \infty[)$ be strictly positive; and

(3) let $\hat{f} \in C_0^\infty(\overline{\Omega})$ be strictly convex and s.t.:

$$\mathrm{Det}(D^2\hat{f}) \geqslant \kappa(1 + \|D\hat{f}\|^2)^{(n+2)/2}.$$

Does there exist a strictly convex function $f \in C_0^\infty(\overline{\Omega})$ s.t.:

$$\mathrm{Det}(D^2 f) = \kappa(1 + \|Df\|^2)^{(n+2)/2}, \qquad f \geqslant \hat{f}? \qquad \text{(A)}$$

We refer to this as the **Plateau Problem** with **upper barrier**.

Part II - Compactness

Recalling Part I, we now show **compactness** for families of solutions.

Theorem 2.30

Let $(\kappa_m)_{m\in\mathbb{N}}, \kappa_\infty \in C^\infty(\overline{\Omega})$ *and* $(\hat{f}_m)_{m\in\mathbb{N}}, \hat{f}_\infty \in C_0^\infty(\overline{\Omega})$ *be such that for all* $m \in \mathbb{N}\cup\{\infty\}$*:*

$$F(D^2\hat{f}_m) - \kappa_m G(D\hat{f}_m) \geqslant 0.$$

For all m, let $f_m \in C_0^\infty(\overline{\Omega})$ *be such that:*

$$F(D^2 f_m) - \kappa_m G(Df_m) = 0, \qquad f_m \geqslant \hat{f}_m.$$

If $(\kappa_m)_{m\in\mathbb{N}}$ *and* $(\hat{f}_m)_{m\in\mathbb{N}}$ *converge to* κ_∞ *and* $\hat{f}_\infty$ *respectively in the* C^∞ *sense, then there exists* $f_\infty \in C_0^\infty(\overline{\Omega})$ *towards which* $(f_m)_{m\in\mathbb{N}}$ *subconverges in the* C^∞ *sense.*

Part II - Arzela-Ascoli Theorem

We recall the classical Arzela-Ascoli theorem:

Theorem

Let $(f_n)_{n\in\mathbb{N}} \in C^\infty(\overline{\Omega})$. *Suppose that, for all* $k \in \mathbb{N}$, *there exists* $B_k > 0$ *such that* $\|D^k f\| \leqslant B_k$. *Then there exists* $f_\infty \in C^\infty(\overline{\Omega})$ *towards which* $(f_n)_{n\in\mathbb{N}}$ *subconverges in the* C^k *sense for all* k.

Compactness thus follows from bounds - i.e. a-priori estimates - on the derivatives of solutions up to all orders.

Part II - A-Priori Estimates

A-priori estimates are obtained in the following order:

(1) C^0-estimates - maximum principle;

(2) C^1-estimates - maximum principle;

(3) C^2-estimates along the boundary - Caffarelli-Nirenberg-Spruck barrier technique;

(4) C^2-estimates - maximum principle;

(5) $C^{2,\alpha}$-estimates - Krylov technique;

(6) C^k-estimates for all k - Schauder technique.

Part II - Zeroeth Order Estimates

Lemma 2.1

If f is a solution of (A), then:

$$\|f\|_{L^\infty} \leqslant \|\hat{f}\|_{L^\infty}.$$

Part II - First Order Estimates

Lemma 2.3

If f is a solution of (A), then:

$$\|Df\|_{L^\infty} \leqslant \|D\hat{f}\|_{L^\infty}.$$

Part II - A More General Framework - Preliminaries

Let $\Gamma \subseteq \mathrm{Symm}(2, \mathbb{R}^n)$ be the open, convex cone of positive-definite, symmetric matrices.

Define $F : \Gamma \to]0, \infty[$ by:

$$F(A) = \mathrm{Det}(A)^{1/n}.$$

Let $G : \mathbb{R}^n \to [1, \infty[$ be **smooth** and **convex**.

Let $\hat{f} \in C_0^\infty(\overline{\Omega})$ be a smooth, strictly convex function s.t.:

$$F(D^2\hat{f}) - \kappa G(Df) \geqslant 0.$$

We consider functions f which are solutions to:

$$F(D^2 f) - \kappa G(Df) = 0, \qquad f \geqslant \hat{f} \qquad \text{(A')}$$

Part II - Properties of F

(1) F extends to a continuous function over $\overline{\Gamma}$ which vanishes over $\partial\Gamma$;

(2) for all $A \in \Gamma$, $DF(A) = \frac{1}{n}F(A)A^{-1}$;

(3) in particular, for all $A \in \Gamma$, $DF(A)(A) = F(A)$;

(4) for all $A \in \Gamma$, $DF(A)(\mathrm{Id}) \geqslant 1$;

(5) if A is diagonal, with eigenvalues $\lambda_1, ..., \lambda_n$, then for any symmetric matrix M:

$$D^2F(A)(M, M) \leqslant -\frac{1}{n}F(A)\sum_{i \neq j}\frac{1}{\lambda_i\lambda_j}M_{ij}M_{ij}.$$

(6) in particular, F is **concave**.

Part II - Important Objects I - The Linearised Operator

Let $f \in C_0^\infty(\overline{\Omega})$ be strictly convex.

We define the operator $\mathcal{L}_f : C_0^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ by:

$$\mathcal{L}_f(g) = DF(D^2 f)(D^2 g) - \kappa DG(Df)(Dg).$$

Lemma 2.10

$\mathcal{L}_f$ is a second-order, linear, elliptic, partial differential operator.

Part II - Important Objects II - The Controlling Term

We define $\Lambda_f : \overline{\Omega} \to]0, \infty[$ by:

$$\Lambda_f = DF(D^2 f)(\mathrm{Id}) = \frac{1}{n} F(D^2 f)\mathrm{Tr}((D^2 f)^{-1}).$$

Lemma 2.13

$$\Lambda_f \geqslant 1.$$

Part II - Barriers I

Strictly subharmonic functions may be derived as variants of the distance function.

For all p, define $d_p : \mathbb{R}^n \to \mathbb{R}$ by:

$$d_p(x) = \|x - p\|.$$

Lemma 2.16

There exists $r > 0$, which only depends on $\|\kappa\|_{C^0}$ and $\|f\|_{C^1}$ such that for all $p \in \partial\Omega$ and for all $x \in \overline{B}_r(p) \cap \overline{\Omega}$:

$$\mathcal{L}_f(d_r^2) \geqslant \Lambda_f.$$

Part II - Barriers IIa

Positive, superharmonic functions may be derived as variants of:

Theorem 2.11

Choose $\delta > 0$ *and let* $g, h \in C_0^\infty(\overline{\Omega})$ *be strictly convex functions such that:*

$$F(D^2 g) - \kappa G(Dg) + \delta \leqslant F(D^2 h) - \kappa G(Dh).$$

Then:

$$\mathcal{L}_g(g - h) \leqslant -\delta.$$

Part II - Barriers IIb

We modify this barrier to obtain a superharmonic function which is strictly negative away from a given boundary point.

For all $p \in \partial\Omega$ and for all $\epsilon > 0$, we define:

$$\hat{f}_{p,\epsilon} = \hat{f} - \epsilon d_p^2.$$

Lemma 2.15

There exists $\epsilon > 0$ which only depends on $\|\kappa\|_{C^0}$, $\|\hat{f}\|_{C^2}$ and $\delta(\hat{f})$ such that, for all $p \in \partial\Omega$:

$$F(D^2\hat{f}_{p,\epsilon}) - \kappa G(D\hat{f}_{p,\epsilon}) \geqslant 0.$$

In particular, by Theorem 2.11:

$$\mathcal{L}_f(f - \hat{f}_{p,\epsilon}) \leqslant 0.$$

Part II - The Test Function

Let X be a C^∞ vector field over $\mathbb{R}^n$ which is tangent to $\partial\Omega$.

Define the function $\phi \in C^\infty(\overline{\Omega})$ by:

$$\phi = Xf = df(X).$$

Lemma 2.14

There exists $C > 0$, which only depends on $\|\kappa\|_{C^1}$, $\|X\|_{C^2}$ and $\|f\|_{C^1}$ such that:

$$|\mathcal{L}_f \phi| \leqslant C\Lambda_f.$$

Part II - Summary of Analytic Properties

Choose $p \in \partial\Omega$. Choose $r > 0$ small. Denote:

$$\alpha = f - \hat{f}_{p,\epsilon}, \qquad \beta = d_p^2.$$

F	$\mathcal{L}_f F$	$F\|_{\partial\Omega}$	$F\|_{\partial B_r(p)}$
Xf	$\in [-C\Lambda_f, C\Lambda_f]$	$= 0$	$\in [-C, C]$
α	$\leqslant 0$	$\geqslant (1/C)d_p^2$	$\geqslant 1/C$
β	$\geqslant \Lambda_f$	$\in [0, Cd_p^2]$	$\in [0, C]$

C only depends on:

$$\|\kappa\|_{C^1},\ \|X\|_{C^2},\ \delta(\hat{f}),\ \|\hat{f}\|_{C^2}\ \&\ \|f\|_{C^1}.$$

Part II - Second Order Boundary Estimates I

Lemma 2.17

There exists $C > 0$ *which only depends on* $\|\kappa\|_{C^1}$, $\|X\|_{C^2}$, $\delta(\hat{f})$, $\|\hat{f}\|_{C^2}$ *and* $\|f\|_{C^1}$ *such that for all* $p \in \partial\Omega$*:*

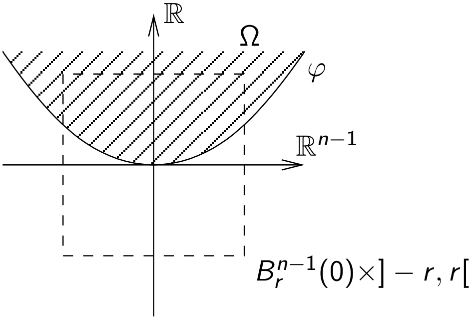
$$\|D(Xf)(p)\| \leqslant C.$$

Part II - Special Coordinates

Suppose $0 \in \partial\Omega$.

Suppose that $e_1, ..., e_{n-1}$ are **tangent** to $\partial\Omega$.

Suppose e_n is **normal** to $\partial\Omega$.



Choose $r > 0$ small. Let $\varphi : B_r^{n-1}(0) \to]-r, r[$ be a smooth function s.t. $(\partial\Omega) \cap (B_r^{n-1}(0)\times]-r, r[)$ is the graph of φ.

Corollary 2.18

There exists $C > 0$ *which only depends on* $\|\kappa\|_{C^1}$, $\delta(\hat{f})$, $\|\hat{f}\|_{C^2}$ *and* $\|f\|_{C^1}$ *such that for all* $(i, j) \neq (n, n)$:

$$\|D^2 f(e_i, e_j)\| \leqslant C.$$

Part II - Towards the Double-Normal Derivative I

Lemma 2.21

If $\kappa \geqslant \kappa_0 > 0$*, then:*

$$(\partial_n f) \leqslant -\kappa_0 R,$$

where R *denotes the radius of the largest closed ball contained in* $\overline{\Omega}$ *tangent to* $\partial\Omega$ *at* $(0, 0)$.

Part II - Towards the Double-Normal Derivative II

Lemma

There exists $C > 0$ which only depends on κ such that for all vectors X tangent to $\partial\Omega$ at 0:

$$D^2 f(0)(X, X) \geqslant (1/C)\|X\|^2.$$

Part II - Second Order Boundary Estimates III

Theorem 2.22

There exists $C > 0$ *which only depends on* $\|\kappa\|_{C^1}$, $\|\hat{f}\|_{C^2}$, $\|f\|_{C^1}$, $\delta(\hat{f})$ *and* $\mathrm{Inf}_{x\in\overline{\Omega}}\,\kappa(x)$ *such that for all* $x \in \partial\Omega$*:*

$$\|D^2 f(x)\| \leqslant C.$$

Part II - Weak Partial Differential Relations

Let U be an open subset of $\mathbb{R}^n$.

Let $\varphi : U \to \mathbb{R}$ be **continuous**.

For $C \in \mathbb{R}$, we say that $\mathcal{L}_f\varphi \geqslant C$ in the **weak sense** whenever, for all $x \in U$, there exists $\psi : U \to \mathbb{R}$ such that:

(1) ψ is smooth;

(2) $\psi \leqslant \varphi$;

(3) $\psi(0) = \varphi(0)$; and

(4) $\mathcal{L}_f\psi \geqslant C$.

Part II - Barriers III

Let $0 < \lambda_1 \leqslant \ldots \leqslant \lambda_n : \overline{\Omega} \to \mathbb{R}$ be the eigenvalues of $D^2 f$.

Observe that, for all i, λ_i is **continuous**.

Lemma 2.26

There exists $C > 0$, which only depends of $\|\kappa\|_{C^2}$, $\|f\|_{C^1}$ and $Inf_{x\in\overline{\Omega}}\,\kappa(x)$ such that:

$$\mathcal{L}_f Log(\lambda_n) + \frac{2}{n\lambda_n}(\partial_n Log(\lambda_n))^2 \geqslant -C,$$

in the weak sense.

Part II - Global Second Order Estimates

Theorem 2.27

There exists $C > 0$ *which only depends on* $\|\kappa\|_{C^2}$, $Inf_{x\in\overline{\Omega}}\,\kappa(x)$, $\delta(\hat{f})$, $\|\hat{f}\|_{C^2}$ *and* $\|f\|_{C^1}$ *such that if* $f \geqslant \hat{f}$, *then:*

$$\|f\|_2 \leqslant C.$$

Part III - Regularity Theory

Let $(K_m)_{m\in\mathbb{N}}$ and K_∞ be compact, convex subsets of $\mathbb{R}^{n+1}$ such that $(K_m)_{n\in\mathbb{N}}$ converges to K_∞ in the **Hausdorff** sense.

Let $k > 0$ be a positive real number and let U be an open subset of $\mathbb{R}^{n+1}$.

Suppose that, for all $m \in \mathbb{N}$, ∂K_m has constant Gauss-Kroenecker curvature equal to k near every point of $(\partial K_m) \cap U$.

This means that ∂K_∞ has constant Gauss-Kroenecker curvature equal to k over $(\partial K_\infty) \cap U$ in the **weak** sense

What is the structure of the singular set of K_∞?

Part III - Barriers III

Lemma 4.25

There exists $C > 0$ *which only depends on* $\|\kappa\|_{C^1}$ *and* $\|f\|_{C^1}$ *such that:*

$$\mathcal{L}_f \|Df\|^2 \geqslant -C + \frac{2}{n}\kappa\lambda_n(f).$$

Part III - Interior a-Priori Estimates

Lemma 4.26

There exists $C > 0$*, which only depends on* $\|\kappa\|_{C^2}$, $\|f\|_{C^1}$ *and* $Inf_{x\in\overline{\Omega}}\,\kappa(x)$ *such that:*

$$\underset{x\in\overline{\Omega}}{Sup}\,|f(x)|^2\,\|D^2 f(x)\| \leqslant C$$

Part III - Supporting Normals of Convex Sets

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$ with non-trivial interior.

Choose $x \in \partial K$ and let N be a unit vector.

We say that N is a **supporting normal** to K at x whenever:

$$\langle y - x, \mathsf{N} \rangle \leqslant 0, \qquad \forall\ y \in K.$$

Lemma 4.6

For every boundary point x of K there exists a supporting normal N to K at x.

Part III - Continuity of Supporting Normals

Lemma 4.3

Let $(K_m)_{m\in\mathbb{N}}$ *and* K_∞ *be compact, convex subsets of* $\mathbb{R}^{n+1}$ *such that* $(K_m)_{m\in\mathbb{N}}$ *converges to* K_∞ *in the Hausdorff sense.*

For all finite m, let x_m *be a boundary point of* K_m *and let* N_m *be a supporting normal to* K_m *at* x_m.

If $(x_m)_{m\in\mathbb{N}}$ *and* $(\mathsf{N}_m)_{m\in\mathbb{N}}$ *converge to* x_∞ *and* N_∞ *respectively, then* x_∞ *is a boundary point of* K_∞ *and* N_∞ *is a supporting normal to* K_∞ *at* x_∞.

Part III - Convex Sets as Graphs

Theorem 4.13

Suppose that $0 \in \partial K$*. Choose* $\theta \in [0, \pi/2[$ *and* $r > 0$ *and suppose that for all* $x \in \partial K \cap B_r(0)$ *and for every supporting normal* N *to* K *at* x*:*

$$\langle \mathsf{N}, -e_{n+1} \rangle \geqslant cos(\theta).$$

Then, denoting $C = tan(\theta)$ *and* $\rho = \frac{r}{\sqrt{1+C^2}}$*, there exists a function* $f : B'_\rho(0) \times] - C\rho, C\rho[$ *such that:*

(1) $f(0) = 0$*;*

(2) f *is convex and* C*-Lipschitz; and*

(3) $(\partial K) \cap (B'_\rho(0) \times] - 2C\rho, 2C\rho[)$ *coincides with the graph of* f*.*

Part III - The Local Geodesic Property

For $x \in \partial K$, we say that K satisfies the **local geodesic property** whenever there exists an open geodesic segment Γ such that:

$$x \in \Gamma \subseteq K.$$

Lemma 4.18

If N *is a supporting normal to K at x, then:*

$$\langle y - x, \mathrm{N} \rangle = 0$$

for all $y \in \Gamma$.

In other words Γ lies in the hyperplane normal to N passing through x.

Part III - The Complementary Property

Let K be a convex subset of $\mathbb{R}^{n+1}$ with non-trivial interior.

Suppose that $0 \in \partial K$ and $-e_n$ is a supporting normal to K at 0.

Lemma 4.23

If K does **not** *satisfy the local geodesic property at* 0*, then, for all* $r > 0$*, there exists a unit vector* N *which we may choose as close to* $-e_n$ *as we wish with the property that for all* $y \in K \cap B_r(0)^c$*:*

$$\langle y - x, \mathrm{N} \rangle < 0.$$

Part III - The Structure of Singularities

Let $(K_m)_{m \in \mathbb{N}}$ and K_∞ be compact, convex subsets of $\mathbb{R}^{n+1}$ such that $(K_m)_{m \in \mathbb{N}}$ converges to K_∞ in the **Hausdorff** sense.

Let $k > 0$ be a real number and let U be an open subset of $\mathbb{R}^{n+1}$

Suppose that, for all $m \in \mathbb{N}$, ∂K_m has constant Gauss-Kroenecker curvature equal to k near every point of $(\partial K_m) \cap U$.

Theorem 4.29

If $y \in (\partial K_\infty) \cap U$, then either:

(1) there exists $r > 0$ such that $(\partial K_\infty) \cap B_r(y)$ is smooth with constant Gauss-Kroenecker curvature equal to k; or

(2) K_∞ satisfies the local geodesic property at y.

Part III - Convex Hulls

Observe that in Theorem 4.29, the singular set is **closed**.

Theorem 4.19

Let X be a closed subset of ∂K. Let Y be the set of all points of ∂K satisfying the local geodesic property.

If $X \cup Y$ is closed, then $Y \subset Conv(X)$.

Part III - Example

Let Ω be a bounded, strictly convex, open subset of $\mathbb{R}^n$ contained in a ball of radius 1.

Theorem

For all $k < 1$, there exists a $C^{0,1}$, strictly convex function $f : \overline{\Omega} \rightarrow]-\infty, 0[$ such that:

(1) $f|_{\partial\Omega} = 0$; and

(2) f is smooth over Ω and the gaussian curvature of Graph(f) is everywhere equal to k.