Dynamical systems and Financial Instability

M. R. Grasselli

Mathematics and Statistics - McMaster University

Research **i**n **O**ptions - Buzios, December 1, 2013

Outline

1. Introduction
   - DSGE models
   - Minskyian views
   - SFC models
2. Goodwin model
   - Original model
   - Inflation
   - Production function
   - Equity and savings
   - Monetary policy
3. Keen model
   - Original model
   - Ponzi financing
   - Noise and Stock Prices
   - Stabilizing government
   - Great Moderation
4. The Ultimate Model
   - Savings and investment
   - Inventories
   - Portfolios
   - Bringing it altogether

Dynamic Stochastic General Equilibrium

- Seeks to explain the aggregate economy using theories based on strong microeconomic foundations.
- Collective decisions of rational individuals over a range of variables for both present and future.
- All variables are assumed to be simultaneously in equilibrium.
- The only way the economy can be in disequilibrium at any point in time is through decisions based on wrong information.
- Money is neutral in its effect on real variables.
- Largely ignores uncertainty by simply subtracting risk premia from all risky returns and treat them as risk-free.

Really bad economics: hardcore (freshwater) DSGE

- The strand of DSGE economists affiliated with RBC theory made the following predictions after 2008:
  1. Increases government borrowing would lead to higher interest rates on government debt because of "crowding out".
  2. Increases in the money supply would lead to inflation.
  3. Fiscal stimulus has zero effect in an ideal world and negative effect in practice (because of decreased confidence).

Wrong prediction number 1

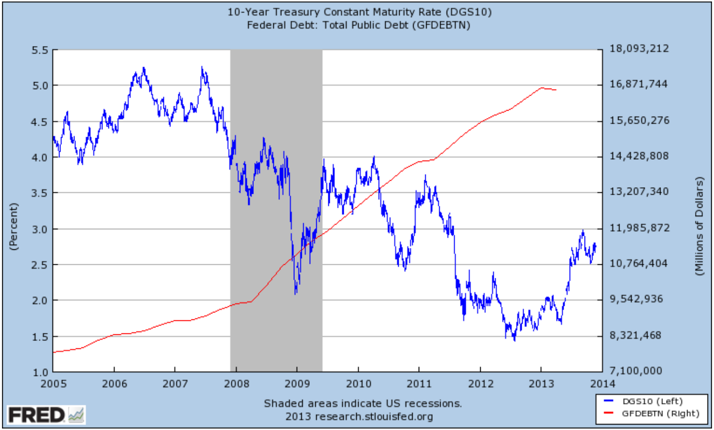


Figure: Government borrowing and interest rates.

Wrong prediction number 2

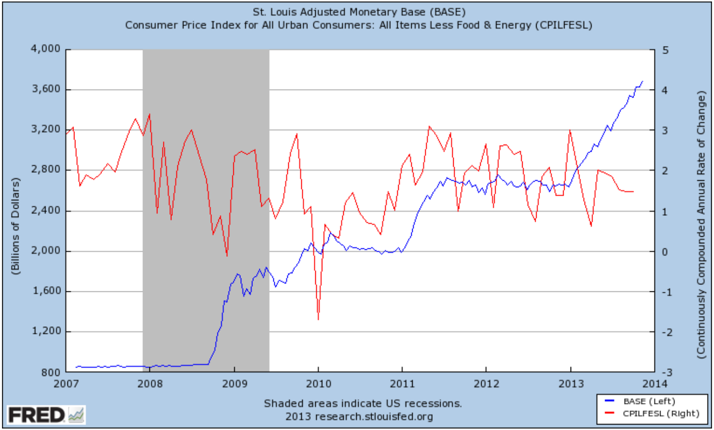


Figure: Monetary base and inflation.

Wrong prediction number 3

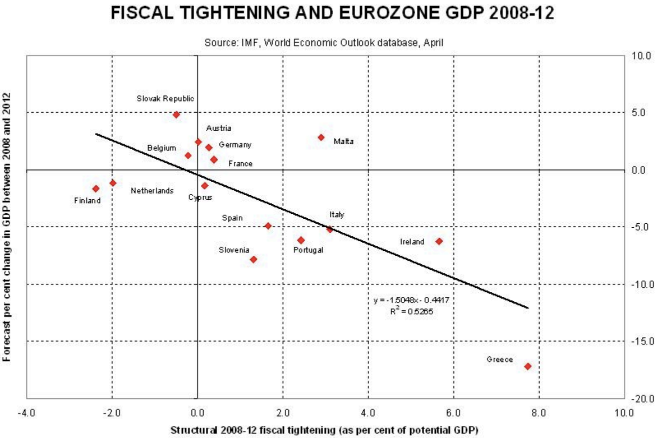


Figure: Fiscal tightening and GDP.

Better (but still bad) economics: soft core (saltwater) DSGE

- The strand of DSGE economists affiliated with New Keynesian theory got all these predictions right.
- They did so by augmented DSGE with 'imperfections' (wage stickiness, asymmetric information, imperfect competition, etc).
- Still DSGE at core - analogous to adding epicycles to Ptolemaic planetary system.
- For example: "Ignoring the foreign component, or looking at the world as a whole, the overall level of debt makes no difference to aggregate net worth – one person's liability is another person's asset." (Paul Krugman and Gauti B. Eggertsson, 2010, pp. 2-3)

Then we can safely ignore this...

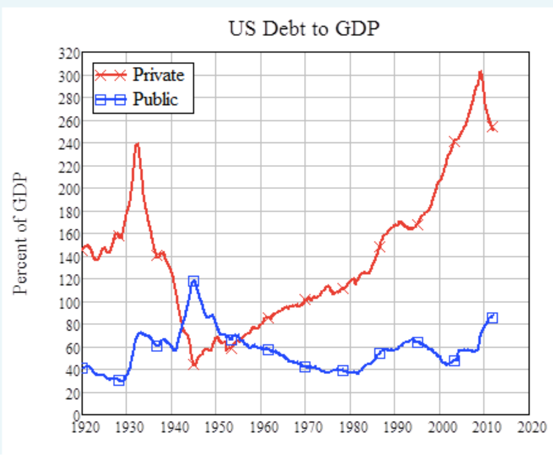


Figure: Private and public debt ratios.

Really?

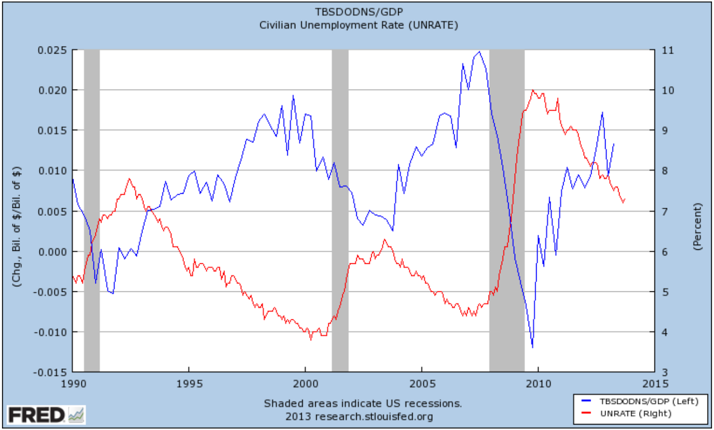


Figure: Change in debt and unemployment.

Minsky's alternative interpretation of Keynes

- Neoclassical economics is based on barter paradigm: money is convenient to eliminate the double coincidence of wants.
- In a modern economy, firms make complex portfolios decisions: which assets to hold and how to fund them.
- Financial institutions determine the way funds are available for ownership of capital and production.
- Uncertainty in valuation of cash flows (assets) and credit risk (liabilities) drive fluctuations in real demand and investment.
- Economy is fundamentally cyclical, with each state (boom, crisis, deflation, stagnation, expansion and recovery) containing the elements leading to the next in an identifiable manner.

Minsky's Financial Instability Hypothesis

- Start when the economy is doing well but firms and banks are conservative.
- Most projects succeed - "Existing debt is easily validated: it pays to lever".
- Revised valuation of cash flows, exponential growth in credit, investment and asset prices.
- Highly liquid, low-yielding financial instruments are devalued, rise in corresponding interest rate.
- Beginning of "euphoric economy": increased debt to equity ratios, development of Ponzi financier.
- Viability of business activity is eventually compromised.
- Ponzi financiers have to sell assets, liquidity dries out, asset market is flooded.
- Euphoria becomes a panic.
- "Stability - or tranquility - in a world with a cyclical past and capitalist financial institutions is destabilizing".

Much better economics: SFC models

- Stock-flow consistent models emerged in the last decade as a common language for many heterodox schools of thought in economics.
- Consider both real and monetary factors from the start
- Specify the balance sheet and transactions between sectors
- Accommodate a number of behavioural assumptions in a way that is consistent with the underlying accounting structure.
- Reject silly (and mathematically unsound!) hypotheses such as the RARE individual (representative agent with rational expectations).
- See Godley and Lavoie (2007) for the full framework.

Balance Sheets

Balance Sheet	Households	Firms current	Firms capital	Banks	Central Bank	Government	Sum
Cash	$+H_h$			$+H_b$	$-H$		0
Deposits	$+M_h$		$+M_f$	$-M$			0
Loans			$-L$	$+L$			0
Bills	$+B_h$			$+B_b$	$+B_c$	$-B$	0
Equities	$+p_f E_f + p_b E_b$		$-p_f E_f$	$-p_b E_b$			0
Advances				$-A$	$+A$		0
Capital			$+pK$				pK
Sum (net worth)	V_h	0	V_f	V_b	0	$-B$	pK

Table: Balance sheet in an example of a general SFC model.

Transactions

Transactions	Households	Firms current	Firms capital	Banks	Central Bank	Government	Sum
Consumption	$-pC_h$	$+pC$		$+pC_b$			0
Investment		$+pI$	$-pI$				0
Gov spending		$+pG$				$-pG$	0
Acct memo [GDP]		$[pY]$					
Wages	$+W$	$-W$					0
Taxes	$-T_h$	$-T_f$				$+T$	0
Interest on deposits	$+r_M.M_h$	$+r_M.M_f$		$-r_M.M$			0
Interest on loans		$-r_L.L$		$+r_L.L$			0
Interest on bills	$+r_B.B_h$			$+r_B.B_b$	$+r_B.B_c$	$-r_B.B$	0
Profits	$+\Pi_d+\Pi_b$	$-\Pi$	$+\Pi_u$	$-\Pi_b$	$-\Pi_c$	$+\Pi_c$	0
Sum	S_h	0	S_f-pI	S_b	0	S_g	

Table: Transactions in an example of a general SFC model.

Flow of Funds

Flow of Funds	Households	Firms		Banks	Central Bank	Government	Sum
		current	capital				
Cash	$+\dot{H}_h$			$+\dot{H}_b$	$-\dot{H}$		0
Deposits	$+\dot{M}_h$		$+\dot{M}_f$	$-\dot{M}$			0
Loans			$-\dot{L}$	$+\dot{L}$			0
Bills	$+\dot{B}_h$			$+\dot{B}_b$	$+\dot{B}_c$	$-\dot{B}$	0
Equities	$+p_f\dot{E}_f + p_b\dot{E}_b$		$-p_f\dot{E}_f$	$-p_b\dot{E}_b$			0
Advances				$-\dot{A}$	$+\dot{A}$		0
Capital			$+pI$				pI
Sum	S_h	0	S_f	S_b	0	S_g	pI
Change in Net Worth	$(S_h + \dot{p}_f E_f + \dot{p}_b E_b)$	$(S_f - \dot{p}_f E_f + \dot{p}K - p\delta K)$		$(S_b - \dot{p}_b E_b)$		S_g	$\dot{p}K + p\dot{K}$

Table: Flow of funds in an example of a general SFC model.

General Notation

- Employed labor force: ℓ
- Production function: $Y = f(K, \ell)$
- Labour productivity: $a = \frac{Y}{\ell}$
- Capital-to-output ratio: $\nu = \frac{K}{Y}$
- Employment rate: $\lambda = \frac{\ell}{N}$
- Change in capital: $\dot{K} = I - \delta K$
- Inflation rate: $i = \frac{\dot{p}}{p}$

Goodwin Model (1967) - Assumptions

- Assume that

$$N = N_0 e^{\beta t} \qquad \text{(total labour force)}$$
$$a = a_0 e^{\alpha t} \qquad \text{(productivity per worker)}$$
$$Y = \min\left\{\frac{K}{\nu}, a\ell\right\} \qquad \text{(Leontief production)}$$

- Assume further that

$$Y = \frac{K}{\nu} = a\ell \qquad \text{(full capital utilization)}$$
$$\dot{\mathrm{w}} = \Phi(\lambda, i, i^e)\mathrm{w} \qquad \text{(Phillips curve)}$$
$$pI = pY - \mathrm{w}\ell \qquad \text{(Say's Law)}$$

- NOTE: In the original paper, Goodwin assumed that w above was the real wage rate, so all quantities were normalized by p.

Goodwin Model - SFC matrix

Balance Sheet	Households	Firms current	Firms capital	Sum
Capital			$+pK$	pK
Sum (net worth)	0	0	V_f	pK
Transactions				
Consumption	$-pC$	$+pC$		0
Investment		$+pI$	$-pI$	0
Acct memo [GDP]		$[pY]$		
Wages	$+W$	$-W$		0
Profits		$-\Pi$	$+\Pi_u$	0
Sum	0	0	0	
Flow of Funds				
Capital			$+pI$	pI
Sum	0	0	Π_u	pI
Change in Net Worth	0	$pI + \dot{p}K - p\delta K$		$\dot{p}K + p\dot{K}$

Table: Balance sheet, transactions, and flow of funds for the Goodwin model.

Goodwin Model - Differential equations

- Define

$$\omega = \frac{\mathrm{w}L}{pY} = \frac{\mathrm{w}}{pa} \quad \text{(wage share)}$$

$$\lambda = \frac{L}{N} = \frac{Y}{aN} \quad \text{(employment rate)}$$

- It then follows that

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{\mathrm{w}}}{\mathrm{w}} - \frac{\dot{p}}{p} - \frac{\dot{a}}{a} = \Phi(\lambda, i, i^e) - i - \alpha$$

$$\frac{\dot{\lambda}}{\lambda} = \frac{1-\omega}{\nu} - \alpha - \beta - \delta$$

- In the original model, all quantities were real (i.e divided by p), which is equivalent to setting $i = i^e = 0$.

Original Goodwin Model - Properties

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model
Original model
Inflation
Production function
Equity and savings
Monetary policy

Keen model

The Ultimate Model

- If we take Φ to be linear, this is a predator-prey model.
- To ensure $\lambda \in (0,1)$ we assume instead that Φ is $C^1(0,1)$ and satisfies

$$
\begin{aligned}
&\Phi'(\lambda) > 0 \text{ on } (0,1) \\
&\Phi(0) < \alpha \\
&\lim_{\lambda \to 1^-} \Phi(\lambda) = \infty.
\end{aligned}
$$

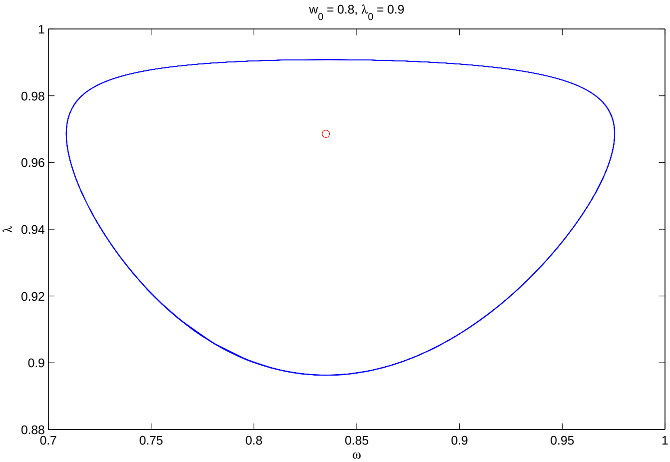
- The only non-trivial equilibrium is

$$(\overline{\omega}_0, \overline{\lambda}_0) = \left(1 - \nu(\alpha + \beta + \delta), \Phi^{-1}(\alpha)\right)$$

is non-hyperbolic.
- Moreover

$$g(\overline{\omega}_0) := \frac{\dot{Y}}{Y}(\overline{\omega}_0) = \frac{1 - \overline{\omega}_0}{\nu} - \delta = \alpha + \beta,$$

Example 1: Goodwin model



Example 1 (continued): Goodwin model

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model

Original model

Inflation

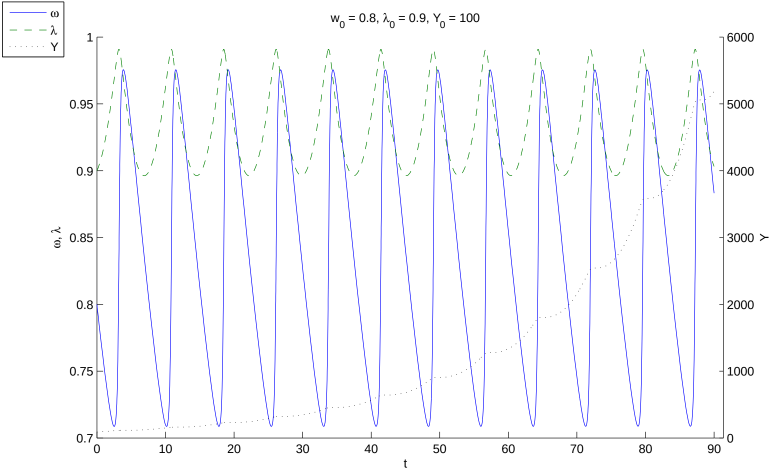
Production function

Equity and savings

Monetary policy

Keen model

The Ultimate Model



Inflation in the Goodwin model

- Assume first that $i^e = i$, so that the Goodwin model with inflation is

$$\begin{aligned} \frac{\dot{\omega}}{\omega} &= \frac{\dot{\mathrm{w}}}{\mathrm{w}} - \frac{\dot{p}}{p} - \frac{\dot{a}}{a} = \Phi\left(\lambda, \frac{\dot{p}}{p}\right) - \frac{\dot{p}}{p} - \alpha \\ \frac{\dot{\lambda}}{\lambda} &= \frac{1-\omega}{\nu} - \alpha - \beta - \delta \qquad (1) \\ \frac{\dot{p}}{p} &= i(p, \omega, \lambda) \end{aligned}$$

- In general, we can define an instantaneous mark-up factor m over unit labour costs by

$$p = m\frac{\mathrm{w}}{a}$$

- Observe that it follows that

$$\omega = \frac{W}{pY} = \frac{\mathrm{w}L}{pY} = \frac{\mathrm{w}}{ap} = \frac{1}{m}$$

Desai (1978) model

- Desai postulates a price dynamics of the form

$$\frac{\dot{p}}{p} = -\eta \log\left(\frac{ap}{\mathrm{w}\overline{m}}\right) = -\eta \log\left(\frac{\overline{\omega}}{\omega}\right)$$

  for an target mark-up factor $\overline{m}$.
- In addition, Desai considers a Philips curve of the form

$$\frac{\dot{w}}{w} = \widetilde{\Phi}\left(\lambda, \frac{\dot{p}}{p}\right) = \Phi(\lambda) + \eta_1\left(\frac{\dot{p}}{p}\right).$$

- This leads to the system

$$\begin{aligned} \frac{\dot{\omega}}{\omega} &= \Phi(\lambda) - \alpha - (1-\eta_1)\eta \log(\omega\overline{m}) \\ \frac{\dot{\lambda}}{\lambda} &= \frac{1-\omega}{\nu} - \alpha - \beta - \delta \\ \frac{\dot{p}}{p} &= \eta \log(\omega\overline{m}) \end{aligned} \tag{2}$$

Equilibrium with inflation and money illusion

- The fixed points for (2) are given by

$$\begin{aligned}\overline{\omega}_1 &= 1 - \nu(\alpha + \beta + \delta) = \overline{\omega}_0 \\ \overline{\lambda}_1(\omega) &= \Phi^{(-1)}\left(\alpha + (1 - \eta_1)\eta \log(\omega \overline{m})\right)\end{aligned}$$

- Define $\overline{\lambda}_1 := \overline{\lambda}_1(\overline{\omega}_1)$. Then

$$\overline{\lambda}_1 - \overline{\lambda}_0 = \Phi^{(-1)}\left(\alpha + (1 - \eta_1)\eta \log(\omega \overline{m})\right) - \Phi^{(-1)}(\alpha)$$

  Therefore provided $\overline{m\omega} > 1$ (positive inflation) and $\eta_1 < 1$ (presence of money illusion), we see that $\overline{\lambda}_1 - \overline{\lambda}_0 > 0$.

- Moreover, this equilibrium is locally stable for $\eta_1 < 1$, a centre for $\eta_1 = 1$ (original Goodwin), and unstable for $\eta_1 > 1$.

Variable capital-to-output ratio

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model

Original model

Inflation

Production function

Equity and savings

Monetary policy

Keen model

The Ultimate Model

- Desai (1978) also considers a cyclical ν of the form

$$\nu = \nu^* \lambda^{-\mu} \tag{3}$$

- This leads to

$$\frac{\dot{\lambda}}{\lambda} = \frac{1}{1-\mu}\left(\frac{(1-\omega)\lambda^{\mu}}{\nu^*} - \alpha - \beta - \delta\right)$$

- The new fixed point relations are

$$\begin{aligned}\overline{\omega}_2(\lambda) &= 1 - \nu^*(\alpha+\beta)\lambda^{-\mu} \\ \overline{\lambda}_2 &= \Phi^{(-1)}\left(\alpha + (1-\eta_1)\eta \log(\overline{\omega m})\right)\end{aligned}$$

- These two curves intersect at two distinct equilibria, which are locally stable for typical parameters, that is, $\eta_1 < 1$, $0 < \mu < 1$ and $\alpha + \beta$ sufficiently small. Higher values of μ or η_1 could make either or both equilibria unstable.

Inflation expectations

- Finally, Desai (1978) consider a Philips curve of the form

$$\frac{\dot{\mathrm{w}}}{\mathrm{w}} = \Phi(\lambda) + \eta_2 \left(\frac{\dot{p}}{p}\right)^e$$

- Moreover, inflation expectations evolve according to

$$\frac{d}{dt}\left(\frac{\dot{p}}{p}\right)^e = \eta_3 \left[\frac{\dot{p}}{p} - \left(\frac{\dot{p}}{p}\right)^e\right],$$

- This leads to

$$\frac{\dot{\lambda}}{\lambda} = -\frac{\alpha+\beta}{1-\mu} + \frac{(1-\omega)\lambda^\mu}{\nu^*(1-\mu)}$$

$$\frac{d}{dt}\left(\frac{\dot{\omega}}{\omega}\right) = \eta_3(\Phi(\lambda)-\alpha) + \Phi'(\lambda)\dot{\lambda} + \eta_2(1-\eta_3)\eta\log(\overline{m}\omega) - (1+\eta_3)\frac{\dot{\omega}}{\omega}$$

CES Production function, var der Ploeg (1985)

- Consider a production function of the form

$$Y = f(K, \ell_{ef}) = A[\mu K^{-\eta} + (1-\mu)\ell_{ef}^{-\eta}]^{-\frac{1}{\eta}} \tag{4}$$

where $\ell_{ef} = A^{-1}a_0 e^{\alpha t}\ell$ is the effective labour.

- This has an elasticity of substitution given by

$$s = \frac{d\log(k/\ell_{ef})}{d\log(f_{\ell_{ef}}/f_K)} = \frac{1}{1+\eta}.$$

- The special cases for this function are

$$\lim_{\eta\to\infty} f(K, \ell_{ef}) = \min(AK, A\ell_{ef}), \quad \text{(Leontief)}$$

$$\lim_{\eta\to 0} f(K, \ell_{ef}) = AK^{\mu}\ell_{ef}^{1-\mu}, \quad \text{(Cobb-Douglas)}$$

Optimal capital-to-output and labour productivity

- Assume that firms optimize profits by setting

$$\frac{\partial Y}{\partial L} = \mathrm{w}.$$

- Using (4), we find that the profit maximizing capital-to-output ratio is

$$\nu(\omega) = \frac{K}{Y} = \frac{1}{A}\left(\frac{1-\omega}{\mu}\right)^{-\frac{1}{\eta}}. \tag{5}$$

- Similarly for the labour productivity we find

$$a(\omega) = \frac{Y}{L} = \left(\frac{\omega}{1-\mu}\right)^{\frac{1}{\eta}} a_0 e^{\alpha t} \tag{6}$$

ODE system for the van der Ploeg 1985 model

- Using (5) and (6) we find

$$\frac{\dot{\omega}}{\omega} = \frac{\Phi(\lambda) - \alpha}{1 + 1/\eta}$$

$$\frac{\dot{\lambda}}{\lambda} = A(1-\omega)\left(\frac{1-\omega}{\mu}\right)^{\frac{1}{\eta}} - \frac{\Phi(\lambda) - \alpha}{(1-\omega)(1+\eta)} - (\alpha + \beta + \delta)$$

- The equilibrium is now

$$\begin{aligned} \overline{\lambda} &= \Phi^{-(1)}(\alpha) \\ \overline{\omega} &= 1 - \left(\frac{\alpha+\beta}{A}\right)^{\frac{\eta}{1+\eta}} \mu^{\frac{1}{1+\eta}}, \end{aligned} \tag{7}$$

and is locally stable for all $0 \leq \eta < \infty$.

- We recover the original Goodwin model when $\eta \to \infty$ and $A = 1/\nu$.

Example 2: van der Ploeg 1985 model

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model

Original model

Inflation

Production function

Equity and savings

Monetary policy

Keen model

The Ultimate Model

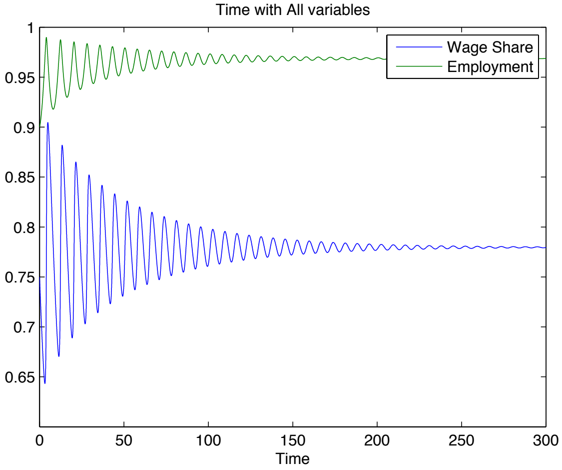


Figure: van der PLoeg 1985 model with $\eta = 500$.

Example 2 (continued): van der Ploeg 1985 model

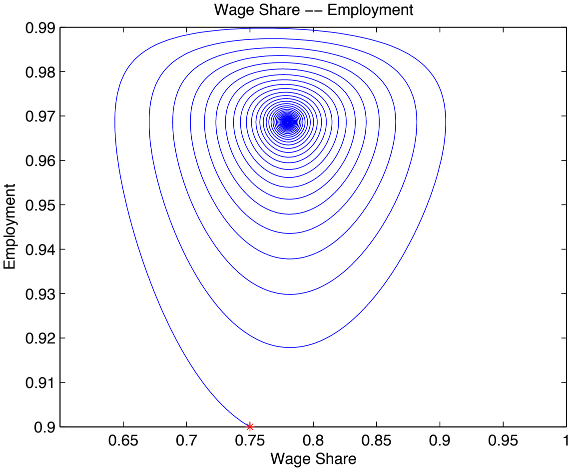


Figure: van der PLoeg 1985 model with $\eta = 500$.

A model with equities

Dynamical systems and Financial Instability
M. R. Grasselli
Introduction
Goodwin model
Original model
Inflation
Production function
Equity and savings
Monetary policy
Keen model
The Ultimate Model

Balance Sheet	Households	Firms current	Firms capital	Sum
Equities	$+p_eE$		$-p_eE$	0
Capital			$+pK$	pK
Sum (net worth)	V_h	0	V_f	pK
Transactions				
Consumption	$-pC$	$+pC$		0
Investment		$+pI$	$-pI$	0
Acct memo [GDP]		$[pY]$		
Wages	$+W$	$-W$		0
Profits	$+\Pi_d$	$-\Pi$	$+\Pi_u$	0
Sum	S_h	0	$S_f - pI$	0
Flow of Funds				
Equities	$+p_e\dot{E}$		$-p_e\dot{E}$	0
Capital			$+pI$	pI
Sum	S_h	0	S_f	pI
Change in Net Worth	$(S_h + \dot{p}_eE)$	$(S_f - \dot{p}_eE + \dot{p}K - p\delta K)$		$\dot{p}K + p\dot{K}$

Table: SFC table for a model with equities.

Saving propensities, van der Ploeg (1983)

- Consider saving functions of the form

$$S_f = \Pi_u = s_p \Pi$$
$$S_h = s_w(W + \Pi_d) = s_w[W + (1 - s_p)\Pi]$$

- We would then have $I = (s_p + s_w - s_w s_p)\Pi + s_w W$, so

$$\frac{\dot{K}}{K} = \frac{s_w + (1 - s_w)s_p(1-\omega)}{\nu} - \delta$$

- This leads to the modified system

$$\frac{\dot{\omega}}{\omega} = \Phi(\lambda) - \alpha$$
$$\frac{\dot{\lambda}}{\lambda} = \frac{s_w + (1 - s_w)s_p(1-\omega)}{\nu} - \alpha - \beta - \delta$$

- Notice how investment is completely determined by savings in this model.

The Di Matteo (1983) model

- Di Matteo considers a model identical to Goodwin's, expect for

$$\frac{\dot{p}}{p} = \frac{\rho}{1+\rho}\left(\frac{\dot{\mathrm{w}}}{\mathrm{w}} - \alpha\right)$$

$$I = Y - \left(\frac{\mathrm{w}\ell}{p}\right) + nK\left(\theta - \mu\frac{\dot{Y}}{Y}\right)$$

$$S = S(Y)$$

$$M^d = F(P, Y, r)$$

$$0 = P(I - S) + v(M^d - M),$$

- The crucial parameter here is an exogenous growth rate θ for the money supply M.

Properties and extensions

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model
Original model
Inflation
Production function
Equity and savings
Monetary policy

Keen model

The Ultimate Model

- Using the new investment and price equations, one finds

$$\frac{\dot{\omega}}{\omega} = \frac{\phi(\lambda) - \alpha}{1 + \rho}$$
$$\frac{\dot{\lambda}}{\lambda} = \frac{1 - \omega + (n\theta - \delta)\nu}{\nu(1 + n\mu)} - \alpha - \beta$$

- For typical parameters, this behaves exactly like the OGM.
- Assuming either variable growth rate in money supply of the form

$$\theta = \theta' + \gamma_1(\overline{\lambda} - \lambda) - \gamma_2(\overline{\omega} - \omega), \quad \gamma_1, \gamma_2 \geq 0$$

  leads to a stable equilibrium provided $\gamma_2 < \frac{1}{n\nu}$.
- A similar result holds for variable interest rates.

SFC table for Keen (1995) model

Balance Sheet	Households	Firms current	Firms capital	Banks	Sum
Deposits	$+D$			$-D$	0
Loans			$-L$	$+L$	0
Capital			$+pK$		pK
Sum (net worth)	V_h	0	V_f	0	pK
Transactions					
Consumption	$-pC$	$+pC$			0
Investment		$+pI$	$-pI$		0
Acct memo [GDP]		$[pY]$			
Wages	$+W$	$-W$			0
Interest on deposits	$+rD$			$-rD$	0
Interest on loans		$-rL$		$+rL$	0
Profits 0		$-\Pi$	$+\Pi_u$		
Sum	S_h	0	$S_f - pI$	0	0
Flow of Funds					
Deposits	$+\dot{D}$			$-\dot{D}$	0
Loans			$-\dot{L}$	$+\dot{L}$	0
Capital			$+pI$		pI
Sum	S_h	0	Π_u	0	pI
Change in Net Worth	S_h	$(S_f + \dot{p}K - p\delta K)$			$\dot{p}K + p\dot{K}$

Table: Balance sheet, transactions, and FOF for the Keen model.

Keen model - Investment function

- Assume now that new investment is given by

$$\dot{K} = \kappa(1 - \omega - rd)Y - \delta K$$

where $\kappa(\cdot)$ is $C^1(-\infty, \infty)$ satisfying

$$\kappa'(\pi) > 0 \text{ on } (-\infty, \infty)$$

$$\lim_{\pi \to -\infty} \kappa(\pi) = \kappa_0 < \nu(\alpha + \beta + \delta) < \lim_{\pi \to +\infty} \kappa(\pi)$$

$$\lim_{\pi \to -\infty} \pi^2 \kappa'(\pi) = 0.$$

Accordingly, total output evolves as

$$\frac{\dot{Y}}{Y} = \frac{\kappa(1 - \omega - rd)}{\nu} - \delta := g(\omega, d)$$

- This leads to external financing through debt evolving according to

$$\dot{D} = \kappa(1 - \omega - rd)Y - (1 - \omega - rd)Y$$

Keen model - Differential Equations

Denote the debt ratio in the economy by $d = D/Y$, the model can now be described by the following system

$$
\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda\left[\frac{\kappa(1-\omega-rd)}{\nu} - \alpha - \beta - \delta\right] \\
\dot{d} &= d\left[r - \frac{\kappa(1-\omega-rd)}{\nu} + \delta\right] + \kappa(1-\omega-rd) - (1-\omega)
\end{aligned}
\tag{8}
$$

Keen model - equilibria

- The system (8) has a good equilibrium at

$$\overline{\omega} = 1 - \overline{\pi} - r\frac{\nu(\alpha+\beta+\delta) - \overline{\pi}}{\alpha+\beta}$$
$$\overline{\lambda} = \Phi^{-1}(\alpha)$$
$$\overline{d} = \frac{\nu(\alpha+\beta+\delta) - \overline{\pi}}{\alpha+\beta}$$

  with

$$\overline{\pi} = \kappa^{-1}(\nu(\alpha+\beta+\delta)),$$

  which is stable for a large range of parameters

- It also has a bad equilibrium at $(0, 0, +\infty)$, which is stable if

$$\frac{\kappa(-\infty)}{\nu} - \delta < r \tag{9}$$

Example 1: convergence to the good equilibrium in a Keen model

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model

Keen model

Original model

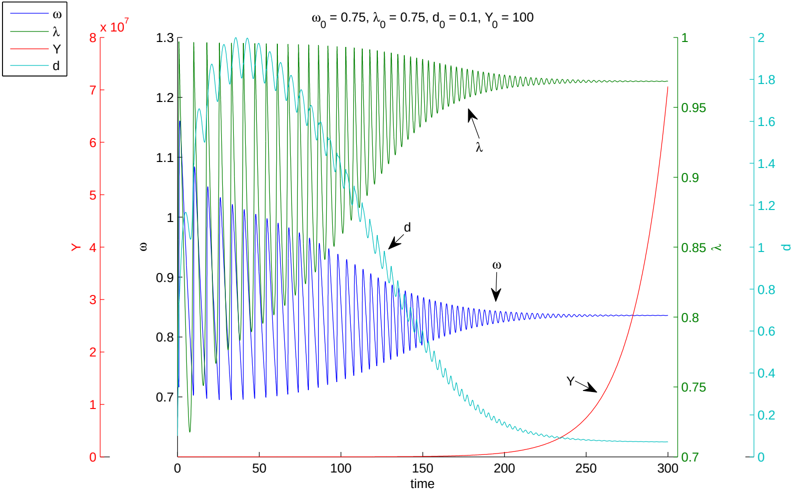
Ponzi financing

Noise and Stock Prices

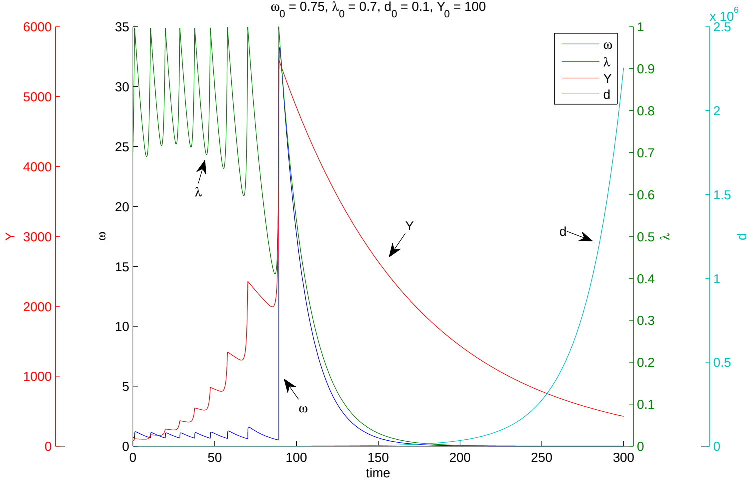
Stabilizing government

Great Moderation

The Ultimate Model



Example 2: explosive debt in a Keen model



Basin of convergence for Keen model

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model

Keen model

Original model

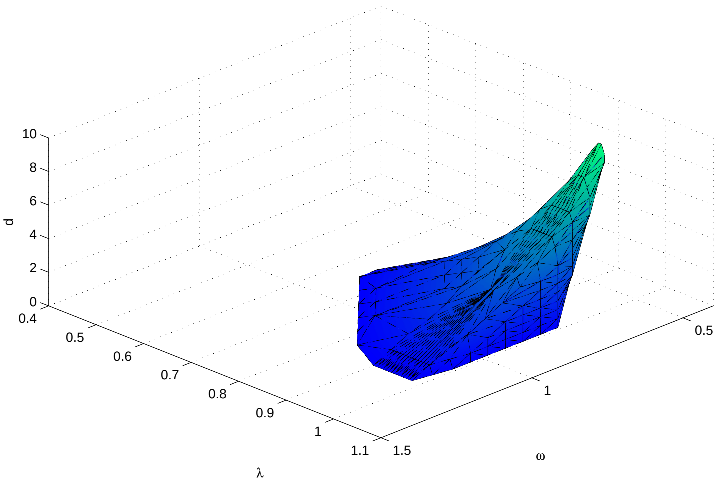
Ponzi financing

Noise and Stock Prices

Stabilizing government

Great Moderation

The Ultimate Model



Ponzi financing

Introduction
Goodwin model
Keen model
Original model
Ponzi financing
Noise and Stock Prices
Stabilizing government
Great Moderation
The Ultimate Model

To introduce the destabilizing effect of purely speculative investment, we consider a modified version of the previous model with

$$\dot{D} = \kappa(1-\omega-rd)Y - (1-\omega-rd)Y + P$$
$$\dot{P} = \Psi(g(\omega,d)P$$

where $\Psi(\cdot)$ is an increasing function of the growth rate of economic output

$$g = \frac{\kappa(1-\omega-rd)}{\nu} - \delta.$$

Ponzi financing - Differential equations

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model

Keen model

Original model

Ponzi financing

Noise and Stock Prices

Stabilizing government

Great Moderation

The Ultimate Model

With Ponzi financing the dynamical system becomes

$$
\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda\left[\frac{\kappa(1-\omega-rd)}{\nu} - \alpha - \beta - \delta\right] \\
\dot{d} &= d\left[r - \frac{\kappa(1-\omega-rd)}{\nu} + \delta\right] + \kappa(1-\omega-rd) - (1-\omega) + p \\
\dot{p} &= p\left[\Psi\left(\frac{\kappa(1-\omega-rd)}{\nu} - \delta\right) - \frac{\kappa(1-\omega-rd)}{\nu} + \delta\right]
\end{aligned}
\tag{10}
$$

Ponzi financing - Equilibria and stability

- We find that $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1, 0)$ is a stable equilibrium iff

$$\Psi(\alpha+\beta) < \alpha+\beta.$$

- Introducing $u = 1/d$ we find that

$$(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_2, \overline{p}) = (0, 0, +\infty, 0)$$

  is stable iff

$$\Psi(g_0) < g_0.$$

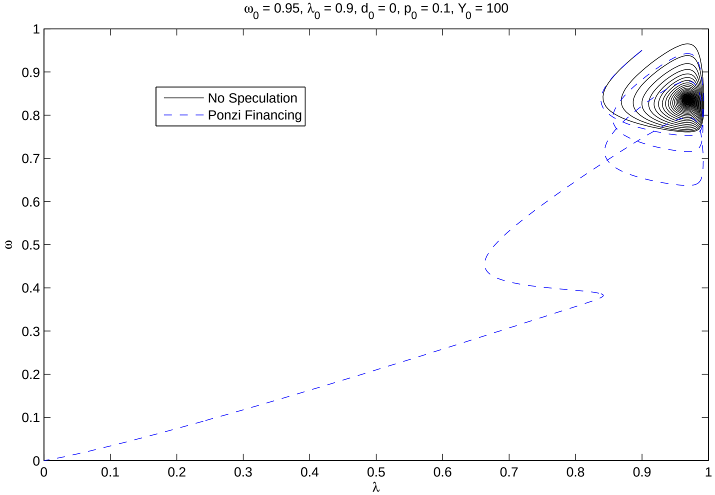
- Moreover, introducing , $x = 1/p$ and $v = p/d$ we find that

$$(\overline{\omega}_3, \overline{\lambda}_3, \overline{d}_3, \overline{p}) = (0, 0, +\infty, +\infty)$$

  is stable iff

$$g_0 < \Psi(g_0) < r.$$

Example 4: effect of Ponzi financing



Stock prices

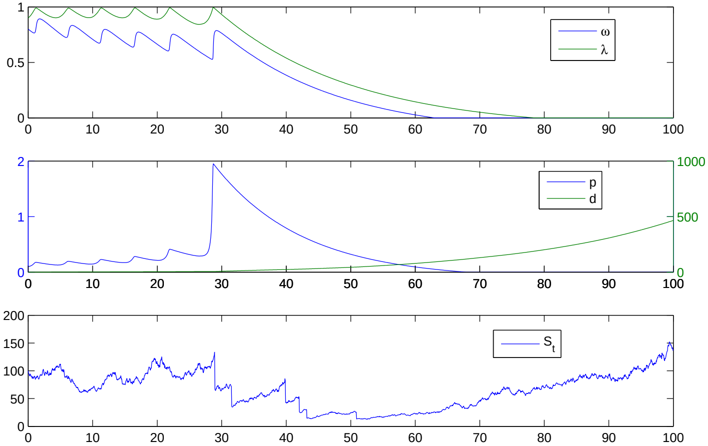
- Consider a stock price process of the form

$$\frac{dS_t}{S_t} = r_b dt + \sigma dW_t + \gamma \mu_t dt - \gamma dN^{(\mu_t)}$$

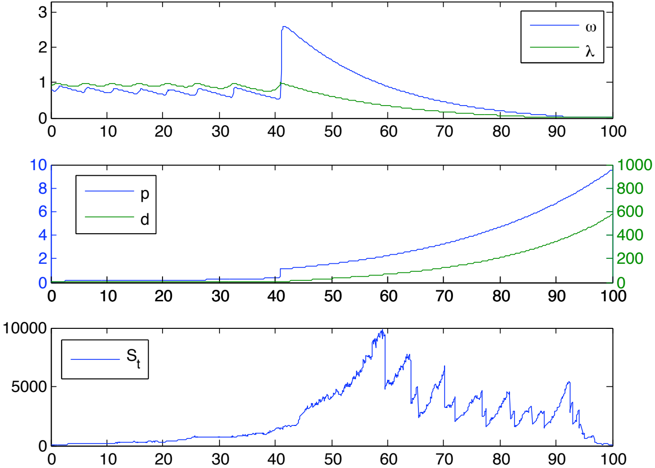
  where N_t is a Cox process with stochastic intensity $\mu_t = M(p(t))$.
- The interest rate for private debt is modelled as $r_t = r_b + r_p(t)$ where

$$r_p(t) = \rho_1 (S_t + \rho_2)^{\rho_3}$$

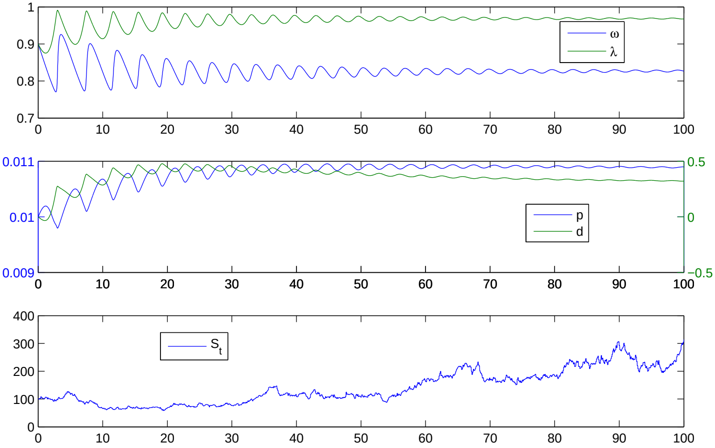
Example 6: stock prices, explosive debt, zero speculation



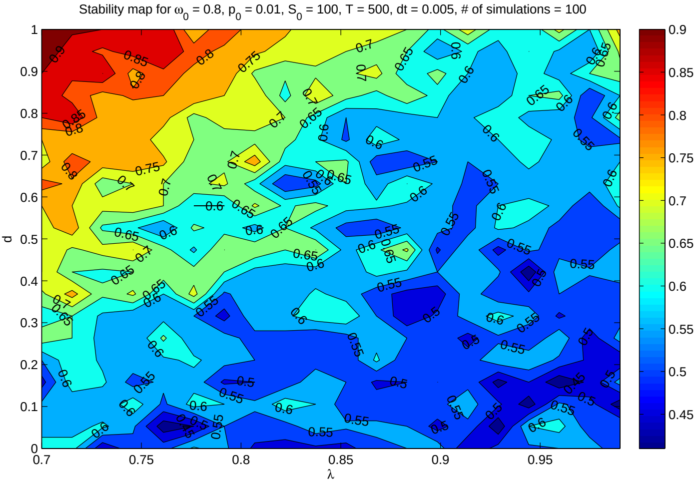
Example 6: stock prices, explosive debt, explosive speculation



Example 6: stock prices, finite debt, finite speculation



Stability map



Introducing a government sector

- Following Keen (and echoing Minsky) we add discretionary government subsidied and taxation into the original system in the form

$$G = G_1 + G_2$$
$$T = T_1 + T_2$$

where

$$\dot{G}_1 = \eta_1(\lambda)Y \qquad \dot{G}_2 = \eta_2(\lambda)G_2$$
$$\dot{T}_1 = \Theta_1(\pi)Y \qquad \dot{T}_2 = \Theta_2(\pi)T_2$$

- Defining $g = G/Y$ and $\tau = T/Y$, the net profit share is now

$$\pi = 1 - \omega - rd + g - \tau,$$

and government debt evolves according to

$$\dot{B} = rB + G - T.$$

Differential equations - full system

Denoting $\gamma(\pi) = \kappa(\pi)/\nu - \delta$, a bit of algebra leads to the following eight–dimensional system:

$$
\begin{aligned}
\dot{\omega} &= \omega[\Phi(\lambda) - \alpha] \\
\dot{\lambda} &= \lambda[\gamma(\pi) - \alpha - \beta] \\
\dot{d} &= \kappa(\pi) - \pi - d\gamma(\pi) \\
\dot{g}_1 &= \eta_1(\lambda) - g_1\gamma(\pi) \\
\dot{g}_2 &= g_2\left[\eta_2(\lambda) - \gamma(\pi)\right] \\
\dot{\tau}_1 &= \Theta_1(\pi) - g_{T_1}\gamma(\pi) \\
\dot{\tau}_2 &= \tau_2\left[\Theta_2(\pi) - \gamma(\pi)\right] \\
\dot{b} &= b[r - \gamma(\pi)] + g_1 + g_2 - \tau_1 - \tau_2
\end{aligned}
\tag{11}
$$

Differential equations - reduced system

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model

Keen model

Original model

Ponzi financing

Noise and Stock Prices

Stabilizing government

Great Moderation

The Ultimate Model

- Notice that π does not depend on b, so that the last equation in (11) can be solved separately.
- Observe further that we can write

$$\dot{\pi} = -\dot{\omega} - r\dot{d} + \dot{g} - \dot{\tau} \tag{12}$$

leading to the five-dimensional system

$$\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right], \\
\dot{\lambda} &= \lambda\left[\gamma(\pi) - \alpha - \beta\right] \\
\dot{g}_2 &= g_2\left[\eta_2(\lambda) - \gamma(\pi)\right] \\
\dot{\tau}_2 &= \tau_2\left[\Theta_2(\pi) - \gamma(\pi)\right] \\
\dot{\pi} &= -\omega(\Phi(\lambda) - \alpha) - r(\kappa(\pi) - \pi) + (1 - \omega - \pi)\gamma(\pi) \\
&\quad + \eta_1(\lambda) + g_2\eta_2(\lambda) - \Theta_2(\pi) - \tau_2\Theta_2(\pi)
\end{aligned} \tag{13}$$

Good equilibrium

- The system (13) has a good equilibrium at

$$\overline{\omega} = 1 - \overline{\pi} - r\frac{\nu(\alpha+\beta+\delta) - \overline{\pi}}{\alpha+\beta} + \frac{\eta_1(\overline{\lambda}) - \Theta_1(\overline{\pi})}{\alpha+\beta}$$

$$\overline{\lambda} = \Phi^{-1}(\alpha)$$

$$\overline{\pi} = \kappa^{-1}(\nu(\alpha+\beta+\delta))$$

$$\overline{g}_2 = \overline{\tau}_2 = 0$$

and this is locally stable for a large range of parameters.

- The other variables then converge exponentially fast to

$$\overline{d} = \frac{\nu(\alpha+\beta+\delta) - \overline{\pi}}{\alpha+\beta}$$

$$\overline{g}_1 = \frac{\eta_1(\overline{\lambda})}{\alpha+\beta}$$

$$\overline{\tau}_1 = \frac{\Theta_1(\overline{\pi})}{\alpha+\beta}$$

Bad equilibria - destabilizing a stable crisis

- Recall that $\pi = 1 - \omega - rd + g - \tau$.
- The system (13) has bad equilibria of the form

$$(\omega, \lambda, g_2, \tau_2, \pi) = (0, 0, 0, 0, -\infty)$$
$$(\omega, \lambda, g_2, \tau_2, \pi) = (0, 0, \pm\infty, 0, -\infty)$$

- If $g_2(0) > 0$, then any equilibria with $\pi \to -\infty$ is locally unstable provided $\eta_2(0) > r$.
- On the other hand, if $g_2(0) < 0$ (austerity), then these equilibria are all locally stable.

Persistence results

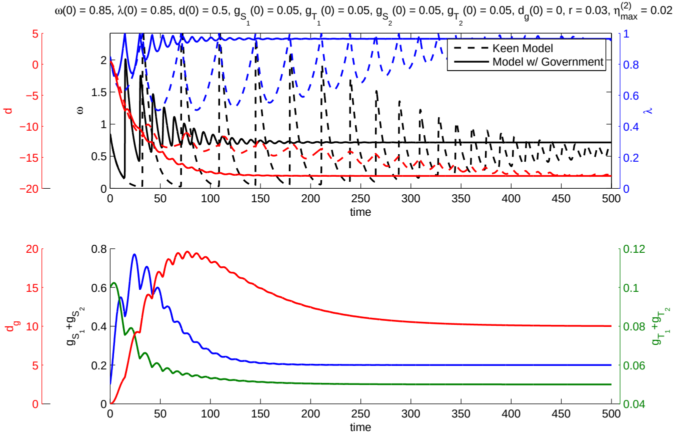
Proposition 1: Assume $g_2(0) > 0$, then the system (13) is e^{π}-UWP if either

1. $\lambda\eta_1(\lambda)$ is bounded below as $\lambda \to 0$, or
2. $\eta_2(0) > r$.

Proposition 2: Assume $g_2(0) > 0$ and $\tau_2(0) = 0$, then the system (13) is λ-UWP if either of the following three conditions is satisfied:

1. $\lambda\eta_1(\lambda)$ is bounded below as $\lambda \to 0$, or
2. $\eta_2(0) > \max\{r, \alpha + \beta\}$, or
3. $r < \eta_2(0) \leq \alpha + \beta$ and $-r(\kappa(x) - x) + (1 - x)\gamma(x) + \eta_1(0) - \Theta_1(x) > 0$ for $\gamma(x) \in [\eta_2(0), \alpha + \beta]$.

Example 3: Good initial conditions



Example 4: Bad initial conditions

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model

Keen model

Original model

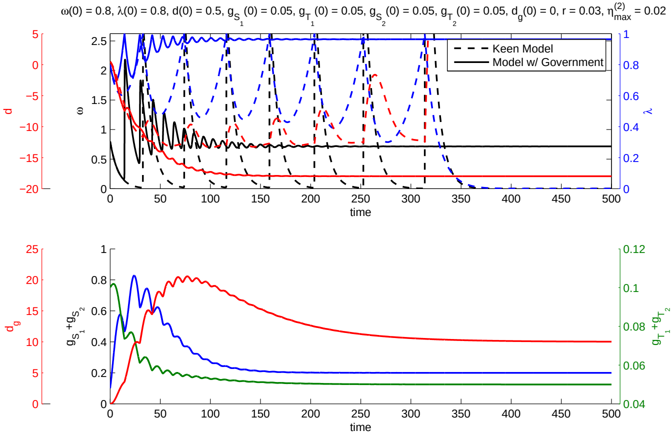
Ponzi financing

Noise and Stock Prices

Stabilizing government

Great Moderation

The Ultimate Model



Example 5: Really bad initial conditions with timid government

Dynamical systems and Financial Instability

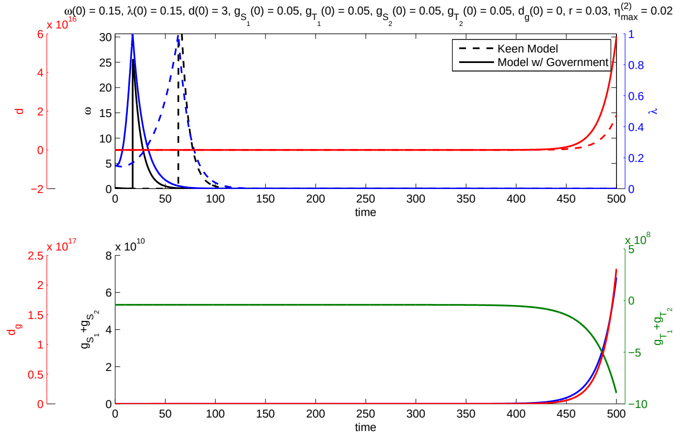
M. R. Grasselli

Introduction

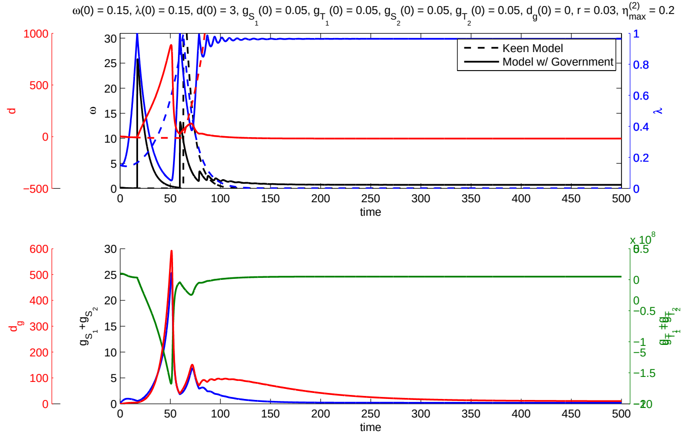
Goodwin model

Keen model
Original model
Ponzi financing
Noise and Stock Prices
Stabilizing government
Great Moderation

The Ultimate Model



Example 6: Really bad initial conditions with responsive government



Example 7: Austerity in good times: harmless

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model

Keen model

Original model

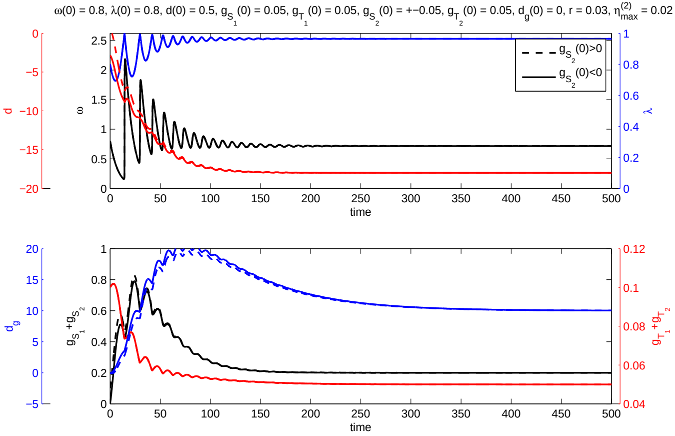
Ponzi financing

Noise and Stock Prices

Stabilizing government

Great Moderation

The Ultimate Model



Example 8: Austerity in bad times: a really bad idea

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model

Keen model

Original model

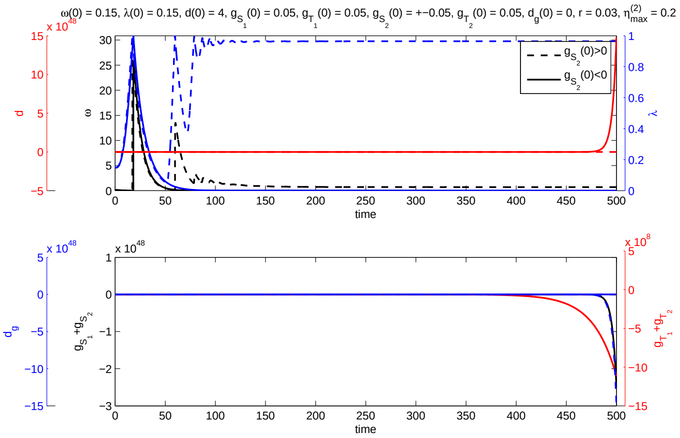
Ponzi financing

Noise and Stock Prices

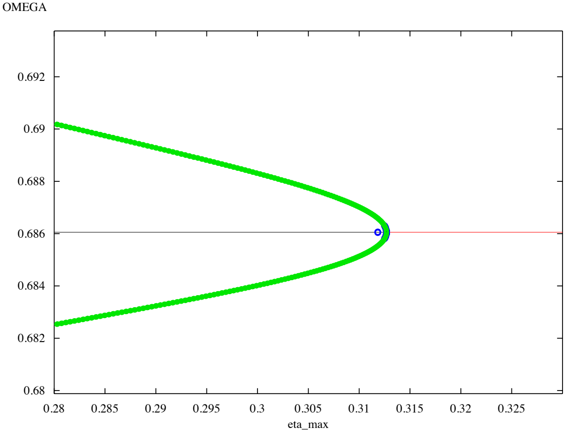
Stabilizing government

Great Moderation

The Ultimate Model



Hopft bifurcation with respect to government spending.



The Great Moderation in the U.S. - 1984 to 2007

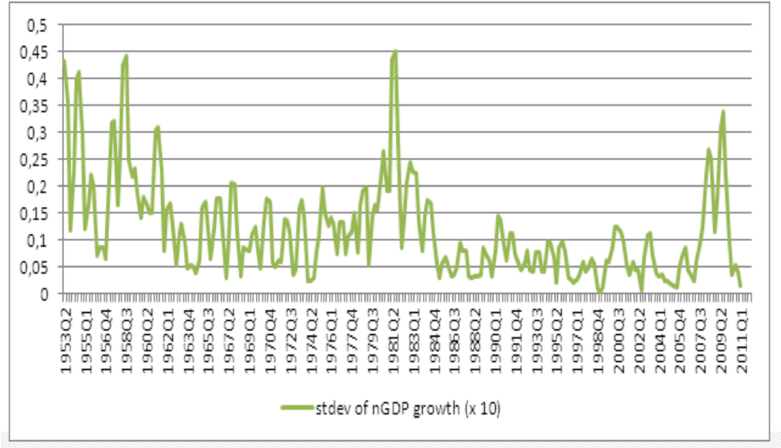


Figure: Grydaki and Bezemer (2013)

Possible explanations

- Real-sector causes: inventory management, labour market changes, responses to oil shocks, external balances , etc.
- Financial-sector causes: credit accelerator models, financial innovation, deregulation, better monetary policy, etc.
- Grydaki and Bezemer (2013): growth of debt in the real sector.

Bank credit-to-GDP ratio in the U.S

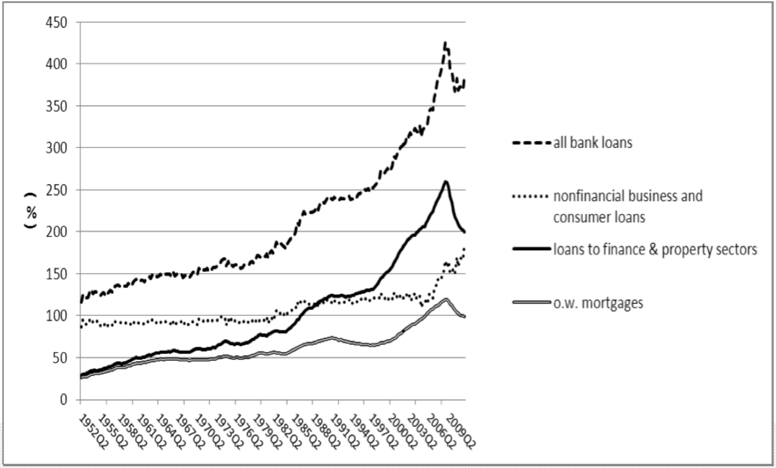


Figure: Grydaki and Bezemer (2013)

Cumulative percentage point growth of excess credit growth, 1952-2008

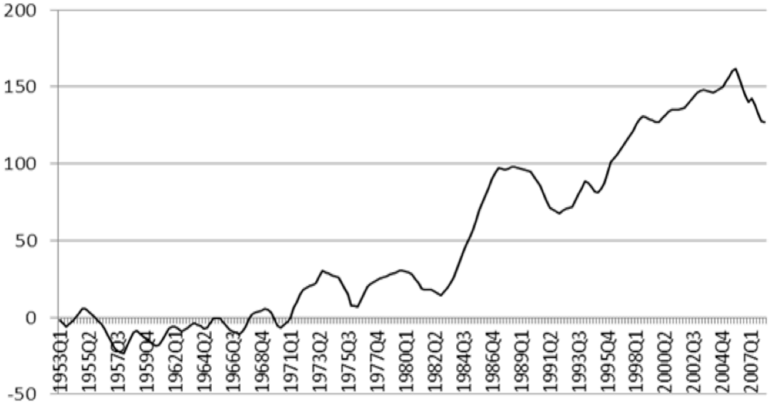


Figure: Grydaki and Bezemer (2013)

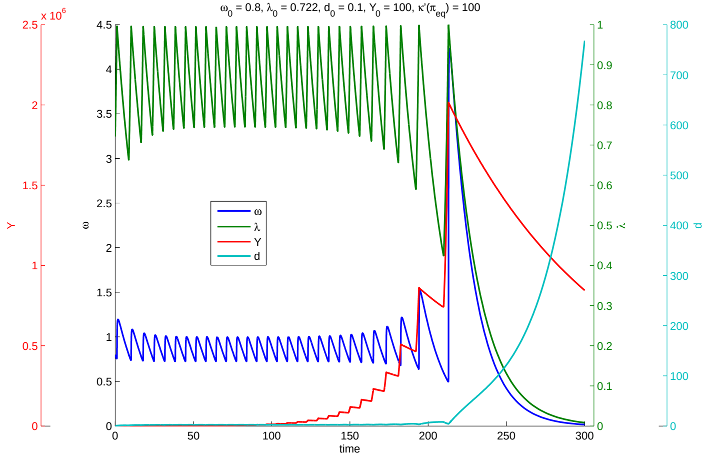
Excess credit growth moderated output volatility during, but not before the Great Moderation

Before the Great Moderation	*During the Great Moderation*
change in interest rate (-) => output volatility	excess credit growth (-) => output volatility
change in interest rate (+) => inflation	output volatility (+) => excess credit growth
excess credit growth (+) => change in interest rate	output volatility (-) => change in interest rate
	excess credit growth (+) => change in interest rate
	inflation (+) => change in interest rate

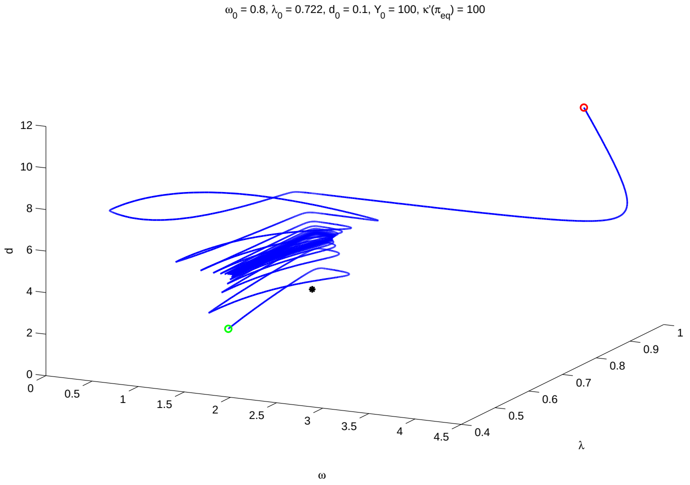
Note: In the table, x (-) => y denotes that a one-standard deviation shock in variable x impacts negatively on the change of variable y. Similarly, x (+) => y indicates a positive impact.

Figure: Grydaki and Bezemer (2013)

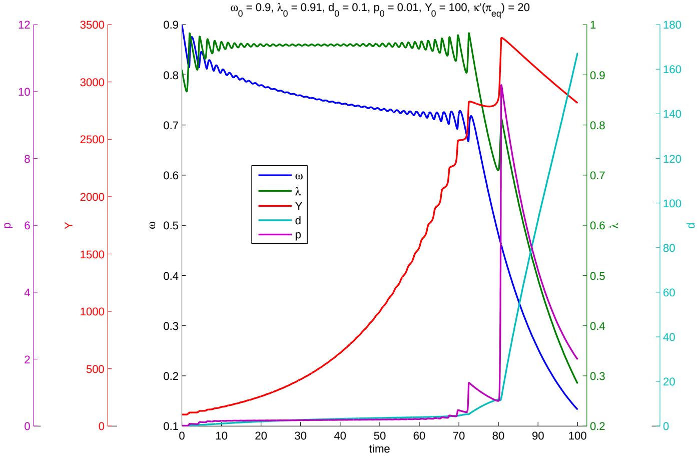
Example 3: weakly moderated oscillations



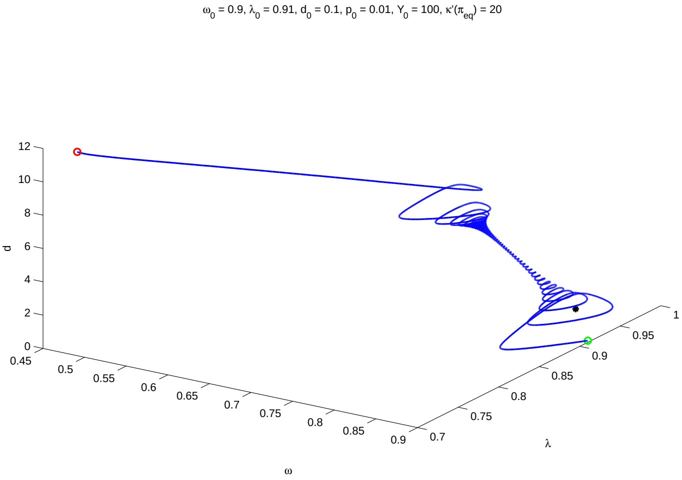
Example 3 (cont): weakly moderated oscillations in 3d



Example 4: strongly moderated oscillations



Example 4 (cont): strongly moderated oscillations in 3d



Shortcomings of Goodwin and Keen models

- No independent specification of consumption (and therefore savings) for households:

$$C = W, \quad S_h = 0 \qquad \text{Goodwin}$$

$$C = (1-\kappa(\pi))Y, \quad S_h = \dot{D} = \Pi_u - I \qquad \text{Keen}$$

- Full capacity utilization.
- Everything that is produced is sold.
- No active market for equities.

Independent savings and investment, Skott (1989)

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model

Keen model

The Ultimate Model

Savings and investment

Inventories

Portfolios

Bringing it altogether

- Skott assumes saving and investment of the form

$$
\begin{aligned}
S &= g(\pi)Y, \quad g' > 0, \\
I &= f(\sigma,\pi)Y, \quad f_\sigma > 0, \quad g' > f_\pi \geq 0, \\
\dot{Y} &= h(\pi,\lambda)Y, \quad h_\pi > 0, h_\lambda < 0
\end{aligned}
$$

  where $\sigma = Y/K = 1/\nu$ is *not* constant.
- Saving and investment decisions of firms are reconciled by an implicit price level p satisfying

$$(W + rD + \Pi_d - pC) + \Pi_u = pI \Leftrightarrow g(\pi) = f(\sigma,\pi)$$

- It then follows that $\pi = \theta(\sigma)$ for some θ and

$$
\begin{cases}
\dfrac{\dot{\sigma}}{\sigma} = h(\theta(\sigma),\lambda) - \sigma f(\sigma,\theta(\sigma)) + \Delta \\[2ex]
\dfrac{\dot{\lambda}}{\lambda} = h(\theta(\sigma),\lambda) - \alpha - \beta
\end{cases}
\tag{14}
$$

Inventory changes, Chiarella, Flaschel and Franke (2005)

- Given a realized demand $Y^d = C + I$, consider an inventory dynamics of the form

$$
\begin{aligned}
N^d &= \alpha_{n^d} Y^e \\
\dot{In} &= nN^d + \beta_n (N^d - N) \\
Y^s &= Y^e + \dot{In} \\
\dot{N} &= Y^s - Y^d \\
\dot{Y}^e &= nY^e + \beta_e (Y^d - Y^e)
\end{aligned}
$$

Portfolio choices, Tobin (1969, 1980)

Dynamical systems and Financial Instability

M. R. Grasselli

Introduction

Goodwin model

Keen model

The Ultimate Model

Savings and investment

Inventories

Portfolios

Bringing it altogether

- Given a choice between holding equities E at a price p_e and money M, households allocate wealth according to

$$
\begin{aligned}
p_e E &= f_e(r_e^e)W \\
M &= f_m(r)W
\end{aligned}
$$

- Here r_e^e is the expected return on equity given by

$$
\begin{aligned}
r_e^e &= \frac{r_k^e pK}{p_e E} = \frac{r_k^e}{q} + \pi_e^e \\
r_k^e &= (Y^e - \delta K - W)/K \\
\dot{\pi}_e^e &= \beta_{\pi_e}\left(\frac{\dot{p}_e}{p_e} - \pi_e^e\right)
\end{aligned}
$$

where the last equation can be further decomposed between chartists and fundamentalists, etc.

A synthesis, Grasselli and Nguyen (2013)

- Consider

$$
\begin{aligned}
Y^s &= \min\{\sigma K; aL\} = \sigma K = a\lambda\ell \\
\dot{N} &= Y^s - Y^d \\
\dot{K} &= I - \delta K, \quad I = \kappa Y^d \\
\dot{Y}^e &= \alpha_E Y^e + \beta_E (Y^d - Y^e) \\
\dot{\mathrm{w}} &= \Phi(\lambda)\mathrm{w}
\end{aligned}
$$

Differential equations for firm

- Defining $\tau := N/Y^d$, $y := Y^s/Y^d$, $\alpha_N := \gamma Y^e/N$, $g := \dot{Y}^d/Y^d$, we obtain the following autonomous system

$$
\begin{aligned}
\frac{\dot\sigma}{\sigma} &= f(1-y-(1-\alpha_N)\tau) \\
\frac{\dot\tau}{\tau} &= \frac{(y-1)}{\tau} - g \\
\frac{\dot\alpha_N}{\alpha_N} &= \alpha_E + \beta_E\left(\frac{\gamma}{\alpha_n\tau} - 1\right) - \frac{(y-1)}{\tau} \\
\frac{\dot y}{y} &= f(1-y-\alpha_N\tau) + \frac{\kappa\sigma}{y} - \delta - g
\end{aligned}
\tag{15}
$$

- Moreover, this system fully determines employment by

$$
\frac{\dot\lambda}{\lambda} = \frac{\dot\sigma}{\sigma} + \frac{\dot K}{K} - \frac{\dot a}{a} - \frac{\dot\ell}{\ell} = f(1-y-\alpha_T\tau) + \frac{\kappa\sigma}{y} - \delta - \alpha - \beta
$$

Prices and wages

- Given a target mark-up factor $\overline{m}$, firms set prices according to

$$i = \frac{\dot{p}}{p} = \frac{\beta_P}{p}\left(p - m\frac{w}{a}\right) = \beta_P\left(1 - m\frac{\omega}{y}\right).$$

- It then follows that

$$\frac{\dot{\omega}}{\omega} = \phi(\lambda) - \alpha + \frac{\dot{y}}{y} - \beta_P\left(1 - m\frac{\omega}{y}\right),$$

which can also be fully determined by system (15).

Firms financing

- Assume that firms retain a fraction s_p of profits and seeks to obtain a fraction ν of its external financing as new debt.
- This leads to

$$\dot{d} = \mu(\kappa - s_p\pi) - d(i+g).$$

- Using $\pi = 1 - \omega - rd$, we then find

$$\dot{\pi} = -\omega\left[\phi(\lambda) + \frac{\dot{y}}{y} - \alpha_L - i\right] - r\left[\nu(\kappa - s_f\pi) - d(i+g)\right].$$

- We now have an autonomous system for $(y, \tau, \sigma, \alpha_N, \lambda, \pi)$.

Household savings

- Assume that households allocate wealth and consumption according to $D = \gamma_D pC$ and $eE = \gamma_E pC$.
- This lead to Tobin's valuation ratio of the form

$$q = \frac{p_e E}{pK} = \frac{p_e E\sigma}{pY^p} = \frac{\gamma_E D\sigma}{\gamma_D pyY^d} = \frac{\gamma_E \sigma(1-\kappa)}{y}, \tag{16}$$

  which is therefore an auxiliary variable.
- Finally, capital gains in this model are given by

$$\mu = \frac{\dot{p}_e}{p_e} = \frac{\kappa - s_p\pi}{1-\kappa}\left[\frac{(1-\nu)}{\gamma_E} - \frac{\nu}{\gamma_D}\right]. \tag{17}$$

  and we can notice that μ is also an auxiliary variable.