WEAK INFEASIBILITY IN SEMIDEFINITE PROGRAMMING

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In this talk, we present an analysis of Semidefinite Feasibility Problems (SDFPs) with a focus on *weak infeasibility*. We denote by (K_n, L, c) the following semidefinite program:

$$\begin{aligned} &\min 0 \\ &\text{s.t. } x \in (L + c) \cap K_n, \end{aligned}$$

where L is a subspace of $\mathbb{S}_n$, the space of $n \times n$ symmetric matrices, $c \in \mathbb{S}_n$ and K_n is the cone of positive semidefinite matrices. Under these conditions, we say that (K_n, L, c) is *weakly infeasible* when $(L + c) \cap K_n = \emptyset$, but the distance between $L + c$ and K_n is 0. As an example, consider the following SDFP:

$$\begin{aligned} &\min 0 \\ &\text{s.t. } \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} \succeq 0. \end{aligned} \tag{1}$$

This problem is clearly infeasible, however as t goes to $+\infty$, the matrices get closer to K_2 without approaching any point in particular. Hence, this problem is weakly infeasible. In our research, we were interested in knowing whether this simple problem have all the properties shared by arbitrary weakly infeasible SDFPs or not. Unfortunately, it turns out that although there are indeed some shared properties, the answer is *no* and what is missing is a *hierarchical structure* that only appears when $n \geq 3$. This structure is an inclusive sequence of nested weakly infeasible subproblems analogous to (1) which ends with a weakly feasible subproblem.

In order to correctly captures this hierarchical structure, we show how to divide certain SDFPs into smaller subproblems in a way that the feasibility properties are mostly preserved. Applying this idea repeatedly, we arrive naturally at the concepts of *hyper feasible partitions* and *sub-hyper feasible directions*. In our talk, we will define them in a proper fashion and discuss some of their properties. In particular, we show that it is always possible to apply a congruence transformation to a weakly infeasible SDFP in a way that the transformed problem has a hyper feasible partition and a set of associated sub-hyper feasible directions.

It follows from our discussion, that if (K_n, L, c) is weakly infeasible, although the dimension of $L + c$ can be up to $\frac{n(n+1)}{2} - 1$, there exists an affine subspace $L' + c' \subseteq L + c$ of dimension at most $n - 1$ such that (K_n, L', c') is also weakly infeasible. Thus, we need relatively few directions to approach the positive semidefinite cone. We also study the situation where a single directions is enough (as in (1)) and present an example to show that this does not need to happen in general.

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