Extragradient methods for stochastic variational inequalities: convergence results and complexity analysis

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Abstract

We consider stochastic variational inequalities with monotone operators. The operator F defining the variational inequality depends both on a variable in the finite dimensional Euclidean space and on a random variable. We are interested in finding solutions for the deterministic variational inequality problem whose operator T is defined as the expected value of F, but we do not assume that T is explicitly available; rather we propose a Stochastic Approximation procedure, meaning that at each iteration, a step similar to some variant of the deterministic projection method is taken after sampling the random variable, choosing thus a specific realization of the operator. We consider three variants of the method. The first variant is a projection method with approximate projections, where the variational inequality satisfies an error bound on the solution set, called *weak sharpness*. We prove that the generated sequence is bounded and its distance to the solution set converges to zero almost surely. In particular, every cluster point of the sequence is, almost surely, a solution. For the case in which the feasible set is compact, we establish a convergence rate and an estimate on the number of iterations required so that any solution of an auxiliary linear program solves the variational inequality.

The second variant uses non-vanishing stepsizes and our convergence results require only pseudo-monotonicity of the operator. We provide convergence and complexity analysis, allowing for an unbounded feasible set, unbounded operator, non-uniform variance of the oracle and, also, we do not require any regularization procedures. In the stochastic approximation procedure, we employ an iterative variance reduction procedure, consisting of taking, in iteration k, not just the value of the operator at one one sample of the random variable, but the average of the operator values at $m(k)$ samples. We attain the optimal oracle complexity $O(1/\epsilon^2)$ (up to a logarithmic term) and achieve an accelerated rate $O(1/K)$ in terms of the mean (quadratic) natural residual and the D-gap function, where K is the number of iterations required for a given tolerance $\epsilon > 0$. The generated sequence also enjoys a new feature: the sequence is bounded in L^p if the stochastic error has finite p-moment. Explicit estimates for the convergence rate, the complexity and the p-moments are given depending on problem parameters and distance of the initial iterate to the solution set.

The third variant uses a linear search for determining the stepsizes. Our convergence results require just pseudo-monotonicity of the operator and assumes no knowledge of the Lipschitz constant L. We provide convergence and complexity analysis, allowing for an unbounded feasible set, unbounded operator, non-uniform variance of the oracle and we do not require any regularization. We also prove the generated sequence is bounded in L^p. Alongside the stochastic approximation procedure, we iteratively reduce the variance of the stochastic error. Our methods cope with stepsizes bounded away from zero and attain the near-optimal oracle complexity $O(\log_{1/\theta} L) \cdot \epsilon^{-2} \cdot [\ln(\epsilon^{-1})]^{1+b}$ and an accelerated rate $O(1/K)$ in terms of the mean (quadratic) natural residual and the mean D-gap function, where K is the number of iterations required for a given tolerance $\epsilon > 0$ for arbitrary $\theta \in (0, 1)$ and $b > 0$. Explicit estimates for the convergence rate, oracle complexity and the p-moments are given depending on problem parameters and the distance of initial iterates to the solution set.

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