XI International Symposium on Generalized Convexity and Monotonicity

List of abstracts

1 *Generalized vector variational-like inequalities and set-valued optimization*

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Abstract: In this talk, we shall present the concept of different kinds of invariant monotonicities and different kinds of invexities. We shall give several relations among these concepts. Under different kinds of invexities, we shall give some characterizations of solutions of a set-valued optimization. We shall also present generalized vector variational-like inequalities defined by means of normal subdifferential of a set-valued mapping. Some necessary and sufficient conditions for an efficient / weak efficient solution of a set-valued optimization problem shall be provided by using generalized vector variational-like inequalities.

2 *Globally convergent algorithms for finding zeros of duplomonotone mappings*

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Abstract: We introduce a new class of mappings, called duplomonotone, which is strictly broader than the class of monotone mappings. We study some of the main properties of duplomonotone functions and provide various examples, including nonlinear duplomonotone functions arising from the study of systems of biochemical reactions. Finally, we present three variations of a derivative-free line search algorithm for finding zeros of systems of duplomonotone equations, and we prove their linear convergence to a zero of the function.

3 *Impulse controls and proper extensions of optimal control problems*

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Abstract: For a control Cauchy problem

$$\dot{x} = f(t, x, u, v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad x(a) = \bar{x}, \tag{1}$$

on an interval $[a, b]$, we rely on the notion of *limit solution* x, that is defined for $\mathcal{L}^1$ impulsive inputs u and for standard, bounded measurable, controls v. Here $\mathcal{L}^1$ denotes the space of everywhere defined Lebesgue integrable functions. A limit solution corresponding to a control u in $\mathcal{L}^1$ is itself a (everywhere defined) function of $\mathcal{L}^1$ and, loosely speaking, it is the limit (in a suitable, non metric topology) of standard Carathéodory solutions associated to absolutely continuous controls approximating u.

We consider a problem in the Mayer form associated to (**??**) and containing final constraints of the type $(x(b), u(b)) \in \mathcal{S}$. We investigate the question whether the concept of limit solution provides a *proper extension* of the standard problem with absolutely controls u, i.e. (i) whether the subset of trajectories of the latter is dense (in a suitable topology) in the set of trajectories of the former, and (ii) whether the two infimum values coincide. In particular, we provide a sufficient condition, which we call *fast reachability,* guaranteeing that the $\mathcal{L}^1$ extension of the solution's concept is in fact proper.

4 *Convex analysis and optimization in Hadamard spaces*

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Abstract: The notion of convexity in geodesic metric spaces is a natural extension of the usual convexity in linear spaces. We will focus on Hadamard spaces, that is, complete geodesic spaces of nonpositive curvature, and present several recent results in convex analysis and optimization in this nonlinear setting. This theory has a direct application into computational phylogenetics, which will be described in detail.

5 *Conditions for zero duality gap in convex programming*

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Abstract: We introduce and study a new dual condition which characterizes zero duality gap in nonsmooth convex optimization. We prove that our condition is less restrictive than all existing constraint qualifications, including the closed epigraph condition. Our dual condition was inspired by, and is less restrictive than, the so-called Bertsekas' condition for monotropic programming problems. We give several corollaries of our result and special cases as applications. We pay special attention to the polyhedral and sublinear cases, and their implications in convex optimization.

6 *Optimality conditions involving generalized convexity and convex analysis*

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Abstract: The aim of this talk is to present some optimality conditions for unconstrained nonlinear problems, where the objective function is defined on a Banach space. Firstly, we use convex analysis tools to derive a necessary and sufficient condition on global optimality. Secondly, generalized convexity is used to reach a similar result. Local-global minimum property of the objective function is also examined. Lastly, an application from calculus of variations is shown.

7 *A unifying approach to solve a class of rank-three programs involving linear and quadratic functions*

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Abstract: The aim of this paper is to propose a solution algorithm for solving a class of low-rank programs involving linear and quadratic functions and having a polyhedral feasible region. In particular, the proposed solution method solves in an unifying approach some classes of rank-three d.c., multiplicative and fractional programs. The algorithm is based on the so called optimal level solutions method. Some optimality conditions are used to improve the performance of the proposed algorithm.

8 *Parallel methods for big data optimization*

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Abstract: We discuss a decomposition framework for the parallel optimization of the sum of a differentiable function and a (block) separable, nonsmooth, convex one. The latter term is usually employed to enforce structure in the solution, typically sparsity. This kind of problem is easily encountered in many fields of engineering, as diverse as compressed sensing, neuroelectromagnetic imaging, machine learning, data mining, and tensor factorization and completion, just to make a few examples. Our framework is very flexible and includes both fully parallel Jacobi schemes and Gauss-Seidel ones, as well as virtually all possibilities “in between” with only a subset of variables updated at each iteration. We also show that many instances of the general framework enjoy powerful identification properties that permit the development of hybrid active-set variants. Our theoretical convergence results improve on existing ones and numerical experiments show that the new method compares favourably to existing algorithms.

9 *Hidden convexity in nonconvex optimization*

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Abstract: The convex world offers the most desirable properties in optimization, and the lack of that provides an interesting challenge in mathematics. In this lecture we show various instances from mathematical programming to calculus of variations where convexity is present in one way or in another. Among the issues to be described lie: KKT optimality conditions; Dines theorem, the S-lemma in quadratic programming; strong duality; optimal value functions; local optimality implies global.

10 *Fixed point methods for a certain class of operators*

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Abstract: We introduce in this paper a new class of nonlinear operators, which contains, among others, the class of operators with semimonotone additive inverse and also the class of nonexpansive mappings. We study this class and discuss some of its properties. Then, we present iterative procedures for computing fixed points of operators in this class, which allow for inexact solutions of the subproblems and relative error criteria. We prove weak convergence of the generated sequences in the context of Hilbert spaces. Strong convergence is also discussed.

11 *Convexity and difference property*

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Abstract: We show that the class of all delta-convex selfmappings of $\mathbb{R}$ (differences of two convex functions) enjoys the difference property in the sense of N.G. de Bruijn. $\mathbb{Q}$-differentiability technique has been applied as a proof tool.

12 *On strongly Wright-convex functions of higher order*

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Abstract: The concept of strongly convex functions was introduced by B. T. Polyak (1966) and they have been investigated by several authors. Related to these investigations, we consider

strongly Wright-convex functions of higher order and we prove decomposition and characterization theorems for them. Our results are closely related to those presented by R. Ger and K. Nikodem (2011).

According to our definition, if c is a positive real number and n is a positive integer, a real valued function f defined on an open interval I is called strongly Wright-convex of order n with modulus c if it satisfies the inequality

$$\Delta_{h_1} \cdots \Delta_{h_{n+1}} f(x) \geq c\,(n+1)!\, h_1 \cdots h_{n+1}$$

for all $x \in I$, $h_1, \ldots, h_{n+1} > 0$ such that $x + h_1 + \cdots + h_{n+1} \in I$. We prove that a function $f : I \to \mathbb{R}$ is strongly Wright-convex of order n with modulus c if and only if it is of the form $f(x) = g(x) + p(x) + cx^{n+1}$, $(x \in I)$, where $g : I \to \mathbb{R}$ is a convex function of order n and $p : I \to \mathbb{R}$ is a polynomial function of degree n. Additionally, we present a characterization of Wright-convex functions of order n with modulus c via generalized derivatives.

13 *Evolutionary variational inequality formulation of time dependent generalized Nash equilibrium problem*

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Abstract: Generalized Nash equilibrium problems (GNEPs) are noncooperative games where the strategy of each player depend on the rival players strategies. This class of problem has gained popularity in recent times because of its capacity to model economical systems, routing problems in communication networks. More recently, Pang and Fukushima (2005) proposed some GNEP formulations of multi-leader-follower games. In order to study the GNEP and to have efficient computational process some reformulation of GNEP have been given in literature. But the best situation corresponds to the case when the GNEP can be reformulated as a variational inequality problem (VIP), thus inheriting all the corresponding theoretical and numerical machinery. In this work, we established a connection between time-dependent version of GNEP to a parametric VIP in particular an evolutionary variational inequality problem without assuming any differentiability and, above all, for quasiconvex loss functions, a very classical and natural hypothesis in mathematical economics.

14 *Second order asymptotic analysis*

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Abstract: Recently, the concepts of second order asymptotic directions and functions have been introduced and applied to global and vector optimization problems. In this talk, we establish some new properties for these two concepts. In particular, in case of a convex set, a complete characterization of the second order asymptotic cone is given. Also, formulas that permit the easy computation of the second order asymptotic function of a convex function are provided. It will be shown that the second order asymptotic function provides a finer description of the behavior of functions at infinity, than the first order asymptotic function. An application is also given for generalized convex functions.

15 *On a new Benson proper ϵ-subdifferential for vector-valued mappings*

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Abstract: In this talk, we define a new notion of Benson proper ϵ-subdifferential for vector-valued mappings through a recent concept of Benson ϵ-proper solution of a vector optimization problem. This approximate subdifferential inherits the main properties of the Benson ϵ-proper solutions. We study these properties and we show that this proper ϵ-subdifferential extends and improves some similar notions of the literature, due to the fact that it is suitable to deal with minimizing sequences.

16 *Concepts and techniques of optimization on the sphere*

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Abstract: In this paper some concepts and techniques of Mathematical Programming are extended in an intrinsic way from the Euclidean space to the sphere. In particular, the notion of convex functions, variational problem and monotone vector fields are extended to the sphere and several characterizations of these notions are shown. As an application of the convexity concept, necessary and sufficient optimality conditions for constrained convex optimization problems on the sphere are derived.

17 *Uniform convexity of paranormed generalizations of L^p spaces*

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Abstract: The results presented in the talk have been obtained jointly with Janusz Matkowski (Zielona Góra, Poland). Given a measure space (Ω, Σ, μ) and a bijective increasing function $\varphi : [0, \infty) \to [0, \infty)$ the formula $P_\varphi(x) = \varphi^{-1}(\int_\Omega \varphi \circ |x| d\mu)$ defines the L^p-like paranorm on the linear space S of μ-integrable simple functions $x : \Omega \to \mathbb{R}$. Main results give general conditions under which this space is uniformly convex. The classical Clarkson theorem on the uniform convexity of L^p-space is generalized. Under some specific assumptions imposed on φ we give not only theorems on the uniform convexity but also formulas of modulus of convexity.

Given a measure space (Ω, Σ, μ) and a bijective increasing function $\varphi : [0, \infty) \to [0, \infty)$ the formula $\mathbf{p}_\varphi(x) = \varphi^{-1}\left(\int_\Omega \varphi \circ |x| \, d\mu\right)$ defines the L^p-like paranorm on the linear space S of μ-integrable simple functions $x : \Omega \to \mathbb{R}$. Main results give general conditions under which this space is uniformly convex. The classical Clarkson theorem on the uniform convexity of L^p-space is generalized. Under some specific assumptions imposed on φ we give not only theorems on the uniform convexity but also formulas of modulus of convexity.

The main theorems presented in the talk reads as follows:

Theorem 1. *Let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing bijection such that $\mathbf{p}_\varphi$ is a paranorm in $S = S(\Omega, \Sigma, \mu)$. If φ is superquadratic, then the space $(S, \mathbf{p}_\varphi)$ is uniformly convex, and the function $\delta : \Delta \to (0, \infty)$, given by*

$$\delta(r, \varepsilon) = r - \varphi^{-1}\left(\varphi(r) - \varphi\left(\frac{\varepsilon}{2}\right)\right),$$

is its modulus of the convexity.

Theorem 2. *Let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing bijection such that $\mathbf{p}_\varphi$ is a paranorm in $S = S(\Omega, \Sigma, \mu)$. Assume that φ is strictly convex and the function $H : [0, \infty)^2 \to (0, \infty)$, given by*

$$H(r, s) = \varphi\left(\varphi^{-1}(r) + \varphi^{-1}(s)\right) + \varphi\left(\left|\varphi^{-1}(r) - \varphi^{-1}(s)\right|\right),$$

satisfies the condition

$$H\left(\sum_{i=1}^{k} a_i (r_i, s_i)\right) \le \sum_{i=1}^{k} a_i H(r_i, s_i)$$

for all $k \in \mathbb{N}$, $r_1, \ldots, r_k$, $s_1, \ldots, s_k \in [0, \infty)$, and $a_1, \ldots, a_k \in [0, \infty)$ such that $a_i = \mu(A_i)$, $i = 1, \ldots, k$, for some pairwise disjoint sets $A_1, \ldots, A_k \in \Sigma$. Then the space $(S, \mathbf{p}_\varphi)$ is uniformly convex and there is a modulus $\delta : \Delta \to [0, \infty)$ of convexity of it such that

$$\varphi\left(r - \delta(r, \varepsilon) + \frac{\varepsilon}{2}\right) + \varphi\left(\left|r - \delta(r, \varepsilon) - \frac{\varepsilon}{2}\right|\right) = 2\varphi(r)$$

for all $(r, \varepsilon) \in \Delta$.

In the proof of these theorems we use some ideas and results from (J. Matkowski, 1989) and (J. Matkowski, 2013).

In the talk we also deal with the special case of a finite set Ω. Then the considered space S is simply $\mathbb{R}^k$ with $k = \#\Omega$. We establish the uniform convexity of the spaces $\left(\mathbb{R}^k, \mathbf{p}_\varphi\right)$, generated by a strictly convex bijection φ of $[0, \infty)$.

Theorem 3. *Let $\varphi : [0, \infty) \to [0, \infty)$ be a strictly convex bijection such that $\mathbf{p}_\varphi$ is a paranorm in $\mathbb{R}^k$. Then the space $\left(\mathbb{R}^k, \mathbf{p}_\varphi\right)$ is uniformly convex.*

A contrario proof of this fact provides no information on a modulus of convexity of these spaces. In some cases it can be done, even an exact formula of a modulus can be proved. We show how to make it in details in the case when $S = \mathbb{R}^2$ and φ is given by $\varphi(t) = \mathrm{e}^t - 1$.

Theorem 4. *Let $\mathbf{p}_\varphi$ be the paranorm in $\mathbb{R}^2$ of the form*

$$\mathbf{p}_\varphi(x) = \varphi^{-1}\left(\varphi(|x_1|) + \varphi(|x_2|)\right),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is defined by $\varphi(t) = \mathrm{e}^t - 1$. Then the function $\delta_0 \colon \Delta_\varphi \to (0, \infty)$, given by

$$\delta_0(r, \varepsilon) = r - \varphi^{-1}\left(\varphi\left(\frac{x_{r,\varepsilon} + r}{2}\right) + \varphi\left(\frac{\varphi^{-1}\left(\varphi(r) - \varphi(x_{r,\varepsilon})\right)}{2}\right)\right),$$

where for every $(r, \varepsilon) \in \Delta_\varphi$ the number $x_{r,\varepsilon} \in [0, r]$ is the unique solution of the equation

$$\varphi(t) - \varphi(r - t) = \varphi(r) - \varphi(\varepsilon),$$

is strictly increasing with respect to second variable. Moreover, the function $\delta \colon \Delta \to (0, \infty)$, defined by $\delta(r, \varepsilon) = \delta_0\left(r, \frac{\varepsilon}{4}\right)$, is a modulus of convexity of the space $\left(\mathbb{R}^2, \mathbf{p}_\varphi\right)$.

18 *Convexity and a Stone-type theorem for convex sets in Abelian semigroup setting*

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Abstract: We introduce two parallel notions of convexity of sets in the Abelian semigroup setting.

Let $(S,+)$ denote an Abelian semigroup throughout the talk. Given a subset $A \subseteq S$ and an $n \in \mathbb{N}$, the sets nA, $n^{-1}A$, $[n]A$ are defined by

$$nA := \{nx \mid x \in A\}, \qquad n^{-1}A := \{x \mid nx \in A\}, \; [n]\,A := \{x_1 + \cdots + x_n \mid x_1, \ldots, x_n \in A\},$$

respectively. Obviously, the inclusions

$$n(n^{-1}A) \subseteq A \subseteq n^{-1}(nA), \qquad A \subseteq n^{-1}([n]A), \qquad nA \subseteq [n]A \tag{2}$$

hold for every $n \in \mathbb{N}$ and $A \subseteq S$.

For a fixed $n \in \mathbb{N}$, a subset $A \subseteq S$ is called *n-convex* and *n-coconvex*, if

$$n^{-1}([n]A) \subseteq A \qquad \text{and} \qquad [n]A \subseteq nA \tag{3}$$

hold, respectively. In fact, in view of the last two inclusions in (**??**), a set A is n-convex and n-coconvex if and only if, in the respective case, there is equality in (**??**).

Observe that if S is the additive group of a vector space over the field $\mathbb{Q}$, then the notions of n-convexity and n-coconvexity coincide and a subset of S is n-convex for all $n \in \mathbb{N}$ if and only if it is closed under rational convex combination, i.e., it is $\mathbb{Q}$-convex in the standard sense.

More generally, if the semigroup S is divisible by n, then n-convexity implies n-coconvexity. In the case when this n-divisibility of S is unique, both the notions of convexity coincide. In general, however, they are different whenever $n \geq 2$. Suitable examples will be presented.

Let $\mathbb{F}$ be a nonvoid subset of $\mathbb{N}$. A set $A \subseteq S$ is said to be $\mathbb{F}$-*convex* (resp. $\mathbb{F}$-*coconvex*) if it is n-convex (resp. n-coconvex) for all $n \in \mathbb{F}$. If A is $\mathbb{N}$-convex (resp. $\mathbb{N}$-coconvex), then it is called *convex* (resp. *coconvex*). Observe that the semigroup S is automatically convex, however, it may not be coconvex. On the other hand, for every $x \in S$, the singleton $\{x\}$ is coconvex and, in general, it is not convex.

We compare both the notions and examine their algebraic and set-theoretical properties. Suitable examples show that some of them fail. It turns out that the notions of n-convexity and n-coconvexity are, in a sense, complementary: a number of the properties is adhered only to one of the notions. In particular, the family of n-convex sets is closed under the intersection, but the family of n-coconvex sets is not in general. For that reason, for every subset A of the semigroup and for any nonempty set $\mathbb{F}$ of positive integers, we may consider only $\mathbb{F}$-convex hull $\operatorname{conv}_{\mathbb{F}}(A)$ of A. Given a multiplicative subsemigroup $\mathbb{F}$ of $\mathbb{N}$ we find the form of the set $\operatorname{conv}_{\mathbb{F}}(A)$. We describe also the $\mathbb{N}$-convex hull of the union of finitely many sets, in particular obtaining a kind of the drop theorem. The equivalence relation, determined by the partition of the semigroup into $\mathbb{F}$-convex hulls of singletons, is also studied.

Next we prove a a Stone-type theorem for the separation of $\mathbb{N}$-disjoint sets by complementary convex subsets of the semigroup.

Two subsets A, B of S are called $\mathbb{N}$-disjoint if

$$[n]\,A \cap [n]\,B = \emptyset$$

for all $n \in \mathbb{N}$. Obviously, the $\mathbb{N}$-disjointness of sets implies their disjointness. However, the converse may not be true.

For the separation of $\mathbb{N}$-disjoint sets we prove the following Stone-type theorem.

Theorem. Let A_0 and B_0 be $\mathbb{N}$-disjoint subsets of S. Then there exist $\mathbb{N}$-disjoint convex subsets A and B such that

$$A_0 \subseteq A, \qquad B_0 \subseteq B, \quad \text{and} \quad A \cup B = S.$$

If, in addition, S is coconvex, then A and B are also coconvex sets.

19 *Solutions of parametric complementarity problem being monotone with respect to parameters*

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Abstract: In many applications, it is important that the solutions to parametric complementarity problems be monotone with respect to the parameters involved. This paper establishes conditions of various types that guarantee the latter property. In the majority of them, the crucial demand is that the mappings forming the problem be monotone (in various senses) by the state variables and antitone with respect to the parameters.

20 *Generalized impulsive control as a result of impulsive convexification of the classical optimal control problem*

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Abstract: We introduce the concept of MP-pseudoinvexity for general nonlinear impul- sive optimal control problems whose dynamics are specified by measure driven control equations. This is a general paradigm in that, both the absolutely con- tinuous and singular components of the dynamics depend on both the state and the control variables. The following impulsive optimal control problem is considered:

$$\begin{cases} \text{Minimize} & \varphi(x(0), x(1)) \\ \text{subject to} & dx = f(x,u,t)dt + G(x,u,t)d\vartheta, \\ & (x(0), x(1)) \in S, \\ & u(t) \in \Omega, \quad t \in [0,1], \end{cases} \tag{IP}$$

where $S \subset \mathbb{R}^{2n}$ is a closed set, $\Omega \subset \mathbb{R}^m$, is a compact set, $\varphi : \mathbb{R}^{2n} \to \mathbb{R}$ is the cost functional and $f : \mathbb{R}^n \times \mathbb{R}^m \times [0,1] \to \mathbb{R}^n$, and $G : \mathbb{R}^n \times \mathbb{R}^m \times [0,1] \to \mathbb{R}^{n\times q}$ are functions specifying the conventional and the singular dynamics, respectively. The function $u(\cdot)$ with values in Ω is called conventional control and it is assumed to be measurable and essentially bounded with respect to both Lebesgue $\mathcal{L}$ (generated by interval length on the real line) and Lebesgue-Stieltjes $|\mu|$ measures. The impulsive control is denoted by ϑ.

The key necessary conditions for optimal control problems are the celebrated Pontryagin Maximum Principle. As regards sufficient conditions of optimality, there are those involving convexity assumptions, second order conditions and the verification function method via Hamilton-Jacobi-Bellman theory. To the best of our knowledge, the literature concerns primarily conventional systems, that is, systems for which the trajectories are absolutely continuous. Moreover, this is the main context in which generalized convexity assumptions, say invexity-like property, has been proposed in the context of optimal control.

The key result of this work consists in showing the sufficiency for optimality of the MP-pseudoinvexity for (IP). It is proved that, if this property holds, then every process satisfying the maximum principle is an optimal one. This result is obtained in the context of a proper solution concept that will be presented and discussed.

21 *Some tools of convex optimization in geodesic spaces*

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Abstract: Concepts and thecniques of convex optimization, traditionally used in the realm of linear spaces, have been extended to metric spaces with nonpositive curvature in connection with generalized harmonic mappings (J. Jost, 1995), gradient flows (I. Stojkovic, 2012) or minimization of convex fuctionals (C. Li, G. López and V. Martín-Márquez, 2009). We present some results of existence and approximation of fixed point on which most of the previously mentioned works could be built.

22 *Some glimpses on convex subdifferential calculus*

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Abstract: In this talk we present some glimpses on convex subdifferential calculus. In particular, we provide a general formula for the optimal set of the so-called relaxed minimization problem in terms of the approximate minima of the data function. More precisely, if X is a (real) Hausdorff locally convex space and h is an extended real valued function on X, the *relaxed problem* associated with

$$(\mathcal{P}) : \qquad \text{minimize } \ h(x) \ \text{ s.t. } \ x \in X$$

is defined as

$$(\mathcal{P}') : \qquad \text{minimize } \ h^{**}(x) \ \text{ s.t. } \ x \in X,$$

where h^{**} denotes the Legendre-Fenchel bi-conjugate of h: It is well-known that the optimal values of both problems coincide:

$$\inf_X h = \inf_X h^{**} \in \mathbb{R} \cup \{\pm\infty\}$$

Our main formula provides the optimal set of $(\mathcal{P}')$, i.e. $\arg\min h^{**}$, in terms of the approximate solutions of $(\mathcal{P})$, i.e. $\varepsilon-\arg\min h$. Other related results presented in this talk are specific formulas for the following objects:

1) The subdifferential of the conjugate of an extended-real-valued function, ∂h^{**}, in terms of the data function.

2) The subdifferential of the supremum of an arbitrary family of functions, $\partial(\sup_{t\in T} f_t)$, yielding well-known formulas (Volle, Brøndsted, etc.) as simple consequences.

3) Alternative approaches for deriving the subdifferential of the supremum function via the involvement of finite subfamilies.

23 *A fixed point method for solving linear variational relation problems*

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Abstract: We consider the following problem (called a variational relation problem): find $\bar{x} \in X$ such that $R(\bar{x}, y)$ holds for all $y \in T(\bar{x} \subseteq Y$, where X and Y are nonempty sets, T is a set-valued map from X to Y and $R(x, y)$ is a relation linking $x \in X$ and $y \in Y$. In a general form, R is given by a subset of the product space $X \times Y$ and $R(x, y)$ holds if and only if (x, y) belongs to that subset. Typical examples of variational relation problems are variationl inequalities or equilibrium problems when R is given by a system of inequalities. In this talk we apply a fixed point theorem of N. Mizoguchi and W. Takahashi (1989) and an error bound of C. Bergthaler and I. Singer (1992) to establish an existence result and construct an algorithm to solve the above problem when T and R are determined by systems of linear inequalities.

24 *Bregman metrics: new characterizations in terms of induced proximal distances*

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Abstract: This talk deals with the proximal distances. We say that a function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ is a proximal distance associated to an open convex nonempty set $C \subset \mathbb{R}^n$ if it satisfies, for all $y \in C$,

1. the function $d(\cdot, y)$ is lsc, proper, C^1 on C;
2. $\mathrm{dom}(d(\cdot, y) \subset \overline{C}$ and $\mathrm{dom}\partial_1 d(\cdot, y) = C$;
3. $d(\cdot, y)$ is level bounded on $\mathbb{R}^n$, i.e., $\lim_{\|u\|\to\infty} d(u, y) = +\infty$;
4. $d(y, y) = 0$.

The proximal distances are used in the exact *Interior Proximal Algorithm* (IPA), we briefly introduce it: let $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, closed and convex function, and consider the optimization problem

$$F_* = \inf_{x\in\overline{C}} F(x).$$

The IPA solves it by generating a sequence $\{x^k\}$ according to the following iterative scheme:

$$x^{k+1} = \arg\min_{x\in\overline{C}} \lambda_k F(x) + d(x, x^k), \;\; k = 0, 1, 2, \dots, \tag{4}$$

In order to study a such of algorithm, the authors Ausslender and Teboule (2006) introduced the induced proximal distance to d, which is a function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ satisfying $H(a, a) = 0$ and

$$\langle c - b, \nabla_1 d(b, a)\rangle \le H(c, a) - H(c, b).$$

When we can chose $H = d$ it is said that d is *self-induced.* During this work we have given a characterization of the Bregman metrics in terms of its gradients, and obtained that all the self-induced distances are Bregman metrics. After this, we have shown that under suitable assumptions, the induced proximal distance to a proximal distance is a Bregman metric. This result permits to obtain a convergence result of the proximal algorithm under minimal assumption on d.

25 *Right Bregman nonexpansive operators in Banach spaces*

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Abstract: Nonexpansive operators in Banach spaces are of utmost importance in Nonlinear Analysis and Optimization Theory. We are concern in this talk with classes of operators which are, in some sense, nonexpansive not with respect to the norm, but with respect to Bregman distances. Since these distances are not symmetric in general, it seems natural to distinguish between left and right Bregman nonexpansive operators. Some left classes have already been studied quite intensively, so this talk is devoted to right Bregman nonex- pansive operators. In particular, we characterize sunny right quasi-Bregman nonexpansive retractions, and as a consequence, we show that the fixed point set of any right quasi-Bregman nonexpansive operator is a sunny right quasi- Bregman nonexpansive retract of the ambient Banach space. It also turns out that the corresponding right Bregman projection is the unique sunny right quasi-Bregman nonexpansive retraction, while the left Bregman pro- jection is not sunny. Then we study the conjugate resolvent of a monotone mapping and provide a characterization of right Bregman firmly nonexpan- sive operators. To conclude, we present diverse examples of right Bregman firmly nonexpansive operators. This talk is based on a joint work with Prof. Simeon Reich and Shoham Sabach from Technion- Israel Institute of Technology, Haifa.

26 *Tilt stability in optimization and its applications*

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Abstract: The talk is devoted to the notion of tilt-stable minimizers in general optimization problems and its applications to various classes in mathematical programming. We present second-order characterizations of tilt stability via second-order growth conditions and generalized Hessians. Local monotonicity of set-valued operators plays a crucial role in deriving the obtained results.

27 *Jensen and Hermite-Hadamard inequalities for strongly convex set-values maps*

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Abstract: Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be real normed spaces, D be a convex subset of X and c be a positive constant. Denote by B the closed unit ball in Y and by $n(Y)$ the family of all nonempty subsets of Y. A set-valued map $F : D \to n(Y)$ is called *strongly convex with modulus* c if

$$tF(x_1) + (1-t)F(x_2) + ct(1-t)\|x_1 - x_2\|^2 B \ \subset\ F(tx_1 + (1-t)x_2),$$

for all $x_1, x_2 \in D$ and $t \in [0,1]$; F is *strongly midconvex with modulus* c if

$$\frac{1}{2}F(x_1) + \frac{1}{2}F(x_2) + \frac{c}{4}\|x_1 - x_2\|^2 B \ \subset\ F\Big(\frac{x_1 + x_2}{2}\Big),$$

for all $x_1, x_2 \in D$.

Some properties of strongly convex and strongly midconvex set-valued maps are presented. In particular, a Bernstein-Doetsch and Sierpiński-type theorems for strongly midconvex set-valued maps, as well as a Kuhn-type result are obtained. Counterparts of the classical integral and discrete Jensen inequalities for strongly convex set-valued maps are given. Finally the following Hermite-Hadamard-type theorem is presented:

Let I be an open interval and Y be a Banach space. If a set-valued map $F : I \to cl(Y)$ is strongly convex with modulus c, then

$$\frac{1}{b-a}\int_a^b F(x)dx + \frac{c}{12}(a-b)^2 B \subset F\left(\frac{a+b}{2}\right)$$

and

$$\frac{F(a)+F(b)}{2} + \frac{c}{6}(a-b)^2 B \subset \frac{1}{b-a}\int_a^b F(x)dx$$

for all $a, b \in I,\ a < b$.

28 *On sufficient optimality conditions for multiobjective control problems*

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Abstract: This work is devoted to present optimality conditions for the sufficiency of the maximum principle for multiobjective optimal control problems with nons- mooth data. An extremal (or stationary) control process satisfies first order optimality conditions, the maximum principle conditions. Naturally, an extremal process may not be an optimal one. In order to distinguish a minimizing process among the extremal ones, we can resort, for example, convex conditions. The sufficient conditions provided in this work go in this direction. They are based on a kind of generalized convexity notion, called invexity. It is well known that the Kuhn-Tucker necessary

conditions turn to be also sufficient under convexity assumptions. Invex functions was designed in 1981 by M.A. Hanson in order to get the sufficiency of the Kuhn-Tucker conditions for nonlinear programming problems. It is well known that a differentiable real-valued function is invex if, and only if, every stationary point is a global minimizer. However, though, as demonstrated by Hanson, every stationary point is a global minimizer for invex mathematical programming problems, the converse is not true. That is, there are non invex problems with the property that every stationary point is a global minimizer. Therefore invex problems are not the most general class of problems which possesses such a property. In 1985, D.H. Martin redesigned Hanson's definition of invexity, but main taining the sufficiency of the Kuhn-Tucker conditions. This generalized concept of invexity was termed KT-invexity. Moreover, Martin proved that if a mathematical programming problem is such that every stationary point is a global minimizer, necessarily it satisfies this weakened invexity condition. Summarizing, Martin showed that every stationary point is a global minimizer if, and only if, the problem is KT-invex. The concept of KT-invexity was brought to the optimal control context in 2009 in a previous work by the authors, where Kuhn-Tucker type optimality conditions were used, instead of the maximum principle. In a recent paper, the maximum principle was employed when we introduced MP-pseudoinvexity for optimal control problems. Similar results were obtained. In this paper, MP-pseudoinvexity is generalized to include nonsmooth multiobjective optimal control problems. It is showed that every extremal process is optimal if, and only if, the problem is MP-pseudoinvex

29 *On the joint range of a pair of inhomogeneous quadratic functions with applications*

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Abstract: The classical result of Dines, perhaps motivated by Finsler's work, stated that the joint range of two homogeneous quadratic functions (defined in $\mathbb{R}^n$) is always a convex cone. This convexity property plays a central role in the development of Farkas-type alternative theorems, known as s-lemmas in the context of quadratic functions. Unfortunately, this result is no longer true when the quadratic functions are inhomogeneous.

The goal of this talk is to describe the directions to be added to the pair of (inhomogeneous) quadratic functions for which convexity is obtained. We will see that it occurs for almost all directions. As a consequence, when a convex cone with nonempty interior is added, the set remains convex. A detailed study in the case $n = 2$ unveals the intimate relationship of this problem with the homogeneous one and with the properties of Non-Degenerate or Simultaneous Diagonalization. Applications to derive new versions of S-lemmas and strong duality results will also be established.

30 *On an extremal property of Wright convex functions*

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Abstract: Given an open convex subset $D \subseteq \mathbb{R}^n$, we consider the cones of convex, Wright-convex and Jensen-convex functions defined by

$$\begin{aligned} K(D) &:= \big\{ f : D \to \mathbb{R} \mid f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \\ &\qquad\qquad ((t, x, y) \in [0,1] \times D^2) \big\}, \\ W(D) &:= \big\{ f : D \to \mathbb{R} \mid f(tx + (1-t)y) + f((1-t)x + ty) \le f(x) + f(y) \\ &\qquad\qquad ((t, x, y) \in [0,1] \times D^2) \big\}, \\ J(D) &:= \big\{ f : D \to \mathbb{R} \mid f(\tfrac{1}{2}x + \tfrac{1}{2}y) \le \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y) \quad ((x, y) \in D^2) \big\}, \end{aligned}$$

respectively. It is obvious that $K(D) \subseteq W(D) \subseteq J(D)$. The main result presented is that $W(D)$ forms an extremal subcone of $J(D)$ in the following sense: if $f, g \in J(D)$ such that $f + g \in W(D)$, then $f, g \in W(D)$ holds. On the other hand $K(D)$ is non-extremal in $W(D)$ and in $J(D)$.

The proof uses Ng's decomposition theorem of Wright-convex functions and Rademacher's differentiablity theorem of locally Lipschitzian functions.

Among the many consequences, we mention that if $f, g \in J(D)$ such that $f + g \in W(D)$ and f is t-convex then g is also t-convex.

The analogous problem for higher-order convex, Wright-convex and Jensen-convex functions is also discussed.

31 *An inexact proximal point method for cohypomonotone operators with a practical relative error criterion*

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Abstract: Eckstein and Silva (2013) introduced a practical relative stopping criterion for the subproblems arising in the classical augmented Lagrangian method for convex optimization. In this work, we propose a variant of this method for cohypomonotone operators in product spaces. Our analysis is based on Pennanen's duality framework (Pennanen 2000) and the new algorithm is an inexact variant of the proximal point method for cohypomonotone operators (Iusem, Pennanen and Svaiter 2003). We analyze the existence of iterates and prove local convergence. As an application, we extend the convergence analysis of the exact version of the augmented Lagrangian method to the cohypomonotone case.

32 *Zero-convex functions, perturbation resilience, and subgradient projections for feasibility-seeking methods*

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Abstract: The convex feasibility problem (CFP) is at the core of the modeling of many problems in various areas of science. Subgradient projection methods are important tools for solving the CFP because they enable the use of subgradient calculations instead of orthogonal projections onto the individual sets of the problem. Working in a real Hilbert space, we show that the method is perturbation resilience. In other words, under appropriate conditions the sequence generated by the method converges weakly, and sometimes also strongly, to a point in the intersection of the given subsets of the feasibility problem, despite certain perturbations which are allowed in each iterative step. Unlike previous works on solving the convex feasibility problem, the involved functions, which induce the feasibility problem's subsets, need not be convex. Instead, we allow them to belong to a wider and richer class of functions satisfying a weaker condition, that we call zero-convexity. This class, which is introduced and discussed here, holds a promise to solve optimization problems in various areas, especially in non-smooth and non-convex optimization. The relevance of this study to approximate minimization and to the recent superiorization methodology for constrained optimization is explained.

33 *Optimality conditions for approximate solutions of unconstrained vector optimization problems*

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Abstract: In this talk we focus on approximate solutions defined through a coradi- ant set of nonsmooth nonconvex unconstrained vector optimization problems between Banach spaces. For this kind of solutions we obtain necessary condi- tions in terms of the coradiant set and the contingent derivative of the objective mapping of the problem. For deriving these conditions we combine a version of the Ekeland variational principle for vector optimization problems where the perturbed mapping is set-valued, and also some calculus rules for contingent derivatives.

34 *A modified projection algorithm for constrained equilibrium problems*

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Abstract: We consider Constrained Equilibrium Problem (CEP), which consists in finding a point x^* in the intersection of two nonempty, closed and convex subsets of a Hilbert space, C and D, such that x^* is a solution of an equilibrium problem on C. It generalizes the Constrained Variational Inequality Problem considered by Censor, Gibali and Reich (Optimization, 2012). In this work, an algorithm using projections is proposed to solve the CEP. Convergence properties of the algorithm are established under mild assumptions. Some numerical results and comparisons are reported.

35 *On the sufficiency of the maximum principle for state constrained optimal control problems*

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Abstract: The search to obtain sufficient conditions of optimality for feasible processes of nonlinear optimal control systems brakes down into two directions. One is via dynamic programming: a solution of the Hamilton-Jacobi partial differential equation associated to the control problem, under certain conditions, is proved to be the value function of the optimization problem. In many cases, the unique solution. This function is then used to verify that a feasible process in question is actually optimal for the control problem. The process to be evaluated can be found by any means, and it is not necessary to certify beforehand that it obeys the maximum principle. Another feature of this approach is that the value function is not the only function that can be used to verify that a certain process is optimal. There may be many verification functions, but the value function has the advantage that it can be computed by solving the Hamilton-Jacobi equation and this can lead to algorithms. The other approach to distinguish a minimizing process among candidates is via convexity of the Hamiltonian or second order conditions. The sufficient conditions provided in this work fall into this direction. An extremal control process is a process that satisfies the maximum principle of Pontryagin. Mainly, an extremal process may not be optimal. It is the case for control problems whose dynamics are linear and the costs are convex functionals. This sufficiency of the maximum principle for linear quadratic problems lead Pontryagin to think, at first, that the maximum principle conditions were sufficient for nonlinear problems. However, as expected, in general, the maxi- mum principle is only a set of necessary conditions for optimality. Since then, a question that remained open until recently was: what are the weakest assump- tions and conditions that should be imposed to the control problem data in order that the maximum principle becomes automatically a set of sufficient conditions of optimality? A complete answer to that question has been addressed for fixed time control problems without unilateral state constraints in a recent work by the authors. It was shown that, the so called class of MP-pseudoinvex optimal control problems, is the largest class of nonsmooth control problems (without state state constraints) in which every process that satisfies the maximum prin- ciple is optimal. Here we generalize this result for nonsmooth control problems with state constraints. It is showed that every extremal process is optimal if, and only if, the problem is MP-pseudoinvex.

36 *Nonsmooth multiobjective fractional bilevel programming problem under generalized d_I-invexity*

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Abstract: Bilevel programming problems are hierarchical optimization problems which combine decisions of two decision makers, namely the leader and the follower. Formally, the bilevel programming problem involves two optimization problems where the constraints region of the upper-level problem is implicitly determined by another second level problem called lower-level problem. Thus, the upper-level decision maker corresponds to the leader and the lower-level decision maker corresponds to the follower. The hierarchical process means that the leader makes a decision first and thereafter the follower chooses his/her strategy according to the leader's action. In this process, the leader can influence but cannot control the decision of the follower. The origin of the bilevel optimization problem is due to Von Stackelberg (1952) who introduced a Stackelberg competition model perfect. The first formulation for bilevel optimization problem appeared in 1976 is owing to Bracken and McGill. The designation bilevel and multilevel programming are first used by Candler and Norton (1977). For bibliographical survey on bilevel programming one can see Anandalingam and Friesz (1992), Vicente and Calamai (1994) and the monographs by Bard (1982) and Dempe (2002, 2003). In this paper, we consider a nonsmooth optimistic bilevel programming problem in which the objective function of the upper-level is nonlinear fractional vector-valued, the objective function of the lower-level is nonlinear vector-valued and the common feasible region is determined by inequality constraints. By using Karush-Kuhn-Tucker type conditions associated to the lower-level problem, we reformulate the bilevel programming problem into an equivalent nonsmooth nonlinear multiobjective fractional single-level programming problem with inequality constraints, under appropriate constraint qualification and generalized d_I-invexity assumptions. We study the obtained problem by considering each involved function is semidirectionally differentiable in its own direction instead of a same direction, and we establish necessary and sufficient optimality conditions for a feasible point to be (weakly) efficient under generalized d_I-invexity.

37 *Equilibrium problem on Hadamard manifolds*

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Abstract: In this work, we present a sufficient condition for the existence of a solution for an equilibrium problem on an Hadamard manifold and under suitable assumptions on the sectional curvature, we propose a framework for the convergence analysis of a proximal point algorithm to solve this equilibrium problem in finite time.

38 *Arrow-Debreu condition in the setting of generalized concavity*

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Abstract: In 1954, Arrow and Debreu introduce an assumption called in the literature as Arrow-Debreu condition, which is very important in Eco- nomic Theory. The concern of this paper is try to introduce char- acterizations of the Arrow-Debreu condition using some generalized concavity notions, as for example the pseudo-concavity introduced by Iusem-Kassay-Sosa in 2009. Of course, we compare other generalized concavity notions appear in the literature, as for example the geomet- ric pseudo-concavity introduced by Crouzeix in 2010. Finally, we in- troduce some new properties for the Arrow-Debreu functions (Arrow- Debreu functions are all functions satisfying the Arrow-Debreu condition), when assumptions of upper-closedness or quasi-concavity are considered.

39 *On exactly expanding Markowitz mean-variance portfolio selection to a third criterion*

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Abstract: Over sixty years ago, Markowitz introduced the mean-variance efficient frontier to finance. While mean-variance is still the predominant model in portfolio selection, it has endured many criticisms. One of its most persistent has been that it does not allow for additional criteria. The difficulty is that with additional criteria, the efficient frontier becomes a surface. With results on how to compute surfaces, a method is specified for exactly translating Markowitz's risk-return (bi-criterion) approach to tri-criteria when the extra objective is linear (such as for dividends, liquidity, social responsibility, etc.). The case of when the third criterion is quadratic is also discussed. With the geometry of the generalization of this paper playing a major role, many graphs are used to illustrate.

40 *Old and new results on enlargements of maximally monotone operators*

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Abstract: In this presentation we will review some existing results on enlargements of maximally monotone operators and will define a new enlargement and describe some of its properties.

41 *Elements of a quantitative theory of cognition*

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Abstract: Concepts like truth, belief, knowledge, perception, information, communication and description all enter in the cognitive process. Quantitative considerations start from the view that knowledge demands effort. Shannon theory addresses some of these aspects in a probabilistic setting. There you find notions like entropy and divergence and optimization principles based on maximization of entropy. Important results on statistical inference are based on so called Pythagorean theorems. Though Shannon theory and also results of statistical physics initiated by Jaynes are strictly probabilistic, certain aspects as the Pythagorean theorems have a strong resemblance to classical results in geometry. We shall free ourselves from the probabilistic setting and develop elements of a general theory of cognition. The approach will be based on philosophical considerations and involve two fictive persons, Nature and Observer. The interaction between the two is steared by an effort function which is an analogue in the abstract setting of the proper scoring rules of the statisticians. Convexity comes into the picture in order to derive existence results for situations of equilibrium.

42 *On vector variational inequality problems and nonsmooth vector optimization problems via higher order strong convexity*

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Abstract: This paper deals with the relations between vector variational inequality problems and nonsmooth vector optimization problems using the concept of efficient and strict minimizer of order m. Gordan alternative theorem is employed to identify the vector critical points, the strict minimizers of order m and the solutions of the weak vector variational inequality problems under generalized strong convexity assumptions. The results of the paper generalize and extend several known results from the literature to the nonsmooth case as well to a more general class of functions.

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Abstract: Let I be a closed interval, $f : I \to I$ and let f^n denotes the n-th iterate of f. A function $g : I \to I$ such that $g^k = f$ is said to be an *iterative root of* f *of order* $k \geq 2$.

A function $f : I \to I$ is said to be *iteratively convex* if f possesses convex iterative roots of all orders.

A family of functions $\{f^t : I \to I\}$ is said to be a *convex semi-flow of* f if

$$f^t \circ f^s = f^{t+s}, \quad t, s \geq 0,$$

$f^1 = f$ and all f^t are convex .

If f is of class C^1 and $f' > 0$, then f is iteratively convex if and only if f possesses a convex semi-flow. In such a case convex iterative roots are uniquely determined .

We accept the analogous definitions for the concave functions.

The problem of description of iterative convexity (concavity) can be reduced to the functions satisfying the condition

(H) $f : [0,1] \to [0,1]$ is strictly increasing of class C^1 and $0 < f(x) < x$ for $x \in (0,1)$.

Further, let f satisfy (H) and $f'(0) \neq 0$. The problem of giving the direct criterion of the iterative convexity is still open, but for concave functions we have the following result. If f and f' are concave then f is iteratively concave.

For convex functions we have the following propositions.

If f of class C^3 is convex and $f''(0) \neq 0$ then f is iteratively convex in a neighbourhood of 0. Moreover, there exists an iteratively convex function $\widetilde{f}$ on $[0,1]$ which coincides with f in a neighbourhood of the origin.

If a convex function f is iteratively convex in $[0,\delta]$ for a $\delta > 0$, then there exist iteratively convex functions g and h in $[0,1]$ such that

$$g \leq f \leq h, \qquad g|_{[0,\delta]} = f|_{[0,\delta]} = h|_{[0,\delta]}$$

and

$$g^t \leq f^t \leq h^t, \quad t > 0,$$

where $\{f^t, t \geq 0\}$ is the C^1 semi-flow of f, however $\{g^t, t \geq 0\}$, $\{h^t, t \geq 0\}$ are convex semi-flows of g and h in I, respectively.

Let $0 < \lambda_0 < 1 < \lambda_1$ and denote by $Conv(\lambda_0, \lambda_1)$ the set of all convex functions f satisfying (H) such that $f(1) = 1$, $f'(0) = \lambda_0$ and $f'(1) = \lambda_1$.

The set of all iteratively convex functions is of the first category in the space $Conv(\lambda_0, \lambda_1)$ endowed with the classical metric in $C^2[0,1]$ space.

If $f \in Conv(\lambda_0, \lambda_1)$ is iteratively convex then for every $p \in (0,1)$ there exists a neighbourhood U of p such that every convex function F for which

$$F|_{[0,1]\setminus U} = f|_{[0,1]\setminus U} \quad \text{and} \quad F|_U > f|_U$$

is not iteratively convex.

We consider also the problem of the iterative convexity of functions satisfying (H) which are affine in neighbourhoods of 0 and 1.