Lectures on Mean Curvature flow

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Preface

This is a first (incomplete) draft on a series of 4 lectures at the XIX School of Differential Geometry 2016 at IMPA (Brazil). The notes are mostly based on the exposition in the references [8, 28].

These notes are by no means a comprehensive survey on the subject. In fact there are many relevant references and results that are not included here.

Lecture 1

Basic definitions and short time existence

1.1 Introduction

Definition 1.1.1 (Wikipedia)**.** *In mathematics, specifically differential geometry, a geometric flow is the gradient flow associated to a functional on a manifold which has a geometric interpretation, usually associated with some extrinsic or intrinsic curvature. They can be interpreted as flows on a moduli space (for intrinsic flows) or a parameter space (for extrinsic flows).*

These are of fundamental interest in the calculus of variations, and include several famous problems and theories. Particularly interesting are their critical points.

A geometric flow is also called a geometric evolution equation.

Examples.

- *Consider an immersion $F_0 : M^n \to \mathbb{R}^{n+1}$ and the functional*

$$\mathit{Area}\ (F_0) = \int_M d\mu,$$

where μ is the canonical measure associated to the metric g induced by the immersion. If we consider a smooth variation F_t of F_0, the first variation of area is given by

$$\frac{d\mathit{Area}\ (F_t)}{dt}\bigg|_{t=0} = -\int_M H\langle X, \nu\rangle d\mu,$$

where H is the mean curvature, ν the outward unit normal and $X = \frac{\partial F_t}{\partial t}\big|_{t=0}$.

Hence, the gradient flow of the area functional is given by solutions to the equation

$$\frac{\partial F_t}{\partial t} = -H\nu.$$

This is the steepest descent direction in the L^2-sense and it is known as Mean Curvature Flow.

- *Consider a Riemannian manifold (M, g) and the energy functional*

$$E(g) = \int_M R d\mu,$$

where R is the scalar curvature of M and μ the measure induced by g.

Consider a variation $g(t)$ of the metric g (i.e. $g(0) = g$ and $\frac{\partial g}{\partial t}\Big|_{t=0} = X$), then

$$\frac{dE\,(g(t))}{dt}\Big|_{t=0} = \int_M (-Ric^{ij} + \frac{1}{2} Rg^{ij}) X_{ij}, d\mu.$$

Then, the gradient flow associated to E is given by

$$\frac{\partial g}{\partial t} = -2Ric(g) + Rg.$$

If a geometric flow depends on intrinsic geometric quantities (such as scalar curvature) the flow is known as an *intrinsic geometric flow*, while if the geometric quantities are extrinsic (i.e. depend on the ambient geometry, such as the mean curvature) the geometric flow is known as extrinsic.

Notice that the first example above, can be written as follows:

$$\frac{\partial F}{\partial t} = \Delta_{g(t)} F,$$

where $\Delta_{g(t)}$ is the Laplace Beltrami operator on the immersion of M induced by F_t. This equation is a quasilinear parabolic system of equations that has some nice analytic properties.

On the other hand, the second example above does not have nice analytic properties as a PDE. This contrasts with the evolution equation

$$\frac{\partial g_t}{\partial t} = -2\text{Ric }(g), \tag{1.1.1}$$

which also has the structure of a quasilinear parabolic system of PDE's , but it is not clearly a gradient flow.

Remark 1.1.2. *Equation* (1.1.1) *is known as the Ricci flow and it was used by Perelman to prove the geometrization conjecture.*

Definition 1.1.3. *We will understand as an extrinsic (resp. intrinsic) geometric flow a family of immersions of a submanifold $N^n \subset M^{n+1}$ (resp. a family of metrics) indexed by a parameter t such that the immersion (resp. the metric) is induced as family of solutions to a partial differential equation that involves extrinsic (resp. intrinsic) geometric quantities.*

Examples. *Consider $F(t) : N^n \to M^{n+1}$ and $g(t)$ a metric on a manifold M*

- *Mean Curvature Flow*
$$\frac{\partial F_t}{\partial t} = -H\nu,$$
where H is the mean curvature of N in M and ν the outward pointing unit normal.

- *Inverse mean curvature flow*
$$\frac{\partial F_t}{\partial t} = \frac{1}{H}\nu,$$
where H is the mean curvature of N in M and ν the outward pointing unit normal.

- *Willmore Flow*
$$\frac{\partial F_t}{\partial t} = -\nabla W(F(t)),$$
where $W = \int_M (H^2 - K)dA$ and K is the Gaussian curvature.

- *Gauss flow*
$$\frac{\partial F_t}{\partial t} = -G\nu,$$
where G denotes the Gaussian curvature of N in M and ν the outward pointing unit normal.

- *Yamabe flow*
$$\frac{\partial g}{\partial t} = R,$$
where R is the scalar curvature

- *Ricci flow*
$$\frac{\partial g}{\partial t} = -2Ric(g),$$
where Ric is the Ricci curvature.

Remark 1.1.4. *The extrinsic flows above can be defined for co-dimension bigger than 1. However, from an analytic point of view the study of such flows becomes far more complicated and in this series of lectures we will assume that the co-dimension is 1.*

We remark that geometric flows are often degenerate parabolic equations. This is not a surprising feature, due to geometric invariances, such as the invariance of geometric quantities under reparametrizations.

1.2 Mean Curvature flow

We start by recalling definitions that we will use throughout the lectures. We note that everything could be defined more generally for a Riemannian manifold (M^{n+1}, g) and a smooth immersion of an k-dimensional submanifold $F : N^k \to M^{n+1}$ for $k \leq n$. However, in our context we will always assume that $M = \mathbb{R}^{n+1}$ and that N is n-dimensional.

Let $F : N^n \to \mathbb{R}^{n+1}$ be a smooth immersion and ν a choice of a unit normal vector field that is smooth. We will denote spatial derivatives as ∂_i (as the derivative respect to the $i-$th parameter). The metric induced by the immersion F will written as g_{ij} and g^{ij} its inverse. We denote the Levi Civita connection associated to g by ∇^M and more precisely, for $h : M \to \mathbb{R}$ the tangential gradient is given by

$$\nabla^M h = g^{ij} \partial_j h \partial_i F.$$

For a smooth tangent vector field $X = X^i \partial_i F = g^{ij} X_j \partial_i F$ the covariant derivative tensor can be computed as

$$\nabla_i^m X^j = \partial_i X^j + \Gamma_{ik}^j X^k,$$

where the Christoffel symbols Γ_{ij}^k are defined by

$$(\partial_i \partial_j F)^T = \Gamma_{ij}^k \partial_k F.$$

The Laplace Beltrami operator can be computed by

$$\Delta_M h = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j h).$$

We define the second fundamental form as

$$\begin{aligned} h_{ij} =& - \nu \cdot \frac{\partial F}{\partial \omega_i \partial \omega_j}, \\ A =& (h_{ij})_{ij}. \end{aligned}$$

The mean curvature can be computed as the trace of the second fundamental form:

$$H = g^{ij} h_{ij}.$$

Finally, we denote the norm of the second fundamental form by

$$|A|^2 = g^{ij} g^{kl} h_{ik} h_{jl}.$$

We say that a hypersurface $F(t)$ evolves under mean curvature flow if

$$\frac{\partial F_t}{\partial t} = -H\nu, \qquad \text{(MCF)}$$

where ν is a smooth choice of unit normal vector field, that points outward if $F(\omega, t)$ is convex.

1.2.1 Examples

- Consider a sphere of radius R_0. Let us assume that the solution is a sphere for every time t, that is $F(t) = R(t)\omega$ with $\omega \in \mathbb{S}^n$. Then the mean curvature is given by $H = \frac{n}{R}$ and the equation is equivalent to

  $$R' = \frac{n}{R}.$$

  Hence, the solution to the equation is

  $$R(t) = \sqrt{R_0 - 2nt}.$$

  Notice that the solutions are well defined only for finite time.

  Exercise: Repeat the computation for cylinders.

- Minimal surfaces are solutions to the equation. All these solutions are non compact.

- Consider the curve $y = -\log\cos x$, then the curve $t(0,1) + (x, y(x))$ is a translating solution to (MCF). This solutions is known as the Grim Reaper and it is defined for every $t \in (-\infty, \infty)$.

1.2.2 Short time existence

First notice that when we consider a time dependent parametrization $\varphi(\omega, t)$ of a solution to (MCF) we have that $\tilde{F}(\omega, t) = F(\varphi(\omega, t), t)$ satisfies

$$\frac{\partial \tilde{F}}{\partial t} = dF\left(\frac{\partial \varphi_i}{\partial t}, t\right) + \frac{\partial F}{\partial t}(\varphi(\omega, t), t) = dF\left(\frac{\partial \varphi}{\partial t}\right) - H\nu.$$

Notice that $\tilde{F}$ is not longer a solution to (MCF) (although is geometrically the same object). However, we have that the following converse result holds:

Theorem 1.2.1. *Assume that a family of immersions satisfies*

$$\frac{\partial F}{\partial t} = - H\nu + X, \tag{1.2.1}$$
$$F(\omega, 0) = F_0, \tag{1.2.2}$$

where X is a smooth (possibly time dependent) tangent vector. Then there is a family of reparametrizations of F that satisfies (MCF)*.*

Proof. Consider $\tilde{F} = F(\varphi(\omega, t), t)$. As before we have that

$$\frac{\partial \tilde{F}}{\partial t} = dF\left(\frac{\partial \varphi}{\partial t}\right) - H\nu + X.$$

By standard ODE theory, we can choose φ that solves

$$\begin{aligned}\frac{\partial\varphi}{\partial t} =& -(dF)^{-1}(X(\varphi,t)),\\ \varphi(\cdot,0) =& Id.\end{aligned}$$

Note that the choice is possible since $X \in TM$. Then we have that $\tilde{F}$ is a solution to (MCF).

□

The previous theorem implies that an equivalent (and more geometric) equation to (MCF) is

$$\frac{\partial F_t}{\partial t}\cdot\nu = -H. \tag{MCF'}$$

Theorem 1.2.2 (Short time existence)**.** *Let M_0 be a compact, smooth hypersurface given by an immersion F_0. Then, there exists a solution to* (MCF') *with initial condition F_0.*

Proof. We will follow the proof in [28] (that follows [25]). We look for a solution that can be written as a graph over the initial condition:

$$F(\omega,t) = F_0(\omega) + f(\omega,t)\nu_0.$$

As before, we denote by $\partial_i F$ the derivative respect to the i-th space parameter The metric on $F(t)$ is given by

$$\begin{aligned}g_{ij}(t) =& \partial_i F(t)\cdot\partial_j F(t)\\ =& (\partial_i F(0) + \partial_i f\nu_0 + f\partial_i\nu_0)\cdot(\partial_j F(0) + \partial_j f\nu_0 + f\partial_j\nu_0)\\ =& g_{ij}(0) - 2fh_{ij} + \partial_i f\partial_j f + f^2\partial_i\nu_0\cdot\partial_j\nu_0.\end{aligned}$$

A simple computation shows that

$$\frac{\partial\nu_0}{\partial\omega_i} = g^{ik}h_{kj}\frac{\partial F(0)}{\partial\omega_j}.$$

Then

$$g_{ij}(t) = g_{ij}(0) - 2fh_{ij} + \partial_i f\partial_j f + f^2 h_{ik}g^{kl}h_{lj}.$$

The tangent space to $F(t)$ is spanned by

$$\frac{\partial F(0)}{\partial\omega_i} + \frac{\partial f}{\partial\omega_i}\nu_0 + f^2 g^{ik}h_{kj}\frac{\partial F(0)}{\partial\omega_j}.$$

Them, the normal vector can be computed as

$$\begin{aligned}\nu(t) =& \frac{\nu_0 - \nu_0\cdot\partial_i F g^{ij}\partial_j F}{|\nu_0 - \nu_0\cdot\partial_i F g^{ij}\partial_j F|}\\ =& \frac{\nu_0 - \partial_i f g^{ij}\partial_j F}{|\nu_0 - \partial_i f g^{ij}\partial_j F|}.\end{aligned}$$

Notice that $f(\omega,0)=\partial_i f(\omega,0)=0$, then if f and its derivatives are continuous in time, then for t small enough, we would have that the norm of $\nu_0-\partial_i f g^{ij}\partial_j F$ is close to 1.

The second fundamental form is given by

$$\begin{aligned}h_{ij}=&-\nu(t)\cdot\partial^2_{ij}F(t)\\=&\partial^2_{ij}f\nu_0\cdot\nu(t)+P_{ij}(F_0,\nabla F_0,D^2F_0,f,\nabla f),\end{aligned}$$

and the mean curvature.

$$\begin{aligned}H=&g^{ij}h_{ij}\\=&g^{ij}\frac{\partial^2 f}{\partial\omega_i\partial\omega_j}\nu_0\cdot\nu(t)+g^{ij}P_{ij}\\=&\nu_0\cdot\nu(t)\Delta_{g(t)}f+Q(\omega,f,\nabla f).\end{aligned}$$

Then (MCF') is equivalent to

$$\begin{aligned}\frac{\partial f}{\partial t}\nu_0\cdot\nu(t)=&\frac{\partial F}{\partial t}\cdot\nu\\=&-H\\=&\nu_0\cdot\nu(t)\Delta_{g(t)}f+Q(\omega,f,\nabla f).\end{aligned}$$

Dividing by $\nu_0\cdot\nu(t)$we have

$$\frac{\partial f}{\partial t}=\Delta_{g(t)}f+\tilde{Q}(\omega,f,\nabla f).$$

We have reduced the system (MCF) to a quasilinear parabolic equation that can be studied using a standard procedure: Study the linearization, obtain appropriate a priori estimates and conclude the result using the implicit function theorem. For details, we refer the reader to [25].

□

Note that the previous theorem does not imply uniqueness, since there was (for instance) a choice of parametrization involved.

Notice also that necessarily the surface is not of class $C^{2,\alpha}$ at the maximal existence time T, since if that was the case, the flow could be started again and extended for a longer time, contradicting the maximality.

Lecture 2

Comparison principle and consequences

The following lemma will be useful throughout this chapter.

Lemma 2.0.1 (Hamilton's trick [22, 28])**.** *Let* $u : M \times (0,T) \to \mathbb{R}$ *be a* C^1 *function such that for every time* t*, there exists a value* $\delta > 0$ *and a compact subset* $K \subset M \setminus \partial M$ *such that at every time* $t' \in (t-\delta, t+\delta)$ *the maximum* $u_{\max}(t') = \max_{\omega \in M} u(\omega, t')$ *is attained at least at one point of* K*.*

Then $u_{\max}$ *is a locally Lipschitz function in* $(0,T)$ *and at every differentiability time* $t \in (0,T)$ *we have*

$$\frac{d}{dt} u_{\max}(t) = \frac{\partial u(\omega, t)}{\partial t},$$

where $\omega \in M \setminus \partial M$ *is any interior point where* $u(\cdot, t)$ *gets its maximum.*

Proof. We first show that $u_{\max}$ is locally Lipschitz. Since u is C^1 we have there is a constant C such that

$$u_{\max}(t+\varepsilon) = u(\omega, t+\varepsilon) \leq u(\omega, t) + C\varepsilon \leq u_{\max}(t) + C\varepsilon.$$

Analogously, we have the converse

$$u_{\max}(t) = u(\omega, t) \leq u(\omega, t+\varepsilon) + C\varepsilon \leq u_{\max}(t+\varepsilon) + C\varepsilon.$$

Hence, $u_{\max}$ is Lipschitz and in consequence differentiable almost everywhere.

Note that for any $\omega \in M \setminus \partial M$

$$u(\omega, t+\varepsilon) = u(\omega, t) + \frac{\partial u(\omega, \xi)}{\partial t}\varepsilon,$$

where $\xi \in (t, t+\varepsilon)$. Then

$$u_{\max}(t+\varepsilon) \geq u(\omega, t) + \frac{\partial u(\omega, \xi)}{\partial t}\varepsilon,$$

Suppose that t is a differentiable point for $u_{\max}(t)$. Choosing ω such that $u_{\max}(t) = u(\omega, t)$, we have if t is a then taking $\varepsilon \to 0$ we have

$$\frac{d}{dt} u_{\max}(t) \geq \frac{\partial u(\omega, t)}{\partial t}$$

The proof of the converse is similar a we leave it as an exercise.

□

2.1 Comparison principle

We start by discussing the uniqueness of solutions, which follows from

Theorem 2.1.1 (Comparison principle, version in [28]). *Let $F_1 : M_1 \times [0, T) \to \mathbb{R}^{n+1}$ and $F_2 : M_2 \times [0, T) \to \mathbb{R}^{n+1}$ two evolutions under* (MCF) *such that M_1 is compact. Then the distance between $F_1(M_1, t)$ and $F_2(M_2, t)$ is increasing.*

Proof. Let $M_t^i = F_i(M_i, t)$ and $d(t) = \text{dist}\ (M_t^1, M_t^2)$. Since M_1 is compact, we have that the distance is attained at points $p_1(t) \in M_t^1$ and $p_2(t) \in M_t^2$.

Note that due to the minimality the tangent planes at p_1 and p_2 are parallel (exercise). Then we can write locally (close to $p_i(t)$) the surfaces as graphs over a common plane. Let $f_i(x, t)$ be the graph function that describes M_t^i near $p_i(t)$. Without lost of generality, we may assume that $F_i(p_i(t), t) = (0, f_i(0, t))$, $f_1(0, t) < f_2(0, t)$ and $M_t^i \cap B_\rho(p_i(t)) = (x, f_i(x, t))$.

A simple computation (exercise) shows that

$$\frac{\partial f_i}{\partial t} = \Delta f_i - \frac{D^2 f_i(\nabla f_i, \nabla f_i)}{1 + |\nabla f_i|^2}.$$

By construction, the function $f_2 - f_1$ has a minimum at $x = 0$. Hence $\nabla f_1(0, t) = \nabla f_2(0, t)$ and $D^2(f_2 - f_1)$ is positive definite. On the other hand, by Lemma 2.0.1 we have that $d(t) = \min_{q_i \in M_t^i} d(q_1, q_2) = |f_1(0, t) - f_2(0, t)|$ satisfies

$$\begin{aligned}
\frac{d}{dt} d(t) =& \frac{f_1(0,t) - f_2(0,t)}{|f_1(0,t) - f_2(0,t)|} \frac{d(f_1 - f_2)}{dt}(0, t) \\
=& \frac{f_1(0,t) - f_2(0,t)}{|f_1(0,t) - f_2(0,t)|} \left(\Delta f_1 - \frac{D^2 f_1(\nabla f_1, \nabla f_i)}{1 + |\nabla f_1|^2} - \Delta f_2 + \frac{D^2 f_2(\nabla f_2, \nabla f_2)}{1 + |\nabla f_2|^2} \right) \\
=& \frac{f_2(0,t) - f_1(0,t)}{|f_1(0,t) - f_2(0,t)|} \left(e_i e_j - \frac{\partial_i f_1 \partial_j f_1}{1 + |\nabla f_1|^2} \right) D^2_{ij}(f_2 - f_1) > 0.
\end{aligned}$$

□

Corollary 2.1.2. *Given a smooth initial condition F_0, there is a unique smooth solution to* (MCF).

Corollary 2.1.3. *Given a smooth compact initial condition F_0 the solution to* (MCF) *with initial condition F_0 develops singularities in finite time.*

Corollary 2.1.4. *Given a smooth compact initial condition F_0 that is embedded, the solution to* (MCF) *with initial condition F_0 is embedded.*

Remark 2.1.5. *The previous results do not hold for co-dimension bigger than 1.*

2.2 Maximum principle and consequences

We start by proving the following general theorem.

Theorem 2.2.1 (Maximum principle, version in [28])**.** *Assume that for $t \in [0, T)$ we have a family of Riemannian metrics $g(t)$ on a manifold (possibly with boundary) such that the dependence on t is smooth. Let u be a function that satisfies*

$$\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u + \langle X(\omega, u, \nabla u, t), \nabla u \rangle_{g(t)} + b(u).$$

Assume that the maximum of $u(\cdot, t)$ is attained in a fixed compact set $K \subset M \setminus \partial M$ for every $t \in [0, T)$. Then

$$\frac{\partial u_{max}}{\partial t} \leq b(u_{max}) \qquad a.e.$$

where $u_{max}(t) = \max_M u(\cdot, t)$.

Remark 2.2.2. *Note that $u_{max}(t)$ is a locally Lipschitz function and hence differentiable almost everywhere.*

Note that if M_t are closed hypersurfaces we can take $K = M_t$. In the case that M_t is not compact, the condition on the set K can be replaced by growth assumptions at infinity ([8, 10]).

Proof. It follows directly from Lemma 2.0.1 and by observing that at a maximum $\Delta_{g(t)} u \leq 0$. □

Corollary 2.2.3. *Let u be as in Theorem 2.2.1. If $h(t)$ satisfies*

$$\begin{aligned} \frac{\partial h}{\partial t} &= b(h) \\ h(0) &= u_{max}(0), \end{aligned}$$

then $u(\cdot, t) \leq h(t)$.

2.3 Geometric estimates

A direct consequence of the previous theorem are short time estimates on the geometric quantities.

We first observe that the following computations hold.

Lemma 2.3.1. *Assume that* $F : M \times [0,T) \to \mathbb{R}^{n+1}$ *satisfies* (MCF). *Then it holds*

1. $\frac{\partial g_{ij}}{\partial t} = -2Hh_{ij}$.
2. $\frac{\partial h_{ij}}{\partial t} = \nabla_i^{M_t}\nabla_j^{M_t} H - Hh_{ik}g^{kl}h_{lj}$.
3. $\frac{\partial H}{\partial t} = \Delta_{g(t)} H + H|A|^2$.
4. $\frac{\partial |A|^2}{\partial t} = \Delta_{g(t)}|A|^2 + 2|A|^4 - 2|\nabla^{M_t} A|^2$.
5. $\frac{\partial \nu}{\partial t} = \nabla^{M_t} H = \Delta_{g(t)}\nu + |A|^2\nu$.
6. $\frac{\partial |\nabla^m A|^2}{\partial t} \leq \Delta_{g(t)}|\nabla^m A|^2 - 2|\nabla^{m+1}A|^4 + C_m(1 + |\nabla^m A|^2)$, *where* C_m *depends on bounds for* $|A|^2, \dots, |\nabla^{m-1}A|^2$.

Proof. We will only compute the equation for h_{ij} and we leave the rest to the reader. We follow the computation in [8] Recall that

$$h_{ij} = -\nu \cdot \partial_i\partial_j F,$$

then

$$\begin{aligned}\frac{\partial h_{ij}}{\partial t} =& - \frac{\partial \nu}{\partial t} \cdot \partial_i\partial_j F - \nu \cdot \partial_i\partial_j \left(\frac{\partial F}{\partial t}\right) \\ =& \nabla^{M_t} H \cdot \partial_i\partial_j F + \nu \cdot \partial_i\partial_j (H\nu)\end{aligned}$$

Assume for now that we are at point where we have chosen normal coordinates. Then we have

$$(\partial_i\partial_j F)^T = 0 \text{ and } g_{ij} = Id.$$

Hence, at this point

$$\frac{\partial h_{ij}}{\partial t} = \partial_i\partial_j H + H\nu \cdot \partial_i\partial_j \nu.$$

In normal coordinates we have that

$$\partial_i\partial_j H = \nabla_i^{M_t}\nabla_j^{M_t} H$$

Additionally

$$\begin{aligned}\nu \cdot \partial_i\partial_j\nu =& - \partial_i\nu \cdot \partial_j\nu \\ =& - g^{kl}h_{li}h_{kj}.\end{aligned}$$

Then we have

$$\frac{\partial h_{ij}}{\partial t} = \nabla_i^{M_t}\nabla_j^{M_t} H - g^{kl}h_{li}h_{kj}.$$

Exercise: Derive the equation for H from here.

Now we use Simmons' identity to conclude

$$\Delta_{g(t)} h_{ij} = \nabla_i^{M_t} \nabla_j^{M_t} H + H g^{kl} h_{ik} h_{lj} - |A|^2 h_{ij}.$$

Note that this implies that the desired equality holds if the coordinates at the point are normal. However, since the quantities involved are invariant under changes of coordinates, we have the desired result. □

Now we have the following estimates as a consequence of the maximum principle and the previous evolution equations

Proposition 2.3.2. *If the initial, compact hypersurface M_0 satisfies $|A| \leq \alpha H$ for some fixed constant α, then the solution M_t, $t \in I$ to* (MCF) *with initial condition M_0 satisfies $|A| \leq \alpha H$ for every $t \in I$.*

Proof. We compute the evolution equation for the function $f = |A| - \alpha H$ and we obtain

$$\begin{aligned} \frac{\partial f}{\partial t} =& \Delta_{g(t)} f + |A|^2 f + \frac{1}{2|A|}(2|\nabla|A|^2 - 2|\nabla A|^2) \\ \leq& \Delta_{g(t)} f + |A|^2 |f| \end{aligned}$$

If $T' < T$, we have that there is a uniform bound for $|A|^2$, and the maximum principle implies that $f_{\max} \leq 0$ □

Note that the previous inequality can only hold for $H > 0$. Recall that by definition we always have $H^2 \leq n|A|^2$.

We leave the proof of the following proposition as an exercise to the reader.

Proposition 2.3.3. *Let M_t, $0 \leq t \leq T'$ be a solution to* (MCF) *that satisfies $|A| \leq C$ then*

$$|\nabla^m A|^2 \leq K e^{C_m T'},$$

where C_m depends on $|A|^2, \ldots |\nabla^{m-1} A|^2$ and K on bounds on the initial condition.

Remark 2.3.4. *It is possible to obtain similar estimates that are interior in time. See [10].*

Proposition 2.3.5. *If the second fundamental form A is not bounded as $t \to T < \infty$, then it mush satisfy the following lower bound for its blow up rate:*

$$|A|^2_{\max}(t) \geq \frac{1}{2(T-t)}.$$

Proof. From Lemma 2.0.1 and Proposition 2.3.1 we have that

$$\frac{d}{dt}|A|^2_{\max} \leq 2|A|^4_{\max}.$$

Then

$$-\frac{d}{dt}\frac{1}{|A|^2_{\max}} \leq 2 \text{ and}$$
$$\frac{1}{|A|^2_{\max}(t)} - \frac{1}{|A|^2_{\max}(s)} \leq 2(s-t).$$

Taking $s \to T$ we obtain the result.

□

Another important consequence is that convexity is preserved by the flow and more generally

Theorem 2.3.6. *If an initial compact hypersurface is k-convex (i.e. the sum of the first k principal curvatures is positive) then it is so for every positive time under mean curvature flow.*

2.4 Some examples of singularities

We finish this section by using the maximum principle to show some examples of singularities. We follow [8], but other examples can be found at [4, 20]

Proposition 2.4.1 (Hyperboloid/Cone comparison)**.** *Let M_t for $t \in I$ be a solution to* (MCF)*. If for $0 \leq \beta \leq n$ and some $\varepsilon > 0$ satisfies*

$$M_0 \subset \{x \in \mathbb{R}^{n+1},\, (n-1-\beta)x_{n+1} \geq |\hat{x}|^2 - \varepsilon^2\}$$

then

$$M_t \subset \{x \in \mathbb{R}^{n+1},\, (n-1-\beta)x_{n+1} \geq |\hat{x}|^2 - \varepsilon^2 + 2\beta t\},$$

for $t < \frac{\varepsilon^2}{2\beta}$ as long as the solution stays smooth for this time.

Using the previous proposition, it is possible to construct dumbbell-shaped initial conditions that will develop singularities in finite time. By using spheres as lower barriers, it is possible to show that at the singular time the evolution will not converge to a lower dimensional object. This contrast with the examples of spheres and cylinders that were previously analyzed.

Lecture 3

Singularities

In the previous lecture we discussed some examples of evolutions that become singular in finite time. It is possible to classify singularities according to the following definition.

Definition 3.0.1. *Let* $F : M \times [0, T) \to \mathbb{R}^{n+1}$ *be a solution to* (MCF) *such that* $\sup_{M_t} |A|(\cdot, t) \to \infty$ *as* $t \to T$*. Then we say that* F *develops a singularity Type I if*

$$\lim_{t \to T} (T - t) \sup_{M_t} |A|^2 < \infty.$$

Otherwise, we say that the singularity is Type II*.*

3.1 Curve Shortening Flow

One of the first cases that was studied in the literature was the evolution of closed embedded curves on the plane. The mean curvature flow of curves is usually known as *curve shortening flow*. One of the most remarkable results in the field was obtained by Gage and Hamilton ([15]) and Grayson ([19]):

Theorem 3.1.1. *Any closed, smoothly embedded planar curve retains these properties under curve-shortening flow and becomes convex in finite time, after which converges into a point in finite time. In the process the solution becomes asymptotically round. That is after appropriate rescaling (e.g. keeping the length fixed) it converges smoothly to a round circle (in infinite time).*

Here we include a proof by Andrews and Bryan [3] that builds on work by Huisken in [24]. The main tool of the proof is the maximum principle.

Remark 3.1.2. *A similar result was proven for convex hypersurfaces by Huisken ([23]). Recall that Theorem 2.3.6 implies that convexity is preserved through the flow.*

Theorem 3.1.3 ([23])**.** *For any smoothly embedded, compact, convex initial hypersurface* M_0 *the solution of mean curvature flow remains smoothly embedded,* *compact and convex until it disappears into a point in finite time. In the process, the solution becomes asymptotically round. That is after appropriate rescaling (e.g. keeping the area fixed) it converges smoothly to a round sphere (in infinite time).*

Proof of Theorem 3.1.1

To prove Theorem 3.1.1 we will proceed as follow

1. Renormalize the flow to have fixed length.
2. Define a function $f(x,t)$ that is non-decreasing in time for every fixed x.
3. Use maximum principle to show that $d(p,q,t) \geq f(l(p,q), t-\bar{t})$, where d is the extrinsic distance between points (on the plane) and l their instrinsic distance.
4. The previous estimates will show that the rescaled flow converges as $t \to \infty$ to a curve of constant curvature 1.

We remark that the comparison of the extrinsic and intrinsic distance was first used by Huisken in [24], where he concluded the same result by combining the maximum principle estimates with a a blow up argument (similar to the one exposed in the coming section) and by using the classification of self-similar solutions by Abresch and Langer ([1]).

Consider $\gamma : S^1 \times [0, \mathcal{T}) \to \mathbb{R}^2$ an embedded curve evolving by curve shortening flow of total length $L(\gamma(\tau))$ and time τ and define

$$F(p,t) = \frac{2\pi}{L(\gamma)}\gamma(p,\tau),$$

where

$$t(\tau) = \int_0^\tau \left(\frac{2\pi}{L(\gamma(\tau))}\right)^2 d\tau' \text{ and } T = t(\mathcal{T}).$$

Then $F : S^1 \times [0,T)$ satisfies

$$\frac{\partial F}{\partial t} = \bar{k}^2 F - k\nu,$$

where $\bar{k}^2 = \frac{1}{2\pi}\int k^2 ds$.

Let $d(p,q,t) = |F(p,t) - F(q,t)|$ and $l(p,q,t)$ the arc length between these points. Consider $f(x,t) = 2e^t \arctan\left(e^{-t}\sin\left(\frac{x}{2}\right)\right)$. A direct (although slightly messy) computation shows that $f(x,t)$ is strictly increasing in t for every fixed x. Moreover, $\lim_{t\to\infty} f(x,t) = 2\sin\left(\frac{x}{2}\right)$ and $\lim_{t\to-\infty} f(x,t) = 0$. Define

$$a(p,q) = \inf\{e^t : \, d(p,q) \geq f(l(p,q), -t)\},$$

for $p \neq q$ in S^1. By the implicit function theorem, we have that a is continuous, smooth and positive for $0 < d < 2\sin\left(\frac{l}{2}\right)$. Using a Taylor expansion, its possible to show (exercise, or see [3]) that a can be extended to a continuous function in $S^1 \times S^1$ that is defined as

$$a(p,p) = \sqrt{\frac{\max\{k^2-1,0\}}{2}}.$$

Let $\bar{a} = \sup_{S^1\times S^1}\{a(p,q) : p \neq q\}$. Then, it holds that

$$d(p,q) > f(l(p,q), -\log a(p,q)) > f(l(p,q), -\log \bar{a})).$$

Now, using the maximum principle we will have that

Lemma 3.1.4.

$$d(p,q,t) > f(l(p,q), t-\bar{t}).$$

Proof. We choose $\bar{t} = -\log \bar{a}$ and define $\mathcal{Z}(p,q,t) = d(p,q,t) - f(l(p,q), t-\bar{t})$ and $\mathcal{Z}_\varepsilon(p,q,t) = \mathcal{Z}(p,q,t) + \varepsilon e^{Ct}$. By the definition of $\bar{t}$, we have that $\mathcal{Z}_\varepsilon(p,q,0) \geq \varepsilon > 0$. Assume that there is a first time t_0 such that

$$\mathcal{Z}_\varepsilon(p_0,q_0,t_0) = \inf_{(p,q)\in S^1\times S^1} \mathcal{Z}_\varepsilon(p,q,t_0) = 0.$$

Then $\frac{\partial \mathcal{Z}_\varepsilon(p_0,q_0,t_0)}{\partial t} \leq 0$, $D\mathcal{Z}_\varepsilon(p_0,q_0,t_0) = 0$ and $D^2\mathcal{Z}_\varepsilon(p_0,q_0,t_0)$ is non-negative.

First observe that

$$\begin{aligned}\frac{\partial \mathcal{Z}}{\partial t}(p_0,q_0,t_0) =& \langle \omega, \frac{\partial F}{\partial t}(p_0) - \frac{\partial F}{\partial t}(q_0)\rangle - \frac{\partial f}{\partial x}\frac{\partial l}{\partial t} - \frac{\partial f}{\partial t} \\ =& \bar{k}^2 d - (k_{p_0}\langle \omega, \nu_{p_0}\rangle - k_{q_0}\langle \omega, \nu_{q_0}\rangle) - \frac{\partial f}{\partial x}\int (\bar{k}^2 - k^2) ds - \frac{\partial f}{\partial t}\end{aligned} \tag{3.1.1}$$

where $\omega = \frac{F(p_0,t_0)-F(q_0,t_0)}{|F(p,t)-F(q,t)|}$.

On the other hand, assume that we are parametrizing respect to arc length at t_0, then if we consider variations $F(p_0+\xi u, q_0+\eta u, t_0)$, we have

$$\left.\frac{\partial \mathcal{Z}_\varepsilon(p,q,t)}{\partial u}\right|_{u=0} = \left(\langle \omega, T_{p_0}\rangle - \frac{\partial f}{\partial x}\right)\xi - \eta\left(\langle \omega, T_{q_0}\rangle - \frac{\partial f}{\partial x}\right).$$

Necessarily at a critical point we have

$$\frac{\partial f}{\partial x} = \langle \omega, T_{p_0}\rangle = \langle \omega, T_{q_0}\rangle. \tag{3.1.2}$$

Since $\frac{\partial f}{\partial x} \neq 1$ at all points, $\omega \neq T_{p_0}$ and $\omega \neq T_{q_0}$. There are two possibilities, either $T_{p_0} = T_{q_0}$ or ω bisects T_{p_0} and T_{q_0}.

In the first case, necessarily the chord defined by ω intersects one of the tangents at an acute angle and the other one at an obtuse angle (by the choice of f it cannot be perpendicular). Then, points near one of the points are inside the curve, while the others are outside. Hence, the chord intersects at least one other point. Let us assume that the additional intersection point is $p_0 < s < q_0$. Then

$$\begin{aligned} d(p_0, q_0) =& d(p_0, s) + d(s, q_0) \\ l(p_0, q_0) =& \min\{l(p_0, s) + l(s, q_0), 2\pi - l(p_0, s) - l(s, q_0)\}. \end{aligned}$$

Since $f(,t)$ is convex for every fixed t we have

$$\mathcal{Z}(p_0, q_0) > \mathcal{Z}(p_0, s) + Z(s, q_0),$$

which yields a contradiction.

Assume now that $T_{p_0} \neq T_{q_0}$. We compute the second variation of $\mathcal{Z}$

$$\begin{aligned} \left.\frac{\partial^2 \mathcal{Z}}{\partial u^2}\right|_{u=0} =& \xi^2 \left[\frac{1}{d}\left(1 - \langle \omega, T_{p_0}\rangle^2\right) + \langle \omega, k_{p_0} N_{p_0}\rangle - f''\right] \\ &+ \eta^2 \left[\frac{1}{d}\left(1 - \langle \omega, T_{q_0}\rangle^2\right) - \langle \omega, k_{q_0} N_{q_0}\rangle - f'\right] \\ &+ 2\xi\eta \left[\frac{1}{d}\left(\langle \omega, T_{p_0}\rangle\langle \omega, T_{q_0}\rangle - \langle T_{p_0}, T_{q_0}\rangle\right) + f'\right] \geq 0 \end{aligned}$$

Let $'$ denote derivatives with respect to the arc-parameter x. We may choose $\xi = -\eta = 1$ and combine it with (3.1.1) to obtain

$$-C\varepsilon e^{Ct_0} \geq 4f'' + \bar{k}^2(\varepsilon e^{Ct_0} + f - f'l) + f' \int k^2 - \frac{\partial f}{\partial t}.$$

A direct computation implies that

$$(f - f'l)' = -f''l > 0,$$

since f is concave. Then $f - f'l > f(0) > 0$. Using Hölder's inequality we have

$$(2\pi)^2 \bar{k}^2 = 2\pi L(F)\bar{k}^2 \geq \left(\int k\, ds\right)^2 = (2\pi)^2.$$

Let ϑ be the angle between T_{p_0} and T_{q_0}. Note that equation (3.1.2) implies that this is twice the angle between T_{p_0} and ω (or equivalently, the angle between T_{q_0} and ω).Then, $\vartheta = 2\arccos f'$. Additionally,

$$\int_{p_0}^{q_0} k^2 ds \geq \frac{\left(\int_{p_0}^{q_0} |k| ds\right)^2}{l} = \frac{\vartheta^2}{l}.$$

Then

$$-C\varepsilon e^{Ct_0} \geq Lf + \bar{k}^2 \varepsilon e^{Ct_0},$$

where

$$Lf = 4f'' + f - f'l + 4\frac{f'}{l}(\arccos(f'))^2 - \frac{\partial f}{\partial t}.$$

By an appropriate choice of C we have that $Lf < 0$. On the other hand, a direct computation shows that

$$Lf \geq \tilde{L}f = 4f'' + f - \sin\left(\frac{l}{2}\right)\left(f' - \cos\left(\frac{l}{2}\right)\right) - \frac{\partial f}{\partial t} = 0,$$

which yields a contradiction. □

Now the main result of this section can be concluded by observing that

$$\sqrt{\frac{\max\{k^2(t) - 1, 0\}}{2}} = a(p, p, t) \leq \sup\{a(p, q, t : p \neq q\} \leq e^{\bar{t} - t}.$$

Then

$$\begin{aligned}\int (k(s) - 1)^2 ds &= \int k^2 ds - 2\int k ds + \int 1 ds \\ &= \int k^2 - 4\pi + L(F) \\ &= \int k^2 - 2\pi \\ &= \int (k^2 - 1) ds \\ &\leq e^{\bar{t} - t} \to 0 \text{ as } t \to \infty\end{aligned}$$

To get smooth convergence we can use Sobolev-Gagliardo-Niremberg inequalities.

Remark 3.1.5. *It is also possible to obtain estimates on the derivatives of k as $t \to \infty$.*

3.2 Monotonicity formula and type I singularities

We start by showing Huisken's monotonicity formula

Theorem 3.2.1 ([23], [8])**.** *Fix (x_0, t_0) and let*

$$\Phi_{(x_0,t_0)}(x, t) = \frac{1}{(4\pi(t_0 - t))^{\frac{n}{2}}} e^{-\frac{|x - x_0|^2}{4(t_0 - t)}}.$$

Consider a smooth solution $\{M_t\}_{t \in I}$ of mean curvature flow for which

$$\int_{M_t} \Phi_{(x_0,t_0)} d\mu < \infty \text{ for all } t \in I \text{ with } t < t_0.$$

Then for these times

$$\frac{d}{dt}\left(\int_{M_t} \Phi_{(x_0,t_0)} d\mu\right) = -\int_{M_t} \left|H - \frac{D^\perp \Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}}\right|^2 \Phi_{(x_0,t_0)}$$

Remark 3.2.2. *The function $\Phi_{(x_0,t_0)}$ is a solution to the backward heat equation. In particular, it satisfies*

$$\frac{\partial \Phi_{(x_0,t_0)}}{\partial t} + \Delta_{\mathbb{R}^{n+1}} \Phi_{(x_0,t_0)} = 0.$$

Proof. The proof is a direct computation:

$$\begin{aligned}\frac{d}{dt}\left(\int_{M_t} \Phi_{(x_0,t_0)} d\mu\right) &= \int_{M_t} \left(\frac{\partial \Phi_{(x_0,t_0)}}{\partial t} + D\Phi_{(x_0,t_0)} \cdot \frac{\partial x}{\partial t} - H^2 \Phi_{(x_0,t_0)}\right) d\mu \\ &= \int_{M_t} \left(-\Delta_{\mathbb{R}^{n+1}} \Phi_{(x_0,t_0)} + HD\Phi_{(x_0,t_0)} \cdot \nu - H^2 \Phi_{(x_0,t_0)}\right) d\mu\end{aligned}$$

Now we use that for any function

$$\Delta_{g(t)} u(F) = \Delta_{\mathbb{R}^{n+1}} u(F) - D^2 u(F)(\nu,\nu) + H\nu \cdot Du(F).$$

Stokes' theorem imply

$$\begin{aligned}\frac{d}{dt}\left(\int_{M_t} \Phi_{(x_0,t_0)} d\mu\right) &= -\int_{M_t} \left(D^2 \Phi_{(x_0,t_0)}(\nu,\nu) - 2HD\Phi_{(x_0,t_0)} \cdot \nu + H^2 \Phi_{(x_0,t_0)}\right) d\mu \\ &= -\int_{M_t} \left| H - \frac{D^\perp \Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}} \right|^2 \Phi_{(x_0,t_0)} d\mu \\ &\quad + \int_{M_t} \left(\frac{|D^\perp \Phi_{(x_0,t_0)}|^2}{\Phi_{(x_0,t_0)}} - D^2 \Phi_{(x_0,t_0)}(\nu,\nu) \right) d\mu.\end{aligned}$$

We leave the rest of the proof to the reader. □

There is a localized version of the monotonicity formula:

Theorem 3.2.3. *Assume that $\Phi_{(x_0,t_0)}$ and $M_{t t\in I}$ are as in the previous theorem. Let f be a sufficiently smooth function defined on M_t that satisfies*

$$\int_{M_t} \left(|f| + |\frac{\partial f}{\partial t}| + |Df| + |D^2 f|\right) \Phi_{(x_0,t_0)} < \infty \text{ for all } t \in I \text{ with } t < t_0.$$

Then for these times

$$\frac{d}{dt}\left(\int_{M_t} f\Phi_{(x_0,t_0)} d\mu\right) = -\int_{M_t} \left(\left(\frac{\partial}{\partial t} - \Delta_{M_t}\right) f - \left| H - \frac{D^\perp \Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}} \right| f \right) \Phi_{(x_0,t_0)}$$

Now we show that Type I singularities can be rescaled to self-similar shrinking solutions:

Consider $F : M_0 \times [0,T) \to \mathbb{R}^{n+1}$ be a solution to (MCF) with singular time T. Assume that the singularity is type I. Let $\hat{p} = \lim_{t\to T} F(\omega,t)$ and define

$$\tilde{F}(\omega,t) = \frac{F(\omega,t) - \hat{p}}{\sqrt{2(T - t(s))}}, \qquad s(t) = -\frac{1}{2}\log(T-t).$$

The main theorem states that

Theorem 3.2.4. *At a singular point $p \in M_t$ the rescaled surfaces given by $\tilde{M}_t = \tilde{F}(M_0, t)$ converge to a limit surface $\tilde{M}_\infty$ that is smooth complete with bounded local volume and bounded curvature. Additionally, $\tilde{M}_\infty$ satisfies $\tilde{H} + \langle y, \tilde{\nu} \rangle = 0$ and is not a unit multiplicity plane through the origin of $\mathbb{R}^{n+1}$.*

Moreover, if the initial surface is embedded then $\tilde{M}_\infty$ is also embedded.

Proof. We will only briefly sketch the main ideas of the proof. Note first that $\tilde{F}$ satisfies

$$\begin{aligned}\frac{\partial \tilde{F}}{\partial s} &= -\frac{H(F)}{\sqrt{2(T-t(s))}} 2(T-t(s))\nu - \tilde{F} \\ &= -\tilde{H}\tilde{\nu} - \tilde{F}.\end{aligned}$$

Moreover, it is direct to compute that

$$\frac{\partial \tilde{H}}{\partial t} = \Delta_{\tilde{g}} \tilde{H} + \tilde{H}|A|^2 - \tilde{H}$$

Notice also that $\tilde{F}$ has second fundamental form $|\tilde{A}|$ uniformly bounded for all times (because the singularities are type I). Similarly to the computations of Section 2.3 we have that all the derivatives are locally bounded.

Note first that

$$\begin{aligned}\frac{\partial \left|H + \langle y, \tilde{\nu}\rangle\right|^2}{\partial s} =& 2\left|\Delta_{\tilde{g}}\tilde{H} + \tilde{H}|\tilde{A}|^2 + \langle y, \tilde{\nu}\rangle - \langle y, \tilde{\nabla}\tilde{H}\rangle\right| |H + \langle y, \tilde{\nu}\rangle| \\ \leq& C(|y|^2 + 1)\end{aligned}$$

A result by Stone ([31]) implies that there is a constant C such that

$$\int_M e^{-|y|} d\tilde{\mu}_s \leq C.$$

Then $\int_{\tilde{M}_t} e^{-\frac{|y|^2}{2}} d\tilde{\mu}$ and $\int_{\tilde{M}_t} e^{-\frac{|y|^2}{2}} \left|H + \langle y, \tilde{\nu}\rangle\right|^2 d\tilde{\mu}$ are finite.

A direct computation shows that $\tilde{F}$ satisfies the following rescaled monotonicity formula:

$$\frac{d}{dt}\left(\int_{\tilde{M}_t} e^{-\frac{|y|^2}{2}} d\tilde{\mu}\right) = -\int_{\tilde{M}_t} e^{-\frac{|y|^2}{2}} \left|H + \langle y, \tilde{\nu}\rangle\right|^2 d\tilde{\mu}.$$

(exercise: prove the rescaled monotonicity formula.)

Then

$$\int_{-\frac{1}{2}\log T}^{\infty} \int_{\tilde{M}_t} e^{-\frac{|y|^2}{2}} \left|H - \langle y, \tilde{\nu}\rangle\right|^2 d\tilde{\mu} \leq \int_{\tilde{M}_{-\frac{1}{2}\log T}} e^{-\frac{|y|^2}{2}} d\tilde{\mu} \leq C < \infty.$$

Then, if there is a sequence of times $s_i \to \infty$ such that $\int_{\tilde{M}_t} e^{-\frac{|y|^2}{2}} \left|H + \langle y, \tilde{\nu}\rangle\right|^2 d\tilde{\mu} > \delta$, we would have a contradiction.

On the other hand, by means of the monotonicity formula we have that $\mathcal{H}^n(\tilde{F} \cap B_R)$ has a uniform upper bound. Combined with the uniform curvature estimates we have that on balls $\tilde{F}_t$ can be written as a graph of a smooth function that converges locally uniformly. Using a diagonal argument we have that $\tilde{M}_\tau \to \tilde{M}_\infty$ as $\tau \to \infty$ and $H + \langle y, \tilde{\nu} \rangle = 0$ on $\tilde{M}_\infty$. To conclude that the limit is not a plane it is possible to use a regularity result proved by White [32]:

Theorem 3.2.5. *There exist constants $\varepsilon = \varepsilon(n) > 0$ and $C = C(n)$ such that if $\Theta(p) < 1 + \varepsilon$ then $|A| < C(n)$ in a ball of $\mathbb{R}^{n+1}$ around p uniformly in time $t \in [0, T)$.*

Here

$$\vartheta(p,t) = \int_M \frac{e^{-\frac{|x-\hat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{\frac{n}{2}}} d\mu_t.$$

and the limit heat density function as

$$\Theta = \lim_{t \to T} \vartheta(p,t).$$

For M compact we also define

$$\sigma(t) = \max_{x_0 \in \mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{\frac{n}{2}}} d\mu_t.$$

and its limit

$$\Sigma = \lim_{t \to T} \sigma(t).$$

□

Remark 3.2.6. *If $\tilde{F}$ satisfies $\tilde{H} + \langle y, \tilde{\nu} \rangle = 0$, then $\sqrt{2(T-t)}\tilde{F}$ is a self-similar shrinking solution.*

All self-similar shrinking curves were classified by Abresch and Langer in [1]. In particular, the only embedded self-similar shrinking curve is the circle.

3.3 Type II singularities

Note that if the singularity is type II the procedure of the previous section does not produce a family of evolutions with bounded curvatures. We describe a different blow-up procedure that was proposed by Hamilton: We first choose sequences $t_k \in [0, T - \frac{1}{k}]$, $p_k \in M_{t_k}$ such that

$$|A(p_k, t_k)|^2 \left(T - \frac{1}{k} - t_k \right) = \max_{t \in [0, T - \frac{1}{k}], p \in M} A(p,t)|^2 \left(T - \frac{1}{k} - t \right).$$

Notice that this quantity goes to infinity, since we assume that the singularity is type II. Now we take

$$F_k(\omega, s) = |A(p_k, t_k)| \left[F\left(\omega, \frac{s}{|A(p_k, t_k)|} + t_k\right) - p_k \right]$$

for $s \in I_k = \left[-|A(p_k, t_k)|^2 t_k, |A(p_k, t_k)|^2 \left(T - \frac{1}{k} - t_k\right)\right]$.

Then the main result is

Theorem 3.3.1. *The family of flows F_k converges (up to subsequence) in the C^∞_{loc} topology to a nonempty smooth evolution by mean curvature of complete hypersurfaces M^∞_s in the time interval $(-\infty, \infty)$.*

It is easy to check that F_k are still solutions to (MCF). From the construction holds that

- $F_k(p_k, 0) = 0 \in \mathbb{R}^{n+1}$ and $|A_k(p_k, 0)| = 1$.
- for every $\varepsilon > 0$ and $\omega > 0$ there exists $\bar{k} \in \mathbb{N}$ such that

  $$\max_{p \in M} |A_k(p, s)| \leq 1 + \varepsilon$$

  for every $k \geq \bar{k}$ and $s \in [-|A_k(p_k, t_k)|^2 t_k, \omega]$.

As before, we have uniform are bounds (by mean of the monotonicity formula), estimates for $|A_k|$ and its derivatives. As before we can take limits and observe that the limit solution exists for all times. Such a solution is known as *eternal solution.*

Remark 3.3.2. *The grim reaper is the only embedded one dimensional eternal solution.*

3.4 Weak solutions

There are several approaches to continue the flow after the first singular time. Here we briefly mention 4 of them.

3.4.1 Brakke flow

The Brakke flow was introduced by K. Brakke in his seminal work in [5]. In his context solutions are varifolds $\{d\mu_t\}$ that satisfy a version of (MCF) in an integral sense:

$$\int \varphi \, d\mu_t \leq \int \left(-H^2 \varphi + \vec{H} \cdot S \cdot D\varphi\right) d\mu_t,$$

where S represents the orthogonal projection onto the tangent plane of the surface. for every $\varphi \in C_0(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ for all $t > 0$.

Brakke's methods rely heavily on geometric measure theory and since he works with a more general class object, it is possible to make sense of solutions that are singular in the classical sense. However, in certain situations, the Brakke flow lacks of uniqueness.

3.4.2 Level set method

The level set method was introduced by Evans and Spruck in [11] and independently by Chen, Giga and Goto in [7]. The main idea is to assume that solutions can be expressed as level sets of a function $u(\cdot,t)$. That is

$$M_t = \{x : u(x,t) = c\}.$$

Notice that if solutions can be in fact expressed in this form, we would have

$$\frac{\partial u}{\partial t} + Du \cdot \frac{\partial x}{\partial t} = 0.$$

Since $\frac{\partial x}{\partial t} = -H\nu$ and M_t is a level set, we have that $\nu = \frac{Du}{|Du|}$ and $H = \text{div}\left(\frac{Du}{|Du|}\right)$. We have that

$$\frac{\partial u}{\partial t} - |Du|\text{div}\left(\frac{Du}{|Du|}\right) = 0. \tag{3.4.1}$$

We say that u (or equivalently $M_t = \{x : u(x,t) = c\}$) is a level set solution to mean curvature flow if it satisfies (3.4.1) in the viscosity sense. In [11, 7] was proved that for Lipschitz initial conditions solutions to (3.4.1) exist for all times and if there is a smooth solution to (MCF) it agrees with the level set solution independently of the representation chosen as initial condition.

On the other hand, the level sets of solutions to (3.4.1) may develop an interior in finite time. This phenomenon is known as fattening.

In contrast with Brakke's flow, the level set has only been formulated for co-dimension 1 evolutions.

3.4.3 Allen-Cahn equation

Consider a proper non-negative function W that attains its minimum 0 at exactly two points 1 and -1 and has only 3 critical points. Let u_ε be a solution to the Allen-Cahn equation.

$$\begin{aligned}\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon + \frac{W'(u)}{\varepsilon^2} &= 0\\ u(x,0) &= u_0(x).\end{aligned}$$

Suppose that $|u_0| \to 1$ a.e. and let $M_0 = \{x \in \mathbb{R}^{n+1} : |u_0|(x) \not\to 1\}$, then $|u_\varepsilon| \to 1$ and if M_0 is a rectifiable varifold, then $M_t = \{x \in \mathbb{R}^{n+1} : |u_\varepsilon(x,t)| \not\to 1\}$ is a Brakke solution to mean curvature flow. For further details see [2, 30]. The set $\{x \in \mathbb{R}^{n+1} : |u_\varepsilon(x,t)| \not\to 1\}$ is usually known as the interface set. In particular, it is possible to define weak solutions to (MCF) as the interfaces developed by solutions to the Allen-Cahn equation. This method is also suited only for co-dimension one evolutions.

3.4.4 Mean Curvature flow with surgery

The mean curvature flow with surgery was first developed by Huisken and Sinistrari in [26] and later reformulated by Haslhofer and Kleiner in [21]. These program is analogous with the surgery procedure proposed by Hamilton a developed by Hamilton and Perelman for the study of Ricci flow. Loosely speaking, the idea is to remove the singularities (by performing a surgery) at the singular time and replace them with spheres.

This procedure requires very precise knowledge of the singularity formation and has only been successfully carried out for the evolution of 2-convex surfaces. On the other hand, it has the advantage that keeps track of topological changes that occur during the evolution, which allows, for instance, topological classification results ([26, 6]).

Lecture 4

Non-compact evolutions

4.1 Complete graphs over $\mathbb{R}^n$

The results of this section discuss the results proved by Ecker and Huisken in [9]. Their work studied evolutions of the form $M_t = \{(x, u(x,t) : x \in \mathbb{R}^n\}$, where u satisfies

$$\frac{\partial u}{\partial t} = \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \text{ in } \mathbb{R}^n \text{ and } t>0, \tag{4.1.1}$$

$$u(x,0) = u_0(x). \tag{4.1.2}$$

Note that (4.1.1) is equivalent to (MCF'). Their main result shows that

Theorem 4.1.1. *Assume that u_0 is locally Lipschitz and satisfies a uniform global gradient bound. Then $u(x,t)$ exists for all times and it is smooth. Moreover, $\frac{M}{\sqrt{2t+1}}$ converges uniformly to a expanding self-similar solution as $t \to \infty$.*

We postpone a proof of this result, but we note that the estimates used to prove this result were first localized in [10].

4.2 Rotationally symmetric surfaces

Notice that not every non-compact surface can have the regularity of graphs. For instance, cylinders have finite time singularities. In [17] Giga, Seki and Umeda studied surfaces a more general class of rotationally symmetric surfaces: $M_t = \{(x, y_1, \ldots, y_n) : \left(\sum_{i=1}^n y_i^2\right)^{\frac{1}{2}} = u(x,t)\}$. Then (MCF') is equivalent to

$$\frac{\partial u}{\partial t} = \frac{u_{xx}}{1+u_x^2} - \frac{n-1}{x} \text{ for } x \in \mathbb{R} \text{ and } t>0, \tag{4.2.1}$$

$$u(x,0) = u_0(x). \tag{4.2.2}$$

Giga, Seki and Umeda showed that in fact open ends may close in finite time. That finite time is called *quenching time* and it may be defined as

$$T(u_0) = \{\sup t > 0 \, : \, \inf_{x\in\mathbb{R}} u(x,t) > 0\}.$$

More precisely, their result states

Theorem 4.2.1. *Assume u_0 is bounded and uniformly continuous in $\mathbb{R}$ and that $m = \inf_{x\in\mathbb{R}} u(x,0) > 0$. Then a solution of the Cauchy problem (4.2.1)-(4.2.2) has a minimal quenching time (that we denote by $T(m)$) if and only if the initial datum satisfies of the following the conditions:*

1. *There exists a sequence $\{x_k\} \subset \mathbb{R}$ such that $x_k \to \infty$ and $u_0(x+x_k) \to m$ a.e. as $k \to \infty$;*
2. *There exists a sequence $\{x_k\} \subset \mathbb{R}$ such that $x_k \to -\infty$ and $u_0(x+x_k) \to m$ a.e. as $k \to \infty$;*

Moreover, the $u(x,t)$ approaches (in a suitable sense) a limit function $u(x, T(m))$ as $t \to T(m)$ that satisfies for any $R > 0$,

$$\lim_{k\to\infty} u(x + x_k, T(m)) = 0 \text{ if } |x| \le R.$$

Furthermore, if the hypothesis 1 (resp. (2)) holds

$$\lim_{x\to\infty} u(x, T(m)) = 0 \text{ (resp. } \lim_{x\to-\infty} u(x, T(m)) = 0).$$

4.3 Graphs over compact domains

In this section we discuss the results in [29] and prove Theorem 4.1.1. The initial conditions that we consider are entire graphs over bounded domains. Informal versions on the main results are

Theorem 4.3.1 (Existence on bounded domains)**.** *Let $A \subset \mathbb{R}^{n+1}$ be a bounded open set and $u_0 \colon A \to \mathbb{R}$ a locally Lipschitz continuous function with $u_0(x) \to \infty$ for $x \to x_0 \in \partial A$.*

Then there exists (Ω, u), where $\Omega \subset \mathbb{R}^{n+1} \times [0,\infty)$ is relatively open, such that u solves graphical mean curvature flow

$$\dot{u} = \sqrt{1 + |Du|^2} \cdot \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) \quad \text{in } \Omega \setminus (\Omega_0 \times \{0\}).$$

The function u is smooth for $t > 0$ and continuous up to $t = 0$, $\Omega_0 = A$, $u(\cdot, 0) = u_0$ in A and $u(x,t) \to \infty$ as $(x,t) \to \partial\Omega$, where $\partial\Omega$ is the relative boundary of Ω in $\mathbb{R}^{n+1} \times [0, \infty)$.

Note that in the previous theorem the domain of definition of the graph changes in time and it is necessary to modify the definition of solution to allow for changes of topology. In fact, the domain can be related to level set solutions by mean curvature flow as follows:

Theorem 4.3.2 (Weak flow)**.** *Let (A, u_0) and (Ω, u) be as in Theorem 4.3.1. Assume that the level set evolution of $\partial\Omega_0$ does not fatten. Then it coincides with $(\partial^\mu\Omega_t)_{t\geq 0}$.*

In the remainder of this section, we are going to sketch the proof of Theorem 4.3.1.

An important tool in the proof of Theorem 4.1.1 and 4.3.1 are the so-called gradient estimates. The gradient function is defined as

$$v = (\nu \cdot e)^{-1},$$

where e is fixed direction. In the graphical case we choose $e = e_{n+1}$, but for local estimates (as in [10]) may be convenient to choose e as another fixed vector. Note that for a graph, $v = \sqrt{1 + |Du|^2}$.

The evolution equations in this context are

Theorem 4.3.3. *Let X be a solution to mean curvature flow. Then we have the following evolution equations.*

$$\begin{aligned}
\left(\tfrac{d}{dt} - \Delta\right) u =& 0,\\
\left(\tfrac{d}{dt} - \Delta\right) v =& -v|A|^2 - \tfrac{2}{v}|\nabla v|^2,\\
\left(\tfrac{d}{dt} - \Delta\right) |A|^2 =& -2|\nabla A|^2 + 2|A|^4,\\
\left(\tfrac{d}{dt} - \Delta\right) |\nabla^m A|^2 \leq& -2\left|\nabla^{m+1} A\right|^2\\
&+ c(m,n) \cdot \sum_{i+j+k=m} |\nabla^m A| \cdot \left|\nabla^i A\right| \cdot \left|\nabla^j A\right| \cdot \left|\nabla^k A\right|,\\
\left(\tfrac{d}{dt} - \Delta\right) \mathcal{G} \leq& -2k\mathcal{G}^2 - 2\varphi v^{-3}\langle\nabla v, \nabla\mathcal{G}\rangle,
\end{aligned}$$

where $\mathcal{G} = \varphi|A|^2 \equiv \frac{v^2}{1-kv^2}|A|^2$ and $k > 0$ is chosen so that $kv^2 \leq \frac{1}{2}$ in the domain considered.

Using the maximum principle we obtain priori estimates. We will assume the following:

Assumption 4.3.4. *Let $\hat{\Omega} \subset \mathbb{R}^{n+1} \times [0, \infty)$ be an open set. Let $u\colon \hat{\Omega} \to \mathbb{R}$ be a smooth graphical solution to*

$$\dot{u} = \sqrt{1 + |Du|^2} \cdot \operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) \quad \text{in} \quad \hat{\Omega} \cap \left(\mathbb{R}^{n+1} \times (0, \infty)\right).$$

Suppose that $u(x,t) \to 0$ as $(x,t) \to (x_0, t_0) \in \partial\hat{\Omega}$. Assume that all derivatives of u are uniformly bounded and can be extended continuously across the boundary for all domains $\hat{\Omega} \cap \left(\mathbb{R}^{n+1} \times [0, T]\right)$ and that these sets are bounded for any $T > 0$.

Theorem 4.3.5 (C^1-estimates)**.** *Let u be as in Assumption 4.3.4. Then*

$$vu^2 \leq \max_{\substack{t=0\\ \{u<0\}}} vu^2$$

at points where $u < 0$.

In consquence,

Corollary 4.3.6. *Let u be as in Assumption 4.3.4. Then*

$$v \leq \max_{\substack{t=0\\ \{u<0\}}} vu^2$$

at points where $u \leq -1$.

Remark 4.3.7.

We remark that is we assume that initial condition has v is bounded everywhere (and u is globally defined), then it is possible to prove that v remains bounded for all times.

The previous result yields higher derivative estimates

Theorem 4.3.8 (C^2-estimates)**.** *Let u be as in Assumption 4.3.4.*

(i) Then there exist $\lambda > 0$, $c > 0$ and $k > 0$ (the constant appearing in the definition of φ and implicitly in the definition of $\mathcal{G}$), depending on the C^1-estimates, such that

$$tu^4\mathcal{G} + \lambda u^2 v^2 \leq \sup_{\substack{t=0\\ \{u<0\}}} \lambda u^2 v^2 + ct$$

at points where $u < 0$ and $0 < t \leq 1$.

(ii) Moreover, if u is in C^2 initially, we get C^2-estimates up to $t = 0$: Then there exists $c > 0$, depending only on the C^1-estimates, such that

$$u^4\mathcal{G} \leq \sup_{\substack{t=0\\ \{u<0\}}} u^4\mathcal{G} + ct$$

at points where $u < 0$.

(iii) There exists $\lambda > 0$, depending on the C^{m+1}-estimates, such that

$$tu^2 \left|\nabla^m A\right|^2 + \lambda \left|\nabla^{m-1} A\right|^2 \leq c \cdot \lambda \cdot t + \sup_{\substack{t=0\\ \{u<0\}}} \lambda \left|\nabla^{m-1} A\right|^2$$

at points where $u < 0$ and $0 < t \leq 1$.

Spatial estimates yield estimates in time:

Lemma 4.3.9. *Let $u\colon \mathbb{R}^{n+1}\times[0,\infty)\to\mathbb{R}$ be a graphical solution to mean curvature flow and $M\geq 1$ such that*

$$|Du(x,t)|\leq M \quad \text{for all } (x,t) \text{ where } \quad u(x,t)\leq 0.$$

Fix any $x_0\in\mathbb{R}^{n+1}$ and $t_1,t_2\geq 0$. If $u(x_0,t_1)\leq -1$ or $u(x_0,t_2)\leq -1$, then $|t_1-t_2|\geq \frac{1}{8(n+1)M^2}$ or

$$\frac{|u(x_0,t_1)-u(x_0,t_2)|}{\sqrt{|t_1-t_2|}}\leq\sqrt{2(n+1)}(M+1).$$

The existence of solutions can be obtained in 2 steps:
The first one was obtained by Huisken in [27]

Lemma 4.3.10 (Existence of approximating solutions)**.** *Let $A\subset\mathbb{R}^{n+1}$ be an open set. Assume that $u_0\colon A\to\mathbb{R}$ is locally Lipschitz continuous and maximal.*

Let $L>0$, $R>0$ and $1\geq\varepsilon>0$. Then there exists a smooth solution $u^L_{\varepsilon,R}$ to

$$\begin{cases}\dot u=\sqrt{1+|Du|^2}\cdot\operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) & \text{in } B_R(0)\times[0,\infty),\\ u=L & \text{on } \partial B_R(0)\times[0,\infty),\\ u(\cdot,0)=\min_\varepsilon\left\{u_{0,\varepsilon},L\right\} & \text{in } B_R(0),\end{cases}$$

where $u_{0,\varepsilon}$ is a standard mollification of u_0. We always assume that $R\geq R_0(L,\varepsilon)$ is so large that $L+1\leq u_{0,\varepsilon}$ on $\partial B_R(0)$.

The previous lemma gives us a family of approximating solutions. We may prove existence by taking limits according to the following lemma

Lemma 4.3.11 (Variation on the Theorem of Arzelà-Ascoli)**.** *Let $B\subset\mathbb{R}^{n+2}$ and $0<\alpha\leq 1$. Let $u_i\colon B\to\mathbb{R}\cup\{\infty\}$ for $i\in\mathbb{N}$. Suppose that there exist strictly decreasing functions r, $-c\colon\mathbb{R}\to\mathbb{R}_+$ such that for each $x\in B$ and $i\geq i_0(a)$ with $u_i(x)\leq a<\infty$ we have*

$$\frac{|u_i(x)-u_i(y)|}{|x-y|^\alpha}\leq c(a) \quad \text{for all} \quad y\in\overline{B_{r(a)}(x)}\cap B.$$

Then there exists a function $u\colon B\to\mathbb{R}\cup\{\infty\}$ such that a subsequence $(u_{i_k})_{k\in\mathbb{N}}$ converges to u locally uniformly in $\Omega:=\{x\in B\colon u(x)<\infty\}$ and $u_{i_k}(x)\to\infty$ for $x\in B\setminus\Omega$. Moreover, for each $x\in\Omega$ with $u(x)\leq a$ we have $B_{r(a+1)}(x)\cap B\subset\Omega$ and

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha}\leq c(a+1) \quad \text{for all} \quad y\in\overline{B_{r(a+1)}(x)}\cap B.$$

We remark that the estimates above imply the existence for infinite time stated in Theorem 4.1.1 (and in particular, the domain $\Omega=\mathbb{R}^n\times(0,\infty)$).

We finish by giving some ideas on the proof the convergence of entire graphs over $\mathbb{R}^n$ with linear growth to a expanding self similar solution.

Let F be an entire graphs over $\mathbb{R}^n$ with linear growth and consider the rescaling

$$\tilde{F} = \frac{F}{\sqrt{2t+1}} \text{ and } s = \frac{1}{2}\log(2t+1).$$

Then $\tilde{F}$ satisfies

$$\frac{\partial \tilde{F}}{\partial s} = \tilde{H} - \tilde{F}.$$

The rescaled estimates are

$$\begin{aligned} \tilde{v}(\tilde{x}, s) \leq & c_1, \\ |\tilde{A}|^2(\tilde{x}, s) \leq & c_2. \end{aligned}$$

As in the previous lecture, we have that the graph converges locally.

Then the result follows by using the maximum principle to prove that

$$\sup_{\tilde{M}_t} \frac{(\tilde{H} + \langle \tilde{x}, \tilde{v} \rangle)^2 \tilde{v}^2}{(1 + \alpha|\tilde{x}|^2)^{1-\varepsilon}} \leq \sup_{\tilde{M}_0} \frac{(H + \langle x, v \rangle)^2 v^2}{(1 + \alpha|\tilde{x}|^2)^{1-\varepsilon}}.$$

The constants α, β and ε need to be choose suitably small.

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