INVARIANT GEOMETRY OF LIE GROUPS

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In this work we study the geometry of Lie groups, with a left invariant metric, analyzing the sectional curvature, Ricci curvature and scalar curvature.

Let G be a Lie group, of dimension n, with a left invariant metric and let $\mathfrak{g}$ its associated Lie algebra. Let $u \in \mathfrak{g}$, following [1], we will show that if $ad(u)$ is skew-adjoint (with respect to the chosen metric) then the sectional curvature of G satisfies

$$k(u, v) \geq 0$$

for all $v \in \mathfrak{g}$, where the equality holds if and only if u is orthogonal to the space $[u, \mathfrak{g}]$. As consequence of this result, one can prove that if u is in the center of the Lie algebra $\mathfrak{g}$, then for all left invariant metric the inequality $k(u, v) \geq 0$ is satisfied for all v.

In the special case of the compact Lie groups, a result of [1] states that it is always possible to choose a left invariant metric (and in fact bi-invariant) such that the sectional curvature satisfies $K \geq 0$. The famous Wallach Theorem shows that examples where $K > 0$ are very scarce.

Besides, we will present a study about the sign Ricci curvature on nilpotent Lie groups and solvable Lie groups, with a left invariant metric. We will see that if G is nilpotent but not commutative, then for any invariant metric there exists a direction of strictly negative Ricci curvature and a direction of strictly positive Ricci curvature. Finally we will study the 3-dimensional case.

References

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