Local convergence analysis of Inexact Newton method with relative residual error tolerance under majorant condition in Riemannian Manifolds

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Abstract

A local convergence analysis of Inexact Newton's method with relative residual error tolerance for finding a singularity of a differentiable vector field defined on a complete Riemannian manifold, based on majorant principle, is presented in this paper. We prove that under local assumptions, the inexact Newton method with a fixed relative residual error tolerance converges Q -linearly to a singularity of the vector field under consideration. Using this result we show that the inexact Newton method to find a zero of an analytic vector field can be implemented with a fixed relative residual error tolerance. In the absence of errors, our analysis retrieve the classical local theorem on the Newton method in Riemannian context.

Keywords: Inexact Newton's method, majorant principle, local convergence analysis, Riemannian manifold.

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Our goal is to prove in Riemannian manifold context the following version of Inexact Newton method with relative residual error tolerance under majorant condition.

Theorem 1. *Let $\mathcal{M}$ be a Riemannian manifold, $\Omega \subseteq \mathcal{M}$ an open set and $X : \Omega \to T\mathcal{M}$ a continuously differentiable vector field. Let $p_* \in \Omega$, $R > 0$ and $\kappa := \sup\{t \in [0, R) : B_t(p_*) \subset \Omega\}$. Suppose that $X(p_*) = 0$, $\nabla X(p_*)$ is invertible and there exists an $f : [0,\ R) \to \mathbb{R}$ continuously differentiable such that*

$$\left\|\nabla X(p_*)^{-1}[P_{\zeta,1,0}\,\nabla X(p) - P_{\zeta,\tau,0}\nabla X(\zeta(\tau))P_{\zeta,1,\tau}]\right\| \leq f'\left(d(p_*, p)\right) - f'\left(\tau d(p_*, p)\right),$$

for all $\tau \in [0, 1]$, $p \in B_\kappa(p_*)$, where $\zeta : [0, 1] \to \mathcal{M}$ is a minimizing geodesic from p_* to p and

h1) *$f(0) = 0$ and $f'(0) = -1$;*

h2) *f' is strictly increasing.*

Let $0 \leq \vartheta < 1/K_{p_*}$, $\nu := \sup\{t \in [0, R) : f'(t) < 0\}$, $\rho := \sup\{\delta \in (0, \nu) : [(1+\vartheta)|t - f(t)/f'(t)|/t + \vartheta] < 1/K_{p_*},\ t \in (0, \delta)\}$ and

$$r := \min\{\kappa,\ \rho,\ r_{p_*}\}.$$

Then the sequence generated by the Inexact Newton method for solving $X(p) = 0$ with starting point $p_0 \in B_r(p_*) \setminus \{p_*\}$ and residual relative error tolerance θ,

$$p_{k+1} = \exp_{p_k}(S_k), \qquad \|X(p_k) + \nabla X(p_k)S_k\| \leq \theta\|X(p_k)\|, \qquad k = 0, 1, \ldots, \tag{1}$$

$$0 \leq cond(\nabla X(p_*))\theta \leq \vartheta / \left[2/|f'(d(p_*, p_0))| - 1\right], \tag{2}$$

is well defined (for any particular choice of each $S_k \in T_{p_k}M$), the sequence $\{p_k\}$ is contained in $B_r(p_*)$ and converges to the point p_* which is the unique zero of X in $B_\sigma(p_*)$, where $\sigma := \sup\{t \in (0, \kappa) : f(t) < 0\}$, and we have that:

$$d(p_*,\ p_{k+1}) \leq K_{p_*}\left[(1+\vartheta)\frac{\left|d(p_*, p_k) - \dfrac{f(d(p_*, p_k))}{f'(d(p_*, p_k))}\right|}{d(p_*, p_k)} + \vartheta\right] d(p_*,\ p_k), \qquad k = 0, 1, \ldots, \tag{3}$$

and $\{p_k\}$ converges linearly to p_*. If, in additional, the function f satisfies the following condition

h3) *f' is convex,*

then there holds

$$d(p_*,\ p_{k+1}) \leq K_{p_*}\left[(1+\vartheta)\frac{\left|d(p_*, p_0) - \dfrac{f(d(p_*, p_0))}{f'(d(p_*, p_0))}\right|}{d^2(p_*, p_0)} d(p_*, p_k) + \vartheta\right] d(p_*,\ p_k), \quad k = 0, 1, \ldots. \tag{4}$$

as a consequence, the sequence $\{p_k\}$ *converges to* p_* *with linear rate as follows*

$$d(p_*,\, p_{k+1}) \leq K_{p_*} \left[(1+\vartheta) \frac{\left| d(p_*, p_0) - \dfrac{f(d(p_*, p_0))}{f'(d(p_*, p_0))} \right|}{d(p_*, p_0)} + \vartheta \right] d(p_*,\, p_k), \quad k = 0, 1, \ldots . \tag{5}$$