A variant of Forward-Backward splitting method for the sum of two monotone operators with a new search strategy

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Abstract

In this paper, we propose variants of Forward-Backward splitting method for finding a zero of the sum of two operators. A classical modification of Forward-Backward method was proposed by Tseng, which is known to converge when the forward and the backward operators are monotone and with Lipschitz continuity of the forward operator. The conceptual algorithm proposed here improves Tseng's method in some instances. The first and main part of our approach, contains an explicit Armijo-type search in the spirit of the extragradient-like methods for variational inequalities. During the iteration process the search performs only one calculation of the forward-backward operator, in each tentative of the step. This achieves a considerable computational saving when the forward-backward operator is computationally expensive. The second part of the scheme consists in special projection steps. The convergence analysis of the proposed scheme is given assuming monotonicity on both operators, without Lipschitz continuity assumption on the forward operator.

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1 Introduction

In this paper, we present a modified method for solving monotone inclusion problems for the sum of two operators. Given the monotone operators, $A : dom(A) \subseteq \mathbb{R}^n \to \mathbb{R}^n$ point-to-point and $B : dom(B) \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ point-to-set, the inclusion problem consists in:

$$\text{Find } \; x \in \mathbb{R}^n \; \text{ such that } \; 0 \in (A + B)(x). \tag{1}$$

The solution set is denoted by $S^* := \{x \in \mathbb{R}^n : 0 \in (A + B)(x)\}$. This problem has recently received a lot attention due, to the fact that many nonlinear problems, arising within applied areas, are mathematically modeled as nonlinear operator equations and/or inclusions, which are decomposed as the sum of two operators.

A classical splitting method for solving problem (1) is the so called Forward-Backward splitting method as proposed in [1]. Assuming that $dom(B) \subseteq dom(A)$, the scheme is given as follows:

$$x^{k+1} = (I + \beta_k B)^{-1}(I - \beta_k A)(x^k), \tag{2}$$

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where $\beta_k > 0$ for all k. The iteration defined by (2) converges when the inverse of the forward mapping is strongly monotone as well as over other undesired assumptions on the stepsize β_k and the operator B; see, for instance, [2] and [4]. An important and promising modification of Scheme (2) was presented by Tseng in [5]. It consists in:

$$J(x^k,\beta_k) = (I+\beta_k B)^{-1}(I-\beta_k A)(x^k) \tag{3}$$

$$x^{k+1} = \mathcal{P}_X\Big(J(x^k,\beta_k) - \beta_k\big[A(J(x^k,\beta_k)) - A(x^k)\big]\Big), \tag{4}$$

where X is a suitable nonempty, closed and convex set, belonging to $dom(A)$. See [5] for more details about the interpretation of (4). The stepsize β_k is chosen to be the largest $\beta \in \{\sigma, \sigma\theta, \sigma\theta^2, \cdots\}$, satisfying:

$$\beta \left\|A\big(J(x^k,\beta)\big) - A(x^k)\right\| \le \delta \left\|J(x^k,\beta) - x^k\right\|, \tag{5}$$

with $\theta, \delta \in (0,1)$ and $\sigma > 0$. The existence of β_k, sufficiently small and satisfying (5), is ensuring by Lipschitz continuity assumption on A assumed in [5]. Note that there exists various choices for the set X. If $dom(B)$ is closed, then the result of Minty in [3], implies that $dom(B)$ is convex, hence we may choose $X = dom(B)$; see [5].

2 The Conceptual Algorithm

Let $A : dom(A) \subseteq \mathbb{R}^n \to \mathbb{R}^n$ and $B : dom(B) \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be two maximal monotone operators, with A point-to-point and B point-to-set. Assume that $dom(B) \subseteq dom(A)$. Choose any nonempty, closed and convex set, $X \subseteq dom(B)$, satisfying $X \cap S^* \neq \emptyset$. Thus, from now on, the solution set, S^*, is nonempty. Also we assume that the operator B satisfies, that for each bounded subset V of $dom(B)$ there exists $R > 0$, such that $B(x) \cap B[0,R] \neq \emptyset$, for all $x \in V$, where $B[0,R]$ is the closed ball centered at 0 with radius R.

Let $\{\beta_k\}_{k=0}^{\infty}$ be a sequence such that $\{\beta_k\} \subseteq [\check{\beta}, \hat{\beta}]$ with $0 < \check{\beta} \le \hat{\beta} < \infty$, and be $\theta, \delta \in (0,1)$. The algorithm is defined as follows:

Conceptual Algorithm

Initialization Step 1: Take

$$x^0 \in X.$$

Iterative Step 1: Given x^k and β_k, compute the forward-backward operator at x^k,

$$J(x^k,\beta_k) := (I+\beta_k B)^{-1}(I-\beta_k A)(x^k). \tag{6}$$

Stop Criteria 1: If $x^k = J(x^k,\beta_k)$ stop.

Inner Loop: Otherwise, begin the inner loop over j.
Put $j = 0$ and chose any $u_j^k \in B\big(\theta^j J(x^k,\beta_k) + (1-\theta^j)x^k\big) \cap B[0,R]$. If

$$\Big\langle A\big(\theta^j J(x^k,\beta_k) + (1-\theta^j)x^k\big) + u_j^k, x^k - J(x^k,\beta_k)\Big\rangle \ge \frac{\delta}{\beta_k}\|x^k - J(x^k,\beta_k)\|^2, \tag{7}$$

then $j(k) = j$ and stop. Else, $j = j+1$.

Iterative Step 2: Set

$$\alpha_k := \theta^{j(k)}, \tag{8}$$

$$\bar{u}^k = u^k_{j(k)}, \tag{9}$$

$$\bar{x}^k := \alpha_k J(x^k, \beta_k) + (1-\alpha_k)x^k \tag{10}$$

and

$$x^{k+1} := \mathcal{F}(x^k). \tag{11}$$

Stop Criteria 2: If $x^{k+1} = x^k$ then stop.

Now we consider three variants on this conceptual algorithm. The difference is given by the definition of the procedure $\mathcal{F}$ in (11).

$$\mathcal{F}_1(x^k) = \mathcal{P}_X\big(\mathcal{P}_{H(\bar{x}^k,\bar{u}^k)}(x^k)\big); \tag{12}$$

$$\mathcal{F}_2(x^k) = \mathcal{P}_{X\cap H(\bar{x}^k,\bar{u}^k)}(x^k); \tag{13}$$

$$\mathcal{F}_3(x^k) = \mathcal{P}_{X\cap H(\bar{x}^k,\bar{u}^k)\cap W(x^k)}(x^0); \tag{14}$$

where

$$H(x,u) := \big\{y \in \mathbb{R}^n : \langle A(x) + u, y - x\rangle \le 0\big\} \tag{15}$$

and

$$W(x) := \big\{y \in \mathbb{R}^n : \langle y - x, x^0 - x\rangle \le 0\big\}. \tag{16}$$

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