Approximation to impulsive optimal control problems using the Euler's discretization

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1 Introduction

We extend the theory of consistent approximations, introduced by [2] for conventional optimal control problems, to impulsive optimal control problems.

2 Consistent Approximations

Let B be a normed space. Consider the problem

$$(P) \quad \min_{x \in X} f(x),$$

where $f : B \to \mathbb{R}$ is continuous and $X \subset B$.

Let $\mathcal{N}$ be an ordered infinite subset of $\mathbb{N}$ and let $\{B_N\}_{N \in \mathcal{N}}$ be a family of finite dimensional subspaces of B such that $B_{N_1} \subset B_{N_2}$ if $N_1 < N_2$ and $\cup B_N$ is dense in B. For all $N \in \mathcal{N}$, let $f_N : B_N \to \mathbb{R}$ be a continous function that aproximates $f(\cdot)$ on B_N, and $X_N \subset B_N$ be an aproximation to X. Consider the family of approximate problems

$$(P_N) \quad \min_{x \in X_N} f_N(x), \;\; N \in \mathcal{N}.$$

Definition 1. *Let B be a normed space, $\{B_N\}_N$ a sequence of subspaces of B that has finite dimension, whose $\cup B_N$ is dense in B and consider the problems (P) and (P_N).*
(i) We say that P_N epi-converges to P if the epigraphs E_N converge to the epigraph E.
(ii) We say that the upper semicontinuous functions $\gamma_N : X_N \to \mathbb{R}$ ($\gamma : X \to \mathbb{R}$) are optimality functions for P_N (P) if they cancel in local minimizers of P_N (P) and $\gamma_N(x_N) \leq 0, \ \forall N \in \mathcal{N}$ ($\gamma(x) \leq 0$).
(iii) We will say that the pairs (P_N, γ_N), in the sequence $\{P_N, \gamma_N\}$ are consistent approximations for the pair (P, γ) if P_N epi-converges to P and for all sequence $\{x_N\}$ such that $x_N \in X_N$ and $x_N \to x \in X$, then $\overline{\lim}\gamma_N(x_N) \leq \gamma(x)$.

3 The impulsive optimal control problem

$$(P) \quad \begin{array}{ll} \min & f^0(x,\Omega) = f^0(x(0), x(T)) \\ & dx = f(x,u)dt + g(x,v)d\Omega, t \in [0,T] \\ & x(0) \in C \end{array}$$

Here $f^0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^r \to \mathcal{M}_{n \times q}$, $\mathcal{M}_{n \times q}$ is the space of $n \times q$ real entries matrix space, $C \subset \mathbb{R}^n \times \mathbb{R}^n$ is a closed, convex set, the functions $u : [0,T] \to \mathbb{R}^m$, $v : [0,T] \to \mathbb{R}^r$ are Borel mensurable and essentially bounded, $d\Omega$ is impulsive control that consists $\Omega := (\mu, \nu, \{v_{t_i}, \psi_{t_i}\})$, where μ is a Borel's measure defined over $[0,T]$ taking values in a closed, convex cone $K \subset \mathbb{R}^q$. The second component of Ω belongs to the set of scalar-valued nonnegative Borel's measures, denoted by $V(\mu)$, such that $\exists\, \mu_N : [0,T] \to K$ and $(\mu_N, |\mu_N|) \to^* (\mu, \nu)$. Here, the symbol $*$ means the convergence in the weak* topology. The functions $v_{t_i} : [0,1] \to \mathbb{R}^r$ are families of Borel measurable functions and essencially bounded with respected to Lebesgue's measure, that are related to the atoms of the measure μ, i.e., $\{v_{t_i}\}_{i\in\mathcal{I}}$, where $\Theta := \{t_i \in [0,T] : \mu(t_i) \neq 0\}$, $\mathcal{I}$ is the set atomic indices, and $\mu(t)$ is the measure vectorial value on K. The functions $\psi_{t_i} : [0,1] \to K$ are measurable and satisfy, for all $t_i \in \Theta$,

$$\begin{array}{ll} (i) & \sum_{j=1}^{q} \|\psi_{t_i}^j(\sigma)\| = |\nu|(t_i) \text{ q.s. } \sigma \in [0,1]; \\ (ii) & \int_0^1 \psi_{t_i}^j(s)ds = \mu^j(t_i), \quad j = 1,2,\ldots,q. \end{array}$$

According [4], the equation

$$dx = f(x,u)dt + g(x,v)d\Omega, t \in [0,T]. \tag{1}$$

can be reparametrized, in order to get the reparametrized problem

$$(P_{rep}) \ \min_{\eta \in S_C} f^0(y^\eta(0), y^\eta(1)),$$

such that $y : [0,1] \to \mathbb{R}^n$ satisfies the following theorem.

Theorem 1. *[3] Suppose Ω is given and x_ϑ is solution of (1). Then, $y_\vartheta = y$ is a reparametrized solution of (1) if only if x_ϑ is a solution of (1).*

Let

$$S_C := C \times L_{\infty,2}^m[0,1] \times L_{\infty,2}^r[0,1] \times \mathcal{P},$$

with $\mathcal{P}$ the set of $p := (\mu, \nu, \psi_{t_i})$ satisfying the assumptions of the problem (P). We shall represent by $y^\eta(s)$ the solution for the reparametrized system obtained by [4], for all η.

We obtain the following reparametrized problem

$$(P_{rep}) \ \min_{\eta \in S_C} f^0(y^\eta(0), y^\eta(1)).$$

Note that (P) and (P_{rep}) have the same solution, up to reparametrization, since the objective functions are the same. So, we shall use the consistent approximation theory in (P_{rep}).

Define the operator $Ext : [0,1] \to \mathbb{R}^l$ by

$$Ext[h,\Omega](s) = \begin{cases} h((\mathcal{X}_{t_i}(\alpha_{t_i}(s)), x(t_i-)), \bar{v}(s) \text{ se } t_i \in \Theta, s \in I_i \\ h(x(\gamma(s)), \bar{v}(s)) \text{ otherwise} \end{cases}$$

and consider $p_1 = (\mu_1, \nu_1, \psi_{t_i}^1)$, $p_2 = (\mu_2, \nu_2, \psi_{t_i}^2) \in \mathcal{P}$. Let ζ_j be solutions of system

$$d\zeta_j = d\Omega_j, \ \zeta_j(0) = 0, \ j = 1,2,$$

and let d_3 be the metric given in [1] by

$$\begin{aligned} d_3(p_1,p_2) = {} & |\nu_1([0,T]) - \nu_2([0,T])| \\ & + \int_0^T |F_1(t,\nu_1) - F_2(t,\nu_2)|dt \\ & + \max_{s\in[0,1]} \|Ext[\zeta_1(\cdot),\Omega_1](s) - Ext[\zeta_2(\cdot),\Omega_2](s)\|. \end{aligned}$$

Note that $S_C \subset \mathbb{R}^n \times \mathbb{R}^n \times L_{\infty,2}^m[0,1] \times L_{\infty,2}^r[0,1] \times \mathcal{P} =: B$ and in B we have the metric

$$d = d_1 + d_1 + d_2 + \hat{d}_2 + d_3,$$

where $d_1, d_2, \hat{d}_2$ and d_3 are metrics respectively of $\mathbb{R}^n$, L_2^m, L_2^r and $\mathcal{P}$.

Define

$$\mathcal{N} = \{2^k\}_{k=0}^\infty \text{ and } S_N = C_N \times L_N^m \times L_N^r \times \mathcal{P}_N,$$

where $C_N = \mathbb{R}^n$, $\forall N \in \mathcal{N}$,

$$L_N^m := \{u \in L_{\infty,2}^m[0,1]; u(s) = \textstyle\sum_{k=0}^{N-1} u_k \pi_{N,k}(s)\}, \ \ L_N^r := \{v \in L_{\infty,2}^r[0,1]; v(s) = \textstyle\sum_{k=0}^{N-1} v_k \pi_{N,k}(s)\}$$

$$\pi_{N,k}(t) := \begin{cases} 1 \; \forall \; s \in [k/N, (k+1)/N[, \text{ se } \; k \leq N-2 \\ 1 \; \forall \; s \in [k/N, (k+1)/N], \text{ se } \; k = N-1 \\ 0, \text{ otherwise,} \end{cases}$$

with $u_k \in \mathbb{R}^m$, $v_k \in \mathbb{R}^r$.

Note that $\cup L_N^m$ and $\cup L_N^r$ are dense in $L_{\infty,2}^m[0,1]$ and $L_{\infty,2}^r[0,1]$, respectively.

Let $\mathcal{P}_N$ be the sets given by

$$\mathcal{P}_N := \{(\mu_N, \nu_N, 0); \mu_N \in \mathcal{Z}_N \text{ e } \nu_N := |\mu_N|\},$$

where $|\mu_N|$ is the total variation of μ_N and

$$\mathcal{Z}_N := \{\mu_N : [0,T] \to K; \mu_N([0,t]) = \textstyle\sum_{j=0}^{N-1} \bar{\pi}_{N,j}(t)\},$$

with

$$\bar{\pi}_{N,j}(t) := \begin{cases} b_j + \frac{t-\bar{t}_j}{\bar{t}_{j+1}-\bar{t}_j}(b_{j+1} - b_j), \forall \; t \in [\bar{t}_j, \bar{t}_{j+1}], \\ \quad j = 0, ..., N-1, 0 = \bar{t}_0 < ... < \bar{t}_N = T \\ 0, \text{ otherwise.} \end{cases}$$

for $b_j \in K$ and $\mu_N : [0,T] \to K$, since K is convex. See that the measures μ_N are absolutely continuous.

Theorem 2. $S_{C,N} \to^{\mathcal{N}} S_C$, $N \to \infty$.

We can discretize the reparametrized equation, given by [4], using Euler's discretization

$$\begin{aligned} y_N^\eta(\tfrac{k+1}{N}) - y_N^\eta(\tfrac{k}{N}) &= f\left(y_N^\eta(\tfrac{k}{N}), u(\tfrac{k}{N})\right)\left(\gamma_N^\eta(\tfrac{k+1}{N}) - \gamma_N^\eta(\tfrac{k}{N})\right) \\ &\quad + g\left(y_N^\eta(\tfrac{k}{N}), v(\tfrac{k}{N})\right)\left(\phi_N^\eta(\tfrac{k+1}{N}) - \phi_N^\eta(\tfrac{k}{N})\right), \\ k = 0, ..., N-1, \; y_N^\eta(0) &= y(0) \text{ e } y_N^\eta(1) = y(1). \end{aligned}$$

in order to get the approximation problems

$$(P_{rep}^{C,N}) \quad \min_{\eta \in S_{C,N}} f_N^0(y_N^\eta(0), y_N^\eta(1)),$$

with $f_N^0(y_N^\eta(0), y_N^\eta(1)) := f^0(y_N^\eta(0), y_N^\eta(1))$.

Theorem 3. *The functions*

$$\gamma_{rep}(\eta) := \min_{\bar{\eta} \in S_C} \left(\langle \nabla f^0(\xi), \bar{\xi} - \xi \rangle + \frac{1}{2}\left[d(\bar{\eta}, \eta)\right]^2 \right), \gamma_{rep}^{C,N}(\eta) = \min_{\bar{\eta} \in S_{C,N}} \left(\langle \nabla f_N^0(\xi_N), \bar{\xi}_N - \xi_N \rangle + \frac{1}{2}\left[d(\bar{\eta}, \eta)\right]^2 \right),$$

with $\xi := (y^\eta(0), y^\eta(1))$, $\bar{\xi} := (y^{\bar{\eta}}(0), y^{\bar{\eta}}(1))$ *and* $\xi_N := (y_N^\eta(0), y_N^\eta(1))$, $\bar{\xi}_N := (y_N^{\bar{\eta}}(0), y_N^{\bar{\eta}}(1))$, *are optimality functions for the problems* (P_{rep}) *e* $(P_{rep}^{C,N})$, *respectively.*

Theorem 4. $\{(P_{rep}^{C,N}, \gamma_{rep}^{C,N})\}_{N \in \mathcal{N}}$ *is a sequence of consistent approximations to the pair* (P_{rep}, γ_{rep}).

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