An Alexandrov–Fenchel-Type inequality in $\mathbb{S}^n$

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Resumo/Abstract:
We use the inverse mean curvature flow to prove an Alexandrov–Fenchel-type inequality for convex hypersurfaces in the sphere $\mathbb{S}^n$, $n \geq 3$. This is a joint work with F. Girão.

In $\mathbb{R}^n$ we have the following result ([5], [1] and [2]):

Theorem 0.1. *If $\Sigma \subset \mathbb{R}^n$ is a star-shaped and strictly mean convex hypersurface then*

$$c_n \int_\Sigma H d\Sigma \geqslant \frac{1}{2}\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}, \tag{0.1}$$

where A is the area of Σ, H is its mean curvature, $c_n = (2(n-1)\omega_{n-1})^{-1}$, and ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

Only in 2013, Dahl, Gicquaud and Sakovich [3], when trying to prove the Penrose inequality for hyperbolic graphs, conjectured the hyperbolic analogous, which was proved by de Lima and Girão in [4].

We say that a closed, embedded hypersurface $\Sigma \subset \mathbb{H}^n$ is star-shaped if it can be written as a radial graph over a geodesic sphere centered at the origin.

Theorem 0.2 (de Lima – Girão [4])**.** *If $\Sigma \subset \mathbb{H}^n$, $n \geq 3$, is a star-shaped and strictly mean convex hypersurface then*

$$c_n \int_\Sigma \rho H d\Sigma \geqslant \frac{1}{2}\left(\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \left(\frac{A}{\omega_{n-1}}\right)^{\frac{n}{n-1}}\right), \tag{0.2}$$

where A, H, c_n and ω_{n-1} are as in the previous theorem and $\rho(r) = \cosh(r)$, with r being the geodesic distance to the origin. Moreover, the equality holds if and only if Σ is a geodesic sphere centered at the origin.

Given a point $q \in \mathbb{S}^n$ and an embedded hypersurface $\Sigma \subset \mathbb{S}^n$ we consider the quantity

$$J(\Sigma) = \left| \int_\Sigma p \, d\Sigma \right|, \tag{0.3}$$

where $p = \langle D\rho, \xi \rangle$, ξ being the unit normal to Σ, and $\rho(r) = \cos r$, r being the geodesic distance to q.

Theorem 0.3. *Let $\Sigma \subset \mathbb{S}^n$, $n \geq 3$, be a closed, strictly convex hypersurface. Then there exists a point $q \in \mathbb{S}^n$, depending on Σ, such that*

$$\int_\Sigma \rho H d\Sigma \geq (n-1)J(\Sigma)\left(\frac{A}{\omega_{n-1}}\right)^{-\frac{n}{n-1}}\left[\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} - \left(\frac{A}{\omega_{n-1}}\right)^{\frac{n}{n-1}}\right], \quad (0.4)$$

where A, H and ω_{n-1} are as in the previous theorem, $\rho(r) = \cos r$, with r being the geodesic distance to q, and $J(\Sigma)$ is given by (0.3). Moreover, if the equality holds, then Σ is a geodesic sphere centered at q.

References

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