CHARACTERIZATION OF CURVES THAT LIE ON A SURFACE IN EUCLIDEAN SPACE

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Resumo/Abstract: In this work we are interested in the solution of the following problem: given a surface Σ in Euclidean space E^3, i.e. R^3 equipped with the standard metric, how can we characterize those (spatial) curves $\alpha : I \to E^3$ that belong to Σ?

Despite the simplicity to formulate the problem, a global understanding is only available for a few examples: when Σ is a plane [5], a sphere [5, 6] or a cylinder [4]. The solution for planar curves is quite easy once we introduce the Frenet frame. On the other hand, the characterization of spherical curves involves an ODE relating the curvature and torsion [5, 6], while the solution for cylindrical curves involves a system of algebro-differential equations [4].

In the 70's, through the idea of equipping a curve with a *relatively parallel moving frame* [1], Bishop was able to characterize spherical curves by using a simple algebraic equation involving the coefficients of the new moving frame. Indeed, if $\{\mathbf{t}, \mathbf{n_1}, \mathbf{n_2}\}$ is an orthonormal moving frame along $\alpha : I \to E^3$ such that

$$\mathbf{t}'(\mathbf{s}) = \kappa_{\mathbf{1}}(\mathbf{s})\mathbf{n_1}(\mathbf{s}) + \kappa_{\mathbf{2}}(\mathbf{s})\mathbf{n_2}(\mathbf{s});\ \mathbf{n_i}'(\mathbf{s}) = -\kappa_{\mathbf{i}}(\mathbf{s})\mathbf{t}(\mathbf{s}), \qquad (1)$$

where $\mathbf{t}(\mathbf{s})$ is the unit tangent vector and s the arc-length, the characterization reads *"α is a spherical curve if and only if* $(\kappa_1(s), \kappa_2(s))$ *lies on a line not passing through the origin"* [1].

By adapting Bishop's technique, we are able to formulate an analogous theorem for surfaces implicitly defined by a smooth function, $\Sigma = F^{-1}(c)$. The main idea consists in equipping R^3 with a pseudo-metric given by $(\cdot, \cdot) = \langle \mathrm{Hess}F\cdot, \cdot\rangle$, i.e., we turn R^3 into a Hessian manifold. In other words, the new inner product should be used to define the orthogonality of a moving frame instead of the standard (euclidean) one. Intuitively, we deform the space R^3 in such a way that the surface Σ becomes a sphere and then follow the steps in

Bishop's proof. However, in order to achieve this goal, one is naturally led to the study of the geometry of Lorentz-Minkowski spaces, E^3_ν [2], since $\langle \mathrm{Hess} F \cdot, \cdot \rangle$ may have a non-zero index ν. This study present some difficulties due to the many possibilities for the casual character of a curve $\beta : I \to E^3_\nu$, i.e., if $(\beta', \beta') > 0$, $= 0$ or < 0.

In this work, we generalize the known results for the existence of Bishop frames in E^3_1 [3] by taking into account the cases where $\mathbf{t}$ or $\mathbf{t}'$ are lightlike vectors and, through the study of the normal plane to $\mathbf{t}$, equipped with the structure inherited from E^3_1, we also interpret the (possible) transition of casual character of $\mathbf{t}$ or $\mathbf{t}'$. Moreover, since one has for the normal curvature in $\Sigma = F^{-1}(c) \subset E^3$ the relation $\kappa_n(p, \mathbf{u}) = \langle \mathrm{Hess}_{\mathbf{p}} F \, \mathbf{u}, \mathbf{u} \rangle / \|\nabla_{\mathbf{p}} F\|$, we can interpret the role of the casual character when passing from E^3 to E^3_1. Finally, we use these results to present a Bishop-like necessary and sufficient characterization of curves that lie on a non-degenerate quadric and a necessary condition for a curve to lie on a non-degenerate level surface $\Sigma = F^{-1}(c)$, i.e. when $\det \mathrm{Hess}_p F \neq 0$.

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