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Abstract

Supermodular functions are widely used in economics to model complementarity. For example, a firm's production function is supermodular if the marginal productivity of each factor increases with the usage of other factors. This in turn guarantees that when the price of a factor falls, the firm's demand for all factors increase. We generalise the notion of supermodular functions so the concept is also applicable to correspondences. Supermodular correspondences arise naturally in a variety of settings. To illustrate the use of the concept and our results, we apply them to study, amongst other things, the optimising behaviour of firms producing multiple output goods and of agents with ambiguity aversion.

Keywords: supermodular correspondence, monotone comparative statics, multi-output production, ambiguity aversion

JEL Classification: C61, D21, D24

1 Introduction

Consider the problem of a profit-maximising firm that uses I inputs to produce J outputs. We denote its production possibility set by $P \subseteq \mathbb{R}^I_+ \times \mathbb{R}^J_+$, where the element (x, y) is in P

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if the firm can produce the output profile $y \in \mathbb{R}^J_+$ with the input vector $x \in \mathbb{R}^I_+$.[1] Suppose the firm faces prices $p \in \mathbb{R}^I_{++}$ and $q \in \mathbb{R}^J_{++}$ for its inputs and outputs, respectively. Its objective is to choose a production plan that maximises its profit, i.e.,

$$\max \Big\{ q \cdot y - p \cdot x : (x, y) \in P \Big\}.$$

How does the solution to this problem vary with input and output prices? In particular, when can we guarantee that inputs are *complements* in the sense that lowering the price of one input raises the firm's demand for *all* inputs?

In the case of single output technologies the answer to the latter question is well-known: inputs are complements whenever the production function $f : \mathbb{R}^I_+ \to \mathbb{R}$ is supermodular (see Topkis, 1978, Milgrom and Roberts, 1990, or Milgrom and Shannon, 1994).[2] Roughly speaking, supermodularity requires that the marginal productivity of every input increases with respect to the remaining factors of production.

Whenever a firm is manufacturing multiple output goods, we can model its production possibilities by a correspondence $\Gamma : \mathbb{R}^I_+ \to \mathbb{R}^J_+$, where $\Gamma(x)$ denotes the set of all output profiles that are feasible at the input vector x. In other words, we have $y \in \Gamma(x)$ if and only if (x, y) is in P. Given x, the firm's decision on what mix of goods to produce will depend on the output prices q. This allows us to characterise the firm's problem by first defining the maximal revenue function f, where

$$f(x) := \max \big\{ q \cdot y : y \in \Gamma(x) \big\}.$$

Since the firm's profit maximisation is equivalent to maximising $f(x) - p \cdot x$ with respect to $x \in \mathbb{R}^I_+$, this problem is formally identical to the one of a single output firm. Therefore, we know that inputs are complements so long as the revenue function f is supermodular. This raises a question which is one of the principal motivations for this paper: what conditions on Γ will guarantee that the revenue function f is supermodular?

For another important motivation, consider a firm that makes its production decision under uncertainty. Observing the input prices $p \in \mathbb{R}^\ell_{++}$, the firm chooses an input vector

[1] Our description of the production possibility set is somewhat non-standard. It would be more usual to use negative entries for inputs and positive entries for outputs. However, the above approach is more convenient for our purposes. Since we pre-assign some goods as inputs and others as outputs, the technology does *not* allow the firm to switch between using a good as an input and producing it as an output in response to prevailing prices. This assumption seems reasonable in most applications.

[2] Specifically, see Theorem 6.1 in Topkis (1978), Theorem 5 in Milgrom and Roberts (1990), as well as Theorems 4 and 10 in Milgrom and Shannon (1994). This material is also surveyed Topkis (1998).

$x \in \mathbb{R}^\ell_+$ and implements a production plan based on x. We assume that the firm is uncertain of its future revenues when those decisions are taken. However, it formulates subjective beliefs over the possible revenue realisations, conditional on x. The firm's multi-prior belief is summarised by a subset $\Gamma(x)$ of probability distributions on $\mathbb{R}_+$. Assuming that the firm is ambiguity averse in the sense of Gilboa and Schmeidler (1989), it chooses an input vector x in order to maximise

$$\min \left\{ \int_S u(s) d\mu(s) : \mu \in \Gamma(x) \right\} - p \cdot x.$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is the firm's Bernoulli index. Clearly, the inputs are complements whenever the function $f : \mathbb{R}^\ell_+ \to \mathbb{R}$, given by

$$f(x) := \min \left\{ \int_S u(s)\ d\mu(s) : \mu \in \Gamma(x) \right\},$$

is supermodular. What conditions on Γ will guarantee this property?

The two problems we have highlighted share a similar structure. In both cases, the function f is the extremum of a linear functional over values of some correspondence Γ. In this paper we characterise those properties of correspondences that lead to the supermodularity of their associated value functions. As we demonstrate in this paper, such a characterisation finds wide application across problems where determining monotone comparative statics is of interest.

In Section 2 we formally introduce the notion of a *supermodular correspondence*. We require the domain of such a mapping to be a lattice X, while its values are subsets of a real vector space Y endowed with a preorder. While our theory holds in this general setting, in our applications, the domain of such a correspondence is typically a Euclidean space endowed with the coordinate-wise partial order. This is mapped into either (i) another Euclidean space endowed with the the same order or (ii) the set of probability distributions, which is contained in the ordered vector space of finite signed measures ranked by first order stochastic dominance. We show in Section 3 that for any supermodular correspondence $\Gamma : X \to Y$, its upper envelope $\sup \{\phi(y) : y \in \Gamma(x)\}$ and its lower envelope $\inf \{\phi(y) : y \in \Gamma(x)\}$ are both supermodular functions of x, for any positive linear functional $\phi : Y \to \mathbb{R}$. Moreover, under some regularity conditions, supermodularity of the correspondence is also necessary for this property to be satisfied. Applications of our general results are discussed in Section 4, with the first motivating example covered in Application 1 and the second in Application 5.

2 Basic concepts

A binary relation on a set X is a *partial order* on X if it is reflexive, transitive and antisymmetric. A *partially ordered set*, or simply a *poset*, is a pair $(X, \geq_X)$ consisting of a set X and a partial order $\geq_X$ on X. We denote the strict counterpart of $\geq_X$ by $>_X$, that is, for any x and x' in X, we have $x' >_X x$, if $x' \geq_X x$ and $x' \ngeq_X x$. Whenever it causes no confusion, we abbreviate our notation by denoting $(X, \geq_X)$ with X.

For any two elements x, x' of a poset X, their *least upper bound*, or the *join*, is denoted by $x \vee x'$, while their *greatest lower bound*, or the *meet*, by $x \wedge x'$, where both elements are defined with respect to the partial order $\geq_X$. A poset X is a *lattice* if, for any x, x' in X, both their join $x \vee x'$ and their meet $x \wedge x'$ belong to the set. A poset Y is a *sublattice* of X if it is a subset of X that contains elements $y \vee y'$, $y \wedge y'$, for any y, y' in Y.

For the purposes of this paper, the most important lattice is the Euclidean space $(\mathbb{R}^\ell, \geq)$, where we say that $x \geq y$ if $x_i \geq y_i$ for all $i = 1, 2, \ldots, \ell$. In this case, for vectors x and y in $\mathbb{R}^\ell$, we have $(x \vee y)_i = \max\{x_i, y_i\}$ and $(x \wedge y)_i = \min\{x_i, y_i\}$.

A function $f : X \to \mathbb{R}$ defined over a lattice X is *supermodular* whenever, for any elements x, x' in X, we have $f(x \wedge x') + f(x \vee x') \geq f(x) + f(x')$. We say that f is *submodular* if and only if $-f$ is supermodular.

A binary relation on a set X is a *preorder* if it is reflexive and transitive. Our generalisation of supermodularity applies to correspondences that map a lattice to what we shall call an *ordered vector space*; this refers to a real vector space endowed with a preorder that is preserved by the vector space operations. In other words $(Y, \geq_Y)$ is an ordered vector space whenever Y is a vector space and $\geq_Y$ is a preorder satisfying the following properties: if $x \geq_Y y$ then $x + z \geq_Y y + z$ and $\alpha x \geq \alpha y$, for any x, y, z in Y and $\alpha \geq 0$. Clearly, the Euclidean space is an ordered vector space. Another important example is the space of signed finite measures defined on a partially ordered measurable space $(S, \mathcal{S})$. This is a real vector space which contains, crucially for our purposes, the set of probability measures. Since S is partially ordered, the signed measures can be ranked with respect to

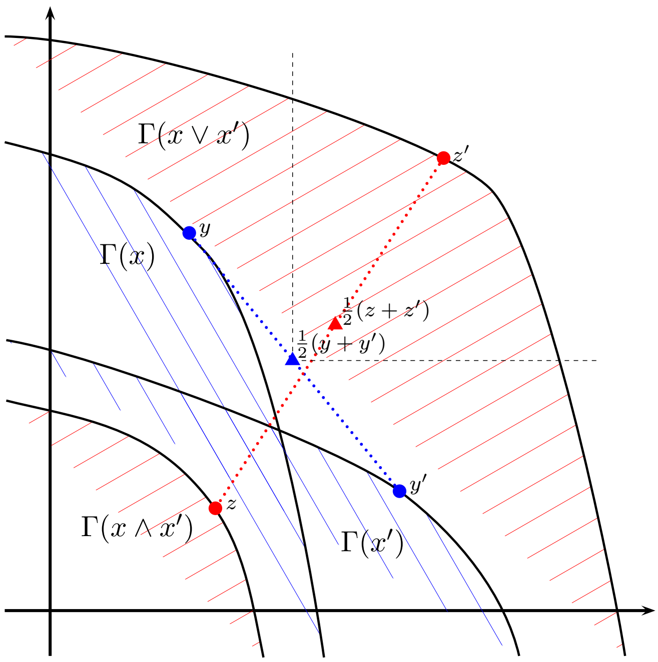

Figure 1: An upper supermodular correspondence for $Y = \mathbb{R}^2$.

first order stochastic dominance, i.e., for any measures μ and ν in Y, we have $\mu \geq_Y \nu$ if $\int_S f d\mu \geq \int_S f d\nu$, for any bounded and measurable function $f : S \to \mathbb{R}$ that is increasing on S with respect to the corresponding partial order.

2.1 Upper and lower supermodularity

Suppose that $(X, \geq_X)$ is a lattice and $(Y, \geq_Y)$ is an ordered vector space. A correspondence $\Gamma : X \to Y$ is said to be *upper supermodular* if for any two elements x, x' in X and $y \in \Gamma(x)$ and $y' \in \Gamma(x')$, there is some $z \in \Gamma(x \wedge x')$ and $z' \in \Gamma(x \vee x')$ such that

$$z + z' \geq_Y y + y'. \tag{1}$$

Equivalently, the vectors need to satisfy $(z + z')/2 \geq_Y (y + y')/2$. See Figure 1 for a graphical interpretation of upper supermodularity. A correspondence $\Gamma : X \to Y$ is *lower supermodular* if for any two elements x, x' in X and $z \in \Gamma(x \wedge x')$, $z' \in \Gamma(x \vee x')$ there are some vectors $y \in \Gamma(x)$ and $y' \in \Gamma(x')$ that satisfy condition (1).[3] Finally, we say that the correspondence is *supermodular* once it is both upper and lower supermodular.

[3]Notice that, the distinction between upper and lower supermodularity disappears if Γ is a function, i.e., Γ is singleton-valued, rather than a set-valued correspondence.

Analogously, the correspondence Γ is *upper submodular* if for any x, x' in X and $y \in \Gamma(x)$, $y' \in \Gamma(x')$, there is some $z \in \Gamma(x \wedge x')$ and $z' \in \Gamma(x \vee x')$ such that (1) holds with the inequality reversed; equivalently, Γ is submodular if $-\Gamma$ is upper supermodular. Similarly, one may define lower submodularity and submodularity.

Clearly, our definition of supermodular correspondences generalises the familiar notion of supermodularity applied to real-valued functions, introduced at the beginning of this section. This notion also extends the concept of *stochastic supermodularity* introduced in Topkis (1968) to correspondences;[4] a function mapping a lattice to the set of probability measures on some measurable space is said to be stochastically supermodular if condition (1) holds with $\geq_Y$ representing first order stochastic dominance.

Suppose that $\Gamma : X \to Y$ has *downward comprehensive* values, i.e., $y \in \Gamma(x)$ and $y \geq_Y y'$ implies $y' \in \Gamma(x)$, for any y, y' in Y and x in X. In this case, it is straightforward to show that Γ is upper supermodular if and only if

$$\Gamma(x \wedge x') + \Gamma(x \vee x') \supseteq \Gamma(x) + \Gamma(x') \text{ for any } x, x' \in X. \tag{2}$$

The fact that (2) implies upper supermodularity is clear and does not even require the downward comprehensiveness of Γ. Conversely, if Γ is upper supermodular, then for any $y \in \Gamma(x)$ and $y' \in \Gamma(x')$ there is some $z \in \Gamma(x \wedge x')$ and $z' \in \Gamma(x \vee x')$ such that $z + z' \geq_Y y + y'$. Hence, $z \geq_Y y + y' - z'$ and, since Γ is downward comprehensive, $(y + y' - z) \in \Gamma(x \wedge x')$. Consequently, $y + y' = (y + y' - z) + z \in \Gamma(x \wedge x') + \Gamma(x \vee x')$.

A special case of property (2) appears in the study of cooperative games with non-transferable utility. In that context, set X is interpreted as the collection of non-empty coalitions of a finite set N of players in a game, endowed with is the set inclusion order $\geq_X$, which makes X a lattice. For any coalition x, set $\Gamma(x) \subseteq \mathbb{R}^N$ consists of utility profiles (across all players in the game) that could result from the formation of that coalition. The game is said to be *cardinally convex* if (2) holds (see Sharkey, 1981, Section 2).

Recall that in the Introduction, we highlighted two problems that we claim motivate the concept of supermodular correspondences. In the first case, a multi-product firm has production possibilities captured by a map from an input vector $x \in \mathbb{R}^I_+$ to $\Gamma(x) \subseteq \mathbb{R}^J_+$, where $\Gamma(x)$ is interpreted as the set of all output combinations that are feasible given

[4]Though Topkis (1968) refers to this property as *stochastic convexity*, the term *stochastic supermodularity* is more commonly used; see Amir (2002) or Balbus, Reffett, and Woźny (2014) for applications.

input x. In this case, we have $X = \mathbb{R}^I_+$ and $Y = \mathbb{R}^J$, were both spaces are endowed with the coordinate-wise ordering. As we show in Section 4, the upper supermodularity of Γ is sufficient to guarantee that inputs are complements. In the second problem, an ambiguity averse firm chooses an input vector x in $X = \mathbb{R}^I_+$, where each x is associated a set $\Gamma(x)$ of revenue distributions. We show in Section 4 that the lower supermodularity of Γ is sufficient to guarantee input complementarity, provided $\geq_Y$ is chosen to be the ranking based on first order stochastic dominance.

2.2 Examples of supermodular correspondences

The following is a list of simple ways to construct supermodular correspondences.

Example 1. Suppose that function $f_i : X \to \mathbb{R}$ is supermodular over a lattice X, for all $i = 1, \dots, \ell$. The map $F : X \to \mathbb{R}^\ell$, given by $F(x) := \big(f_1(x), \dots, f_n(x)\big)$ is a supermodular function, i.e., we have $F(x \wedge x') + F(x \vee x') \geq F(x) + F(x')$, for all x, x' in X, where $\geq$ denotes the coordinate-wise partial order on $\mathbb{R}^\ell$.

Example 2. Consider correspondence $\Gamma_i : X_i \to Y$, where $X_i \subseteq \mathbb{R}$ and Y is an ordered vector space, for $i = 1, 2$. The map $\Lambda : X_1 \times X_2 \to Y$, where $\Lambda(x_1, x_2) := \Gamma_1(x_1) + \Gamma_2(x_2)$, is a supermodular correspondence (in fact, it is also submodular).

Example 3. For any subset Z of an ordered vector space Y, a supermodular function $f : X \to Y$ over a lattice X, and positive scalars α and β, the mapping $\Gamma : X \to Y$, given by $\Gamma(x) := \big\{\alpha y + \beta f(x) : y \in Z\big\}$, is a supermodular correspondence.

Suppose that Y is the space of finite signed measures endowed with the first order stochastic dominance order. Whenever $Z \subset Y$ is a set of probability measures, while $f(x)$ is a probability measure for all $x \in X$, then for any scalars α and β such that $\alpha + \beta = 1$, set $\Gamma(x)$ is a subset of probability measures. This provides an easy way of constructing supermodular correspondences that map a lattice to the space of probability measures.

Example 4. Let Z be a convex subset of an ordered vector space Y, such that $z \geq 0$, for all $z \in Z$. For any supermodular function $f : X \to \mathbb{R}_+$ over a lattice X, the correspondence $\Gamma : X \to Y$ given by $\Gamma(x) := \big\{f(x)z : z \in Z\big\}$ is supermodular.

This claim requires a short proof. Since Z is convex and non-negative, Lemma 5.27 in Aliprantis and Border (2006) guarantees that $\alpha Z + \beta Z = (\alpha + \beta)Z$, for any positive scalars α and β. To show that Γ is upper supermodular, take any $f(x)y \in \Gamma(x)$ and $f(x')y' \in \Gamma(x')$. Given the above property of set Z, there is some vector $v \in Z$ such that $f(x)y + f(x')y' = \big[f(x) + f(x')\big]v$. Moreover, by supermodularity of function f,

$$\big[f(x) + f(x')\big]v \leq \big[f(x \wedge x') + f(x \vee x')\big]v.$$

Since $f(x \wedge x')v \in \Gamma(x \wedge x')$ and $f(x \vee x')v \in \Gamma(x \vee x')$, this concludes the proof. An analogous argument guarantees that Γ is also lower supermodular.

Example 5. Let Y be a space of finite, signed measures endowed with the first order stochastic dominance ordering. Suppose that $P \subset Y$ is a convex subset of probability measures, while ν is a probability measure (not necessarily in P) such that $\mu \geq_Y \nu$ for all $\mu \in P$. Notice that, for an arbitrary function $f : X \to [0, 1]$, correspondence $\Gamma : X \to Y$, given by $\Gamma(x) := f(x)P + \big[1 - f(x)\big]\{\nu\}$, maps elements x in X to a set of probability measures. Furthermore, if f is supermodular, then so is Γ. This follows from an application of Example 4, which says that the correspondence mapping x to $f(x)Z$, where $Z := P - \{\nu\}$, is supermodular. Consequently, so is $\Gamma(x) = f(x)Z + \{\nu\}$.

Example 6. Let Z be a sublattice of $\mathbb{R}^n \times \mathbb{R}^m$. Denote a typical element of Z by (x, t), where $x = (x_1, \ldots, x_n)$ and $t = (t_1, \ldots, t_m)$. It is easy to check that the set

$$X := \big\{x \in \mathbb{R}^n : (x, t) \in Z \text{ for some } t\big\}$$

is a sublattice of $\mathbb{R}^n$. Suppose that $f : Z \to Y$ is a supermodular function, where Y is an ordered real vector space. Then, the correspondence $\Gamma : X \to Y$ given by

$$\Gamma(x) := \big\{f(x, t) : (x, t) \in Z\big\}$$

is upper supermodular. Indeed, take any $y \in \Gamma(x)$ and $y' \in \Gamma(x')$. By definition of Γ, there is some t and t' in $\mathbb{R}^m$ such that $y = f(x, t)$ and $y' = f(x', t')$. Moreover, by the supermodularity of function f, we obtain

$$f\big((x, t) \wedge (x', t')\big) + f\big((x, t) \vee (x', t')\big) \geq f(x, t) + f(x', t').$$

Since Z is a subset of a Euclidean space, we have $(x, t) \wedge (x', t') = \big((x \wedge x'), (t \wedge t')\big)$ and $(x, t) \vee (x', t') = \big((x \vee x'), (t \vee t')\big)$. Hence, the element $f\big((x, t) \wedge (x', t')\big)$ belongs to $\Gamma(x \wedge x')$ and $f((x, t) \vee (x', t'))$ is in $\Gamma(x \vee x')$, which concludes our argument.

It is straightforward to show that upper supermodularity is preserved by downward comprehensive transformations of correspondences. That is, for any upper supermodular correspondence $\Gamma : X \to Y$, mapping $\Lambda(x) := \big\{y \in Y : y \leq_Y z, \text{ for some } z \in \Gamma(x)\big\}$ is an upper supermodular correspondence. Analogously, lower supermodularity is preserved by upward comprehensive transformations.

Last but not least, it is clear that both upper and lower supermodularity are also preserved by weighted sums. That is, for any upper (lower) supermodular correspondences Γ, $\Lambda : X \to Y$, mapping $\Omega(x) := \alpha\Gamma(x) + \beta\Lambda(x)$ is a upper (lower) supermodular correspondence, for any positive scalars α and β.

3 Value functions of supermodular correspondences

In this section we present our main results on supermodular correspondences. While the proofs are simple, we find the results especially important as they very naturally lead to a wide range of applications, which we discuss in the following section.

Main Theorem. *Suppose that X is a lattice and Y is an ordered vector space. For any positive linear functional $\phi : Y \to \mathbb{R}$,*

(i) *if correspondence $\Gamma : X \to Y$ is upper supermodular then the function $f : X \to \mathbb{R}$, given by $f(x) := \sup\big\{\phi(y) : y \in \Gamma(x)\big\}$, is supermodular;*

(ii) *if correspondence $\Gamma : X \to Y$ is lower supermodular then the function $f : X \to \mathbb{R}$, given by $f(x) := \inf\big\{\phi(y) : y \in \Gamma(x)\big\}$, is supermodular.*[5]

Proof. To show (i), take any $y \in \Gamma(x)$ and $y' \in \Gamma(x')$. By the upper supermodularity of Γ, there is some z in $\Gamma(x \wedge x')$ and z' in $\Gamma(x \vee x')$ such that $z + z' \geq_Y y + y'$. Therefore, for any positive linear functional $\phi : Y \to \mathbb{R}$, we have

$$\begin{aligned}\phi(y) + \phi(y') &= \phi\big(y + y'\big)\\ &\leq \phi\big(z + z'\big)\\ &= \phi(z) + \phi(z')\\ &\leq \sup\big\{\phi(v) : v \in \Gamma(x \wedge x')\big\} + \sup\big\{\phi(v) : v \in \Gamma(x \vee x')\big\}\\ &= f(x \wedge x') + f(x \vee x'),\end{aligned}$$

[5]A linear functional $\phi : Y \to \mathbb{R}$ is *positive*, whenever $y \geq_Y z$ implies $\phi(y) \geq \phi(z)$, for all y, z in Y.

where the first inequality follows from ϕ being positive, while the second is implied by the definition of supremum. By taking the supremum over the left side of the inequality, we conclude that $f(x)+f(x') \leq f(x \wedge x')+f(x \vee x')$. Hence, function f is supermodular.

To prove (ii), take any $z \in \Gamma(x \wedge x')$ and $z' \in \Gamma(x \vee x')$. By the lower supermodularity of Γ, there is $y \in \Gamma(x)$ and $y' \in \Gamma(x')$ such that $z + z' \geq_Y y + y'$. Therefore,

$$\begin{aligned} \phi(z) + \phi(z') \;=\; \phi(z+z') \;\geq\; \phi(y+y') \;=\; \phi(y)+\phi(y') \\ \geq \; \inf\{\phi(v) : v \in \Gamma(x')\} + \inf\{\phi(v) : v \in \Gamma(x)\} \;=\; f(x)+f(x'), \end{aligned}$$

where the first inequality follows from ϕ being positive and the second is implied by the definition of supremum. Once we take the infimum on the left of this inequality, we obtain $f(x \wedge x') + f(x \vee x') \geq f(x) + f(x')$, which concludes the proof. □

In some applications one would like to investigate submodular properties of the value functions; in those instances the following analogue to the Main Theorem may apply. We shall skip the proof since it is similar to the one for the Main Theorem.

Main Theorem ($*$)**.** *Suppose that X is a lattice and Y is an ordered vector space. For any positive linear functional $\phi : Y \to \mathbb{R}$,*

(i) *if correspondence $\Gamma : X \to Y$ is upper submodular then function $f : X \to \mathbb{R}$, given by $f(x) := \inf\{\phi(y) : y \in \Gamma(x)\}$, is submodular;*

(ii) *if correspondence $\Gamma : X \to Y$ is lower submodular then function $f : X \to \mathbb{R}$, given by $f(x) := \sup\{\phi(y) : y \in \Gamma(x)\}$, is submodular.*

We now give a flavour of how the Main Theorem can be applied to the analysis of firm behaviour. A fuller discussion is provided in Section 4. Suppose that $Y = \mathbb{R}^J$; in this case, any positive linear functional on Y is of the form $\phi(y) = q \cdot y$, where q is a vector in $\mathbb{R}^J_+$. Part (i) of the Main Theorem tells us that, so long as $q \geq 0$, then

$$f(x) := \sup\{q \cdot y : y \in \Gamma(x)\}$$

is a supermodular function if Γ is upper supermodular. Recall the first motivating example we considered in the Introduction, concerning a firm producing J different output goods using I inputs. In that case, the correspondence $\Gamma : \mathbb{R}^I_+ \to \mathbb{R}^J$, where $\Gamma(x) \subseteq \mathbb{R}^J_+$, gives all combinations of output goods that could be produced using the input vector x. If $q \in \mathbb{R}^J_+$

are the output prices, then $f(x)$ is the maximum revenue obtainable by the firm, given the employment of x. A related but slightly different interpretation of f is to suppose that the firm is operating in a risky environment with J states of the world. Then the set $\Gamma(x) \subseteq \mathbb{R}^J_+$ gives all the contingent revenues that the firm may choose, when the input vector x is employed. If q is the probability distribution over different states, then $f(x)$ is the greatest expected revenue achievable under x. Regardless of the interpretation, the Main Theorem guarantees that *function f is supermodular whenever the production correspondence Γ is upper supermodular.*

Example A. For a specific example of an upper supermodular output correspondence, suppose there are three inputs and two outputs (or state contingent revenues), where

$$\Gamma(x_1, x_2, x_3) := \Big\{(y_1, y_2) \in \mathbb{R}^2_+ : y_1 \le \sqrt[3]{x_1\, x_2\, t},\ y_2 \le \sqrt{x_1} + \sqrt{x_3 - t},\ \text{for some } t \in [0, x_3]\Big\}.$$

In the above example, input 1 is non-rivalrous in the sense that it can be used, in its entirety, to produce both outputs. On the other hand, input 3 has to be shared between the two productions, while input 2 is only used in the production of good 1.

To see that the above correspondence is upper supermodular, first notice that set

$$Z := \Big\{(x_1, x_2, x_3, t) \in \mathbb{R}^4 : x_i \ge 0, \text{ for } i = 1, 2, 3, \text{ and } t \in [0, x_3]\Big\}$$

is a sublattice of $\mathbb{R}^4$. Moreover, $h : Z \to \mathbb{R}^2$, where $h(x,t) := \big(\sqrt[3]{x_1\, x_2\, t}, \sqrt{x_1} + \sqrt{x_3 - t}\big)$, is a supermodular function. Therefore, by the claim made in Example 6, the correspondence $\Lambda(x) := \{h(x,t) : (x,t) \in Z\}$ is upper supermodular. As Γ is a downward comprehensive transformation of Λ, it is also upper supermodular.

It is not hard to see that the assumptions in the Main Theorem are essentially tight. The following result (which we prove in the Appendix) gives a converse to the theorem in the case where Y is an Euclidean space. The assumption that Γ is convex- and compact-valued is standard, at least when we interpret Γ as a production correspondence.

Proposition 1. *Suppose X is a lattice and Y is an Euclidean space. Moreover, let correspondence $\Gamma : X \to Y$ have compact and convex values.*

(i) *If function $f : X \to \mathbb{R}$, given by $f(x) := \max\{\phi(y) : y \in \Gamma(x)\}$, is supermodular for any positive linear functional $\phi : Y \to \mathbb{R}$, then Γ is upper supermodular.*

(ii) *If function $f : X \to \mathbb{R}$, given by $f(x) := \min\{\phi(y) : y \in \Gamma(x)\}$, is supermodular for any positive linear functional $\phi : Y \to \mathbb{R}$, then Γ is lower supermodular.*

4 Applications

In this section we provide applications of our results. Before we do that, it is convenient to explain some basic concepts and results in monotone comparative statics. For any two subsets Y, Y' of a lattice X, we say that Y' dominates Y in the *strong set order* if for any $y' \in Y'$ and $y \in Y$, we have $(y \wedge y') \in Y$ and $(y \vee y') \in Y'$. In particular, for any singletons $Y := \{y\}$ and $Y' := \{y'\}$, the set Y' dominates Y in this sense if and only if $y' \geq_X y$. Moreover, once Y and Y' both contain their greatest elements, denoted by y and y' respectively, then $y' \geq_X y$ if Y' dominates Y in the strong set order. Similarly, if Y and Y' contain their least elements z and z' respectively, then $z' \geq z$. Finally, the strong set order is transitive over the subsets of X (see Theorem 2.1 in Topkis, 1978).

Suppose that function $f : X \times T \to \mathbb{R}$, where $T \subseteq \mathbb{R}$, is supermodular with respect to the product order on $X \times T$. For any sublattice Z of X, define

$$G(t) := \arg\max_{x \in Z} f(x, t).$$

The importance of supermodularity arises from the following fundamental result: *if function f is supermodular, then the set $G(t')$ dominates $G(t)$ with respect the strong set order, for any parameters t, t' in T such that $t' \geq t$.* (See Theorem 6.1 in Topkis, 1978.) Whenever this occurs, we say that correspondence G is *increasing with respect to t*.

As a basic application of this result, suppose a firm is endowed with a supermodular production function $f : \mathbb{R}^I_+ \to \mathbb{R}_+$ and can sell its output at the price 1. Given input prices $p = (p_1, \ldots, p_I)$ in $\mathbb{R}^I_{++}$ and an input profile x, the firm's profit is

$$\pi(x, p) := f(x) - p \cdot x.$$

It is straightforward to verify that π is a supermodular function of $(x, -p_i)$, for any factor i (keeping the prices of other inputs fixed at p_{-i}). Therefore, the correspondence $G(p) := \arg\max_{x \in \mathbb{R}^I_+} \pi(x, p)$ is increasing with respect to the strong set order in $-p_i$, which implies that the factors of production are *complements* since the demand for *all* inputs increase when the price of any factor i falls. As the strong set order is transitive, repeated applications of this observation will guarantee that $G(p')$ dominates $G(p)$ in the strong set order whenever $p' \leq p$.

Application 1: Generalised complementarity in production

Both in the Introduction and in Section 3, we discuss how to guarantee that a firm's demand for inputs exhibits complementarity. Here, we give a more formal and general treatment of the problem of complementarity in a firm's choice behaviour.

Consider a firm endowed with a technology that employs I inputs to manufacture J output goods. We represent this technology by a production possibility set $P \subseteq \mathbb{R}^I_+ \times \mathbb{R}^J_+$. Abusing the notation, we denote the disjoint sets of inputs and outputs by I and J, respectively. An arbitrary element z of P denotes a feasible production profile that uses $(z_i)_{i \in I}$ units of input to produce an output profile $(z_i)_{i \in J}$. Conditional on prices $p_i > 0$, for each good $i \in I \cup J$, the firm chooses $z \in P$ in order to maximise

$$\sum_{i \in J} p_i z_i - \sum_{i \in I} p_i z_i.$$

In Section 3, we provided conditions under which the set of inputs are all complements. We now consider the problem of guaranteeing complementarity among an arbitrary subset C of commodities in $I \cup J$, which may consist of both input and output goods.

First, we need to explain what we mean by complementarity in this more general context. Given a subset C of $I \cup J$, denote its complement by $C' := (I \cup J) \setminus C$. Let $X := \{(z_i)_{i \in C} : z \in P\}$, which is a subset of $\mathbb{R}^C_+$.[6] We say that *the goods in C are complements* whenever correspondence $G : \mathbb{R}^{I+J}_{++} \to X$, given by

$$G(p) := \left\{(z_i)_{i \in C} : z \text{ maximises } \sum\nolimits_{i \in J} p_i z_i - \sum\nolimits_{i \in I} p_i z_i \text{ over } P\right\},$$

increases in the strong set order with respect to prices p_j, $j \in C \cap J$, and decreases with respect to p_i, $i \in C \cap I$. This definition captures the idea that the firm increases the volume z_i of all goods $i \in C$, whenever the prices p_j of an output $j \in C \cap J$ increases or the price p_i of an input $i \in C \cap I$ decreases. In other words, if the price of an input good in C decreases or the price of an output good increases, then the firm's demand for inputs in C as well as the production of output goods in the set will increase.

Define the production possibility correspondence $\Gamma : X \to \mathbb{R}^{C'}$, that maps vectors $x \in \mathbb{R}^C_+$ to those combinations of goods in C' that are feasible given the firm's technology,

[6] We abuse notation by denoting the sets C and C' and their cardinalities in the same manner.

with the proviso that inputs in C' enter with the negative sign. To be precise, let

$$\Gamma(x) := \Big\{ y \in \mathbb{R}^{C'} : \text{there is some } z \in P \text{ such that } z_i = x_i, \text{ for all } i \in C,$$
$$\text{while } z_i = y_i, \text{ for all } i \in C' \cap J, \text{ and } z_i = -y_i, \text{ for all } i \in C' \cap I \Big\}.$$

We claim that the goods in C are complements so long as Γ is upper supermodular. Notice that the correspondence Γ gives a full description of the firm's production possibilities, and it enables us to formulate the firm's profit maximisation problem as a two step procedure. First, for each $x \in X$, we determine the maximal revenue of the firm, i.e,

$$f(x) := \max \left\{ \sum_{i \in C'} p_i \cdot y_i : y \in \Gamma(x) \right\},$$

where by construction $y_i \leq 0$, for all $i \in I \cap C'$. Second, the firm chooses x to maximise

$$f(x) - \sum_{i \in C \cap I} p_i x_i + \sum_{i \in C \cap J} p_i x_i.$$

Observe that, the set of elements x in X that maximise the above expression, given prices p_i, for $i \in I \cup J$, coincides precisely with the set $G(p)$, defined previously. In particular, given our discussion in the introduction to Section 4, it is straightforward to verify that the goods in C are complements so long as function f is supermodular. At the same time, the Main Theorem guarantees that this holds whenever correspondence Γ is upper supermodular. The following are two instances where Γ is upper supermodular.

Example B. We are interested in conditions under which all outputs are complements (i.e., $C = J$) when the firm's production possibility set is given by

$$P := \big\{ (y, x) \in \mathbb{R}^I_+ \times \mathbb{R}^J_+ : g(y) \geq h(x) \big\},$$

where $g : \mathbb{R}^I_+ \to \mathbb{R}_+$ and $h : \mathbb{R}^J_+ \to \mathbb{R}_+$ are strictly increasing functions. In this case, for each output vector x in $X = \mathbb{R}^J_+$, we have

$$\Gamma(x) := \big\{ y \in \mathbb{R}^I_- : g(-y) \geq h(x) \big\}.$$

We claim that, whenever function h is submodular and g is concave and homogeneous of degree 1, then Γ is supermodular, and thus upper supermodular. Indeed, define the set $Z := \big\{ z \in \mathbb{R}^I_+ : g(z) \geq 1 \big\}$, which is positive and convex. By our claim in Example 4, the correspondence $\Lambda(x) := -h(x)Z$ is supermodular. Furthermore, the homogeneity of function g guarantees that $\Lambda(x) = \Gamma(x)$. Therefore, whenever the price of one output $j \in J$ increases, the firm will raise its production of every output.

Example C. Consider a firm producing outputs a and b using capital k and labour ℓ. The use of capital is non-rivalrous but each unit of labour can be assigned to the production of either a or b, but not both. Suppose that input (k, ℓ_a) allows to produce up to $g(k, \ell_a)$ units of output a, while $h(k, \ell_b)$ is the output of good b when (k, ℓ_b) is employed. Hence, the firm's production possibility set is given by

$$P := \Big\{(k, \ell, a, b) \in \mathbb{R}^4_+ : \ a \leq g(k, \ell_a), \, b \leq h(k, \ell_b), \text{ with } \ell_a + \ell_b = \ell\Big\}.$$

We claim that capital k and output a are complements whenever $h : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a supermodular function, $g : \mathbb{R}^2_+ \to \mathbb{R}_+$ is continuous, strictly increasing, and supermodular, while $g(k, \cdot)$ is unbounded and concave, for all $k > 0$. We also assume that both factors are essential, i.e., $g(k, 0) = g(0, \ell_a) = 0$ for all $k \geq 0$ and $\ell_a \geq 0$.

Let $X := \big\{(k, a) \in \mathbb{R}^2 : \ (k, \ell, a, b) \in P \text{ for some } \ell \text{ and } b\big\}$. Since g is continuous, $g(k, \cdot)$ is unbounded, while $g(k, 0) = 0$, for any $k > 0$ and $a \geq 0$, there is a unique $\phi(k, a) \geq 0$ such that $g\big(k, \phi(k, a)\big) = a$. In other words, $\phi(k, a)$ is the least amount of labour needed to produce a when k units of capital are used. As ϕ is well defined for all $k > 0$ and $a \geq 0$, while $g(k, 0) = g(0, \ell_a) = 0$, for all $k \geq 0$ and $\ell_a \geq 0$, this implies that $X = (\mathbb{R}_{++} \times \mathbb{R}_+) \cup \big\{(0, 0)\big\}$, which is a lattice. For any $(k, a) \in X$, let

$$\Gamma(k, a) = \big\{(-\ell, b) \in \mathbb{R}^2 : -\ell \leq -\ell_b - \phi(k, a) \text{ and } b \leq h(k, \ell_b), \text{ for any } \ell_b \geq 0\big\}.$$

It is not difficult to check that since g is monotone, supermodular, and concave in ℓ, the function ϕ is submodular.[7] This implies that $H(k, a, \ell_b) := \big(-\ell_b - \phi(k, a), h(k, \ell_b)\big)$ is a supermodular function over a sublattice of $\mathbb{R}^3$. Given the claim in Example 6, correspondence $\Lambda(k, a) := \big\{H(k, a, \ell_b) : \ell_b \geq 0\big\}$ is upper supermodular. Since Γ is a downward comprehensive transformation of Λ, it is upper supermodular. It follows that an increase in the price of a or a decrease in the price of capital k will cause the firm to raise the volume of output of a and its demand for capital k.

[7] A quick way of verifying this is to assume that g is sufficiently smooth and show that $\partial^2\phi/\partial a \partial k \leq 0$, but it is not difficult to give a discrete proof that dispenses with differentiability.

Application 2: Choosing among technologies

Suppose a firm has access to technologies drawn from a finite set $\mathcal{F}$. Each element of $\mathcal{F}$ is characterised by a function $f : \mathbb{R}^I_+ \to \mathbb{R}$ that maps an input vector x to $f(x)$, where the latter can be interpreted either as the level of physical output or revenue. For any x, the firm can switch costlessly among the technologies in $\mathcal{F}$, so its production function is

$$F(x) := \max\big\{f(x) : f \in \mathcal{F}\big\}. \tag{3}$$

As we know, all factors are complements when F is a supermodular function. What conditions on the set $\mathcal{F}$ guarantee that F is a supermodular function?

One answer to this question was provided in Theorem 4.3 of Topkis (1978). For any lattices X and T and a function $g : X \times T \to \mathbb{R}$, consider a family $\mathcal{F} := \big\{g(\cdot, t) : t \in T\big\}$. Topkis' result states that whenever function g is supermodular over $X \times T$ with respect to the product order, then $F(x) := \max\big\{g(x, t) : t \in T\big\}$ is a supermodular function.

Our theory suggests an alternative set of conditions. Namely, by the Main Theorem, function F is supermodular whenever the correspondence $\Gamma : \mathbb{R}^I_+ \to \mathbb{R}$, given by

$$\Gamma(x) := \big\{f(x) : f \in \mathcal{F}\big\} \tag{4}$$

is upper supermodular. The following result gives conditions under which this holds.

Proposition 2. *Let $\mathcal{F}$ be a set of continuous and supermodular functions $f : \mathbb{R}^I_+ \to \mathbb{R}$ such that for any f, g in $\mathcal{F}$, either $(f - g)$ or $(g - f)$ is weakly single crossing. Then the correspondence Γ, given by (4) is upper supermodular.*

Remark. A function $h : \mathbb{R}^I_+ \to \mathbb{R}$ is *weakly single crossing* if $h(x) > 0$ implies $h(x') > 0$ for any $x' > x$. It is straightforward to verify that h satisfies this property if, for all $i = 1, \ldots, I$, $h(x_i, x_{-i}) > 0$ implies $h(x'_i, x_{-i}) > 0$, for any $x'_i > x_i$ and x_{-i}. Thus, it suffices to check the weak single crossing property for each dimension separately.[8]

The following is a very natural collection of production functions that satisfies the conditions of Proposition 2 but violates the restrictions imposed by Topkis (1978).

[8] The standard definition of the *single crossing property* is that $h(x) > 0$ implies $h(x') > 0$ and $h(x) \geq 0$ implies $h(x') \geq 0$, for any $x' > x$.

Example D. Consider a collection $\mathcal{F} = \{f_t\}_{t=1}^T$ of functions $f_t : \mathbb{R}^2 \to \mathbb{R}$ given by $f_t(k, \ell) := A_t k^{\alpha_t} \ell^{\beta_t}$, where A_t is a positive number, while $0 < \alpha_t \leq \alpha_{t+1}$ and $0 < \beta_t \leq \beta_{t+1}$, for all $t = 1, \ldots, T-1$. Therefore, $\mathcal{F}$ consists of Cobb-Douglas functions with capital and labour as inputs, that are weakly ordered with respect to the output elasticities. Each function f_t is supermodular and, for any $r > s$, the difference

$$\Delta(k, \ell) := A_r k^{\alpha_r} \ell^{\beta_r} - A_s k^{\alpha_s} \ell^{\beta_s}$$

is weakly single crossing. Indeed, for any k, function $\Delta(k, \cdot)$ is weakly single crossing in ℓ; similarly, holding ℓ fixed, $\Delta(\cdot, \ell)$ is weakly single crossing in k. We conclude that F, as defined by (3), is a supermodular function.

However, $f_t(k, \ell)$ is *not* jointly supermodular in (k, ℓ, t) and therefore it does not obey the condition in Topkis' result. To see this, fix $k > 0$ and notice that $\Delta(k, \cdot)$ takes the value zero at $\ell = 0$ and (at most one) other point $\ell^* > 0$, with $\Delta(k, \ell) < 0$ for $0 < \ell < \ell^*$ and $\Delta(k, \ell) > 0$ for $\ell < \ell^*$. Hence, $\Delta(k, \cdot)$ is a weakly single crossing function but it is not increasing. Consequently, $f_t(k, \ell)$ is not jointly supermodular in (k, ℓ, t).

Application 3: Factor prices and output

Suppose a firm produces a single output using I inputs and has the production function $f : \mathbb{R}_+^I \to \mathbb{R}_+$. We assume that the firm derives some benefit from output q, which we denote by $B(q)$ in $\mathbb{R}$. The objective of the firm is to choose inputs x in order to maximise $B\big(f(x)\big) - p \cdot x$, where we denote the input prices by p in $\mathbb{R}_{++}^I$.

The benefit $B(q)$ may be interpreted as the revenue derived from selling q units of the good, which is generally non-linear in q if the firm has monopoly power. Alternatively, the firm could face a risky output price s, in which case

$$B(q) := \int_0^\infty u(sq) d\mu(s),$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is the Bernoulli index that summarises the firm's attitude towards risk, while μ is the probability distribution over the output price s.

We know that the factors are complements if the map from x to $B\big(f(x)\big)$ is supermodular. However, unless we make further assumptions about B, we cannot obtain such a conclusion even if f is supermodular. Nonetheless, with suitable assumptions on f alone, we can guarantee the the firm will *raise its output as the price of a factor falls.*

Given the production function f, the firm's cost function $C : \mathbb{R}^I_{++} \times \mathbb{R}_+ \to \mathbb{R}$ is

$$C(p,q) := \min\big\{p \cdot y : f(y) \geq q\big\}.$$

To keep the exposition short, suppose that, for any $q \geq 0$, there is some $x \in \mathbb{R}^I_+$ such that $f(x) = q$. Moreover, let C be well-defined, for all $p \in \mathbb{R}^I_{++}$ and $q \geq 0$. Hence, the firm's optimisation is equivalent to choosing an output $q \geq 0$ that maximises $B(q) - C(p,q)$.

We wish to find conditions on function f under which the firm's output increases when the p_i price of factor i falls.[9] Notice that, it suffices for the cost function C to be supermodular in (p_i, q). It is clear from the argument in Quah (2007) that the crucial condition to imply this property on C is supermodularity and i-concavity of the production function f.[10] However, the argument in that paper is rather roundabout — it uses the Envelope Theorem and relies on the differentiability of C, as well as various (rather strong) ancillary assumptions. Instead, we provide a direct proof of this result.

Formally, our claim is the following. *Suppose function f is continuous, strictly increasing, supermodular, and i-concave for some factor i; then the cost function C is supermodular in (p_i, q), for any prices p_{-i} of the remaining factors.*[11] Given our previous results, there is a natural proof strategy for this claim. We need to show is that under the above set of assumptions the correspondence $\Gamma_i : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}^I_+$, given by

$$\Gamma_i(p_i, q) := \big\{(p_i y_i, y_{-i}) : f(y) \geq q\big\},$$

is lower supermodular. Then, by the Main Theorem, function $v : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$,

$$v(p_i, q) := \min\big\{(1, p_{-i}) \cdot (z_i, z_{-i}) : z \in \Gamma_i(p_i, q)\big\},$$

is supermodular. Since $v_i(p_i, q) = C\big((p_i, p_{-i}), q\big)$, the cost function C is supermodular in (p_i, q). We provide a formal argument in the Appendix.

[9] We are grateful to Eddie Dekel, whose queries inspired us to look at this issue more closely.

[10] A function f is i-concave if, for any fixed x_i, it is concave function of x_{-i}.

[11] If we wish to guarantee that the firm raises its output whenever the price of any factor falls, then we require f to be i-concave for all $i = 1, \ldots, I$. Note that it is possible for a function to be i-concave for all i without being concave. For example, $f(x_1, x_2) = x_1 x_2$ is i-concave for $i = 1, 2$, but it is not concave.

Application 4: Technological change and output

When does technological change raise a firm's output? To answer this question, suppose that the firm produces a single output using I inputs and a production functions parametrised by some t in $T \subseteq \mathbb{R}$. Formally, consider function $f : T \times \mathbb{R}^I_+ \to \mathbb{R}$. Given parameter t, the firm may produce up to $f(t, x)$ units of output when input vector x is employed. This production function induces a cost function $C : T \times \mathbb{R}_+ \to \mathbb{R}$, given by

$$C(t, q) = \min \{p \cdot y : f(t, y) \geq q\},$$

where $p \in \mathbb{R}^I_{++}$ denotes input prices. Assuming that an output of q leads to a benefit of $B(q)$, the firm chooses q to maximise its profit $B(q) - C(q, t)$. Irrespective of the precise shape of B, we know that the profit-maximising output increases with t whenever C is a submodular function. In fact, this guarantees that the profit function is supermodular in (t, q).[12] By the Main Theorem $(*)$, this holds if correspondence $\Gamma : \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}^I_+$,

$$\Gamma(t, q) := \{y \in \mathbb{R}^I_+ : f(t, y) \geq q\}$$

is upper submodular. In the Appendix we show that *correspondence* Γ *is upper submodular whenever function* f *is increasing and supermodular over its domain, while* $f(t, \cdot)$ *is continuous and concave, for all* $t \in T$. As a simple example of technological change of this type, suppose that $T \subseteq \mathbb{R}_+$, with $f(t, x) := tg(x)$, where $g : \mathbb{R}^I_+ \to \mathbb{R}_+$ is a continuous, increasing, concave, and supermodular function. Then, f satisfies the conditions guaranteeing the upper submodularity of correspondence Γ, from which it follows that C is submodular, and thus an increase in t leads to higher output.

Application 5: Factor complementarity under uncertainty

We return to the second motivating example in the Introduction. At date 0, a firm acquires inputs $x \in \mathbb{R}^I_+$ at prices $p \in \mathbb{R}^I_{++}$ so as to use it in production that delivers revenue at date 1. We assume that date 1 revenue is uncertain, perhaps because of uncertainty over market conditions or over the reliability of the firm's technology.

Let $\triangle_S$ denote the space of probability distributions over the set of all possible revenues $S \subseteq \mathbb{R}_+$. We assume that the firm has multi-prior beliefs over S, given by the

[12] Furthermore, it is straightforward to check that the submodularity of C is also necessary for this to hold if the benefit function B is allowed to take different shapes.

correspondence $\Gamma : \mathbb{R}^I_+ \to \triangle_S$ that maps inputs x to the the set $\Gamma(x)$ of subjective probabilities over date 1 revenue. Assuming that the firm is ambiguity averse at date 0, its objective at date 0 is to choose $x \in \mathbb{R}^I_+$ in order to maximise

$$\min \left\{ \int_S u(s) d\mu(s) : \mu \in \Gamma(x) \right\} - p \cdot x,$$

where the strictly increasing Bernoulli index $u : S \to \mathbb{R}$ captures the firm's attitude towards risk and $p \in \mathbb{R}^I_{++}$ are the input prices. Therefore, the firm evaluates its expected revenues with respect to the least favourable probability belief $\mu \in \Gamma(x)$. Clearly, inputs x are complements whenever the function

$$f(x) := \min \left\{ \int_S u(s) d\mu(s) : \mu \in \Gamma(x) \right\}$$

is supermodular. By the Main Theorem, this holds whenever correspondence Γ is lower supermodular. Indeed, we have $f(x) = \min \{\phi(\mu) : \mu \in \Gamma(x)\}$, where $\phi : \triangle_S \to \mathbb{R}$, given by $\phi(\mu) := \int_S u(s)\, d\mu(s)$, is a positive linear functional with respect to first order stochastic dominance ordering.

Application 6: Shifts in multi-prior beliefs

Consider an agent who chooses an action $x \in X$ before the realisation of some state $s \in S$, where X and S are both subsets of $\mathbb{R}$. Given x, the agent's utility is $f(x, s)$ whenever state s is realised. Assuming that λ is a probability distribution over S, the agent chooses x to maximise the expected utility $\int_S f(x, s) d\lambda(s)$. If f is a supermodular function and the agent is allowed to choose his action *after* observing the state, then we know that his action will increase with the state. Therefore, it is intuitive that under the same condition on f, if the agent has to make a decision *before* the state is realised, then he will pick a higher action if he thinks that higher states are more likely. This turns out to be true. It is not difficult to establish that a first order stochastic dominance shift in the distribution will indeed lead to a higher optimal action.

In the following application we extend this result to the case where the agent is ambiguity averse and his belief over S is represented by a multi-prior set of probability distributions. Let $\triangle_S$ denote the space of probability distributions over S. As $S \subseteq \mathbb{R}$, the set $\triangle_S$, ranked by the first order stochastic dominance, constitutes a lattice. In particular, for any λ, λ' in $\triangle_S$, their meet $\lambda \wedge \lambda'$ is given by $(\lambda \wedge \lambda')(s) = \max\{\lambda(s), \lambda'(s)\}$ and their join $\lambda \wedge \lambda'$ by $(\lambda \vee \lambda')(s) = \min\{\lambda(s), \lambda'(s)\}$, for all $s \in S$. The notion of strong set order, defined at the beginning of this section, gives us a convenient way of comparing sets of distributions. Namely, for any subsets A and B of $\triangle_S$, we say that A dominates B in the strong set order if, for any $\lambda \in A$ and $\lambda' \in B$, we have $(\lambda \vee \lambda') \in A$ and $(\lambda \wedge \lambda') \in B$. Whenever $A = \{\lambda\}$ and $B = \{\lambda'\}$ are singletons, the strong set order simply requires that λ first order stochastically dominates λ'.

In the result that follows, we parametrise shifts in the agent's beliefs so that a higher parameter value is associated with greater optimism. Formally, the beliefs are represented by a correspondence $\Lambda : T \to \triangle_S$. We assume that Λ is increasing with respect to the strong set order, induced by first order stochastic dominance, i.e., set $\Lambda(t')$ dominates $\Lambda(t)$ with respect to the strong set order, for any $t' \geq t$.[13]

For any parameter $t \in T$, an *ambiguity averse* agent, in the sense of Gilboa and Schmeidler (1989), chooses action x in order to maximise

$$v(x,t) := \inf\left\{\int_S f(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}. \tag{5}$$

Alternatively, the agent could *ambiguity loving* and chooses action x to maximise

$$w(x,t) := \sup\left\{\int_S f(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}. \tag{6}$$

The following result concludes that, in either case, supermodularity of function f in (x,s) induces supermodularity of the objective function in (x,t). This in turn implies that the optimal action is increasing with respect to the parameter t.

Proposition 3. *Let X, T, and S be subsets of $\mathbb{R}$, with S being finite. If function $f : X \times S \to \mathbb{R}$ is supermodular and correspondence $\Lambda : T \to \triangle_S$ is increasing with respect to the strong set order, induced by the first order stochastic dominance, then functions v and w, defined in (5) and (6), are supermodular.*

Proof. The idea of the proof is to show that the optimisation problem presented above is equivalent to determining the infimum (supremum) of some positive linear functional

[13] For instance, let P be a convex subset of $\triangle_S$, and $\nu \in \triangle_S$ be dominated in the first order stochastic sense by any $\mu \in P$. It is straightforward to verify that correspondence $\Lambda(t) := h(t)P + [1 - h(t)]\{\nu\}$ is increasing in the strong set order for any increasing function $h : T \to [0, 1]$.

over a submodular correspondence. The rest follows from the Main Theorem (∗). First, denote $S = \{s_i\}_{i=1}^{\ell+1}$ such that $s_1 < s_2 \ldots < s_{\ell+1}$. For any $x \in X$ and $\lambda \in \Lambda(t)$, we have

$$\begin{aligned}\int_S f(x,s)d\lambda(s) &= f(x,s_1)\lambda(s_1) + \sum_{i=1}^{\ell} f(x,s_{i+1})\big[\lambda(s_{i+1}) - \lambda(s_i)\big] \\ &= f(x,s_{\ell+1})\lambda(s_{\ell+1}) + \sum_{i=1}^{\ell}\big[f(x,s_i) - f(x,s_{i+1})\big]\lambda(s_i) \\ &= f(x,s_{\ell+1}) - \sum_{i=1}^{\ell}\delta_i(x)\lambda(s_i),\end{aligned}$$

since $\lambda(s_{\ell+1}) = 1$, while $\delta_i(x) := \big[f(x,s_{i+1}) - f(x,s_i)\big]$, for all $i = 1,\ldots,\ell$. This observation allows us to represent function v by

$$v(x,t) = f(x,s_{\ell+1}) - \sup\left\{\sum_{i=1}^{\ell}\delta_i(x)\lambda(s_i) : \lambda \in \Lambda(t)\right\},$$

which is supermodular whenever $\bar{v}(x,t) := \sup\big\{\sum_{i=1}^{\ell}\delta_i(x)\lambda(s_i) : \lambda \in \Lambda(t)\big\}$ is submodular. By the Main Theorem (∗), this holds once the correspondence $\Gamma : X \times T \to \mathbb{R}^{\ell}$,

$$\Gamma(x,t) := \Big\{y \in \mathbb{R}^{\ell} : y_i = \delta_i(x)\lambda(s_i) \text{ for some } \lambda \in \Lambda(t)\Big\},$$

is upper submodular, with respect to the coordinate-wise partial order on $\mathbb{R}^{\ell}$ — to be precise. Using an analogous argument, we show that the function w is supermodular if $\bar{w}(x,t) := \inf\big\{\sum_{i=1}^{\ell}\delta_i(x)\lambda(s_i) : \lambda \in \Lambda(t)\big\}$ is submodular. By the Main Theorem (∗), this holds once Γ is a lower submodular correspondence. See the Appendix. □

Example E. Consider the classical problem of an investor who divides his wealth $m > 0$ between a *safe asset*, that pays out $r > 0$ for sure, and a *risky asset* with an uncertain gross payout of s in $S \subset \mathbb{R}_+$. The investor's beliefs over the return of the risky asset is captured by the correspondence $\Lambda : T \to \triangle_S$, where $\triangle_S$ is the space of probability distributions over S. We want to analyse the case in which a higher value of t corresponds to greater optimism. Formally, we assume that Λ is increasing in t with respect to the strong set order induced by first order stochastic dominance on $\triangle_S$.

The agent chooses to invest $x \in [0,m]$ in the risky asset, with the rest of his wealth invested in the safe security. Assuming that his Bernoulli index is $u : \mathbb{R}_+ \to \mathbb{R}$ and the investor is ambiguity averse, the investor's utility at $x \in [0,m]$ is

$$v(x,t) := \inf\left\{\int_S u\big(xs + (m-x)r\big)d\lambda(s) : \lambda \in \Lambda(t)\right\}. \quad (7)$$

Clearly, the investment in the risky asset increases with t whenever function v is supermodular. By Proposition 3, this is true if $f(x,s) := u\big(xs+(m-x)r\big)$ is supermodular. Assuming that u is strictly increasing and twice continuously differentiable, it is straightforward to show that f is supermodular if the coefficient of relative risk aversion of u is less than 1.[14] Moreover, the result still holds for an ambiguity loving investor whose objective has the same form as (7), except that the infimum operator is replaced with the supremum.

Example F. A firm operating in uncertain market conditions must decide on how much to produce and how much to spend on promoting its product via advertising. The marginal cost of producing each unit of output is $c > 0$ and the cost of each unit of advertising is $a > 0$. Whenever the firm chooses t units of advertising, its belief over the clearing price for its output is given by a multi-prior set $\Lambda(t)$ of probability distributions over space $S \subseteq \mathbb{R}_+$ of all possible output prices. We assume that higher advertising leads to greater optimism in the sense that the correspondence $\Lambda : T \to \triangle_S$ is increasing in the strong set order induced by first order stochastic dominance. Given a Bernoulli utility index $u : \mathbb{R}_+ \to \mathbb{R}$, an ambiguity averse firm chooses a level of output $x \geq 0$ and advertising $t \geq 0$ in order to maximise

$$v(x,t,c,a) := \inf\left\{\int_S u(sx)d\lambda(s) : \lambda \in \Lambda(t)\right\} - cx - at.$$

It is easy to check that if u has a coefficient of relative risk aversion lower than 1, then the function $f(x,s) := u(xs)$ is supermodular. If this holds, Proposition 3 guarantees that v is a supermodular function of (x,t). Given this, it is clear that v is also supermodular in $(x,t,-c,-a)$. This allows us to conclude that, for this firm, output and advertising are complements in the following sense: *more advertising and higher output will ensue from either a fall in the cost of advertising or a fall in the marginal cost of production.*

Appendix

In this section we provide proofs and auxiliary results that were excluded from the main body of the paper. Unless stated otherwise, the notation employed in the following

[14] It may be worth mentioning that risk loving behaviour is not excluded by this assumption on u, since u can still be convex in a part or all over its domain.

arguments corresponds to the one in the core part of the article.

Proof of Proposition 1. We only prove (i); the proof of (ii) analogous. Given that Γ has compact values and ϕ is continuous, function f is well-defined. If Γ is *not* upper supermodular, there exist some x, x' in X and $y \in \Gamma(x)$, $y' \in \Gamma(x')$ such that there are no $z \in \Gamma(x \wedge x')$ and $z' \in \Gamma(x \vee x')$ with $z + z' \geq y + y'$. Define set

$$U := \big\{u \in Y : u \leq v, \text{ for some } v \in \Gamma(x \wedge x') + \Gamma(x \vee x')\big\},$$

which is closed, convex, and downward comprehensive.[15] Given that U is closed and $(y + y') \notin U$, there is some $v^* \notin U$ such that $y + y' \gg v^*$. Moreover, as U is downward comprehensive, we have $\{z \in Y : z \geq v^*\} \cap U = \emptyset$. Let $V := \{z \in Y : z \geq v^*\}$. Since both U and V are non-empty and convex, by the Separating Hyperplane Theorem, there is a non-zero, linear functional ϕ^* such that $\phi^*(v) \geq \phi^*(u)$, for all $v \in V$ and $u \in U$. As $\phi^*(V)$ is bounded below, ϕ^* is also positive.[16] Finally, $y + y' \gg v^*$ implies $\phi^*(y + y') > \phi^*(v^*)$.

We claim that function $f(x) := \max\big\{\phi^*(y) : y \in \Gamma(x)\big\}$ is *not* supermodular. Indeed,

$$\begin{aligned} f(x \wedge x') + f(x \vee x') \;&=\; \max\big\{\phi^*(y) : y \in \Gamma(x \wedge x')\big\} + \max\big\{\phi^*(y) : y \in \Gamma(x \vee x')\big\} \\ &=\; \max\big\{\phi^*(y) : y \in \Gamma(x \wedge x') + \Gamma(x \vee x')\big\} \;\leq\; \phi^*(v^*) \;<\; \phi^*(y + y') \\ &\leq\; \max\big\{\phi^*(z) : z \in \Gamma(x)\big\} + \max\big\{\phi^*(z) : z \in \Gamma(x')\big\} \;=\; f(x) + f(x'), \end{aligned}$$

which violates the supermodularity of f. □

Proof of Proposition 2. Take any x, x' in X, as well as $y \in \Gamma(x)$ and $y' \in \Gamma(x')$. In particular, there are some f, g in $\mathcal{F}$ such that $y = f(x)$ and $y' = g(x')$. Without loss of generality, suppose that $(f - g)$ is weakly single crossing. We need to show that there is some $z \in \Gamma(x \wedge x')$ and $z' \in \Gamma(x \vee x')$ such that $z + z' \geq y + y'$.

First, suppose that $f(x') \geq g(x')$. Then,

$$y + y' \;\leq\; f(x) + f(x') \;\leq\; f(x \wedge x') + f(x \vee x'),$$

[15] This is the only instance where we use the assumption that Γ is compact-valued. In fact, we only require that Γ satisfies the following property: for any x, x' in X, set $\Gamma(x) + \Gamma(x')$ is closed. This holds whenever Γ is compact-valued. However, alternative restrictions exist (see Debreu, 1975, p. 41).

[16] Otherwise, there would be some $w > 0$ such that $\phi^*(w) < 0$ and $(v^* + \lambda w) \in V$, for any $\lambda \geq 0$. However, this would imply that $\phi^*(v^* + \lambda w)$ tends to $-\infty$ as λ tends to ∞.

where the last inequality follows from the supermodularity of f. As $f(x \wedge x') \in \Gamma(x \wedge x')$ and $f(x \vee x') \in \Gamma(x \vee x')$, the aforementioned condition is satisfied. Similarly, supermodularity of g implies that the condition holds if $g(x) \geq f(x)$.

Alternatively, suppose that $g(x) > f(x)$ and $f(x') < g(x')$. We claim that

$$g(x) + f(x') \leq g(x \wedge x') + f(x \vee x'). \tag{A1}$$

As $(f - g)$ is weakly single crossing and $f(x) < g(x)$, it must be that $f(x \wedge x') \leq g(x \wedge x')$. On the other hand, we have $f(x') > g(x')$. Therefore, by continuity of f and g, there is some $\lambda \in [0, 1)$ such that $f(x \wedge x' + \lambda w) = g(x \wedge x' + \lambda w)$, where $w := (x' - x \wedge x')$. Let λ^* be the largest number satisfying the above property and let $u^* := (x \wedge x') + \lambda^* w$. Take any sequence $\{\lambda_n\}$ that tends to λ^* with $1 > \lambda_n > \lambda^*$. By the construction of λ^*, we have $f(u_n) > g(u_n)$, for all $u_n := (x \wedge x') + \lambda_n w$. Consider $v_n := x + \lambda w$. Clearly, it must be that $v_n > u_n$, for all n. By the weak single crossing property, this implies that $f(v_n) > g(v_n)$, for all n. By taking limits, we obtain $f(v^*) \geq g(v^*)$, where $v^* := x + \lambda^* w$. Moreover, supermodularity of f guarantees that $f(x') - f(u^*) \leq f(x \vee x') - f(v^*)$, while by supermodularity of g we obtain $g(u^*) - g(x \wedge x') \leq g(v^*) - g(x)$. Since $f(u^*) = g(u^*)$ and $f(v^*) \geq g(v^*)$, the two inequalities imply $f(x') - g(x \wedge x') \leq f(x \vee x') - g(x)$. Hence, condition (A1) holds. Given that $g(x \wedge x') \in \Gamma(x \wedge x')$ and $f(x \vee x') \in \Gamma(x \wedge x')$, this concludes our argument. Therefore, correspondence Γ is upper supermodular. □

Proof of the claim in Application 3. Clearly, correspondence Γ_i is well-defined. Take any $p_i' \geq p_i$ and $q' \geq q$. We need to show that, for any $(p_i z_i, z_{-i})$ in $\Gamma_i(p_i, q)$ and $(p_i z_i', z_{-i}')$ in $\Gamma_i(p_i', q')$, there are some $(p_i' y_i, y_{-i}) \in \Gamma_i(p_i', q)$ and $(p_i y_i', y_{-i}') \in \Gamma_i(p_i, q')$ such that

$$(p_i z_i, z_{-i}) + (p_i z_i', z_{-i}') \geq (p_i' y_i, y_{-i}) + (p_i y_i', y_{-i}'). \tag{A2}$$

By definition of correspondence Γ_i, we have $f(z) \geq q$ and $f(z') \geq q'$. We discuss two cases separately. First, suppose that (i) $z_i' \geq z_i$. Given that

$$p_i z_i' - p_i z_i \;=\; p_i(z_i' - z_i) \;\leq\; p_i'(z_i' - z_i) \;=\; p_i' z_i' - p_i' z_i,$$

we have $p_i z_i + p_i' z_i' \geq p_i' z_i + p_i z_i'$. Choose vectors $y := z$ and $y' := z'$. As $f(y) \geq q$ and $f(y') \geq q'$, by construction, element $(p_i' y_i, y_{-i})$ must belong to $\Gamma_i(p_i', q)$, while $(p_i y_i', y_{-i}')$ is in $\Gamma_i(p_i, q')$. Moreover, the two vectors satisfy condition (A2).

Alternatively, suppose that (ii) $z'_i < z_i$. If (a) $f(z) \geq f(z')$, let $y' := z$ and $y := z'$. By construction, we have $f(y') \geq f(y) \geq q' \geq q$. Hence, element $(p_i y'_i, y'_{-i})$ belongs to $\Gamma_i(p_i, q')$, while $(p'_i y_i, y_{-i})$ is in $\Gamma_i(p'_i, q)$. Moreover, condition (A2) holds trivially.

Whenever (b) $f(z') > f(z)$, define $v := z' - (z \wedge z')$. Since $z'_i = (z \wedge z')_i$, it must be that $v_i = 0$. Moreover, monotonicity of function f implies that $f(z \wedge z') \leq f(z) < f(z')$. By continuity of f and the Intermediate Value Theorem, there is some $\lambda \in [0, 1]$ such that $f(z \wedge z' + \lambda v) = f(z)$. Since function f is supermodular and i-concave, by Proposition 2 in Quah (2007), it is $\mathcal{C}_i$-supermodular. In particular, this implies that

$$0 \;=\; f(z) - f(z \wedge z' + \lambda v) \;\leq\; f(z \vee z' - \lambda v) - f(z'),$$

and so $f(z \vee z' - \lambda v) \geq f(z') \geq q'$. Let $y := (z \wedge z') + \lambda v$ and $y' := (z \vee z') - \lambda v$, where $(p' y_i, y_{-i}) \in \Gamma_i(p', q)$ and $(p y'_i, y'_{-i}) \in \Gamma_i(p, q')$. Hence, $y + y' = (z \wedge z') + (z \vee z') = z + z'$. Since $y_i = z'_i$ and $y'_i = z_i$, we conclude that condition (A2) holds, which suffices for correspondence Γ_i to be lower supermodular. □

Proof of the claim in Application 4. Take any $t' \geq t$ and $q' \geq q$. We need to show that for all $y \in \Gamma(t', q)$, $y' \in \Gamma(t, q')$ there is some $z \in \Gamma(t, q)$ and $z' \in \Gamma(t', q')$ such that $z + z' \leq y + y'$. By definition, we have $f(t', y) \geq q$ and $f(t, y') \geq q' \geq q$. Hence, $y' \in \Gamma(t, q)$. Moreover, monotonicity of f implies $f(t', y') \geq f(t, y') \geq q'$, so $y' \in \Gamma(t', q')$.

Whenever $f(t, y) \geq q$, let $z := y$ and $z' := y'$. If $f(t', y) \geq q'$, define $z := y'$ and $z' := y$. In either case $z \in \Gamma(t, q)$ and $z' \in \Gamma(t', q')$, while $z + z' = y + y'$.

Alternatively, suppose $f(t, y) < q \leq q'$. Consider two cases. If (i) $y' \geq y'$, define $v := y' - y$. Since $f(t, y) < q \leq f(t, y')$, by continuity of f and the Intermediate Value Theorem, there is some $\lambda \geq 0$ such that $f(t, y' - \lambda v) = q$. Moreover, we obtain

$$\begin{aligned} q' - q \;&\leq\; f(t, y') - f(t, y' - \lambda v) \;\leq\; f(t', y') - f(t', y' - \lambda v) \\ &\leq\; f(t', y + \lambda v) - f(t', y) \;\leq\; f(t', y + \lambda v) - q, \end{aligned}$$

where the second inequality is implied by supermodularity of f and the third follows from concavity of $f(t', \cdot)$ and the fact that $y' \geq y + \lambda v$. Hence, we have $f(t', y + \lambda v) \geq q'$. Denote $z := y' - \lambda v$ and $z' := y + \lambda v$. Clearly, both $z \in \Gamma(t, q)$ and $z' \in \Gamma(t', q')$, as well as $z + z' = y + y'$. Therefore, the condition is satisfied in this case.

Whenever (ii) y and y' are unordered, there is some $i = 1, \dots, I$ such that $y'_i > y_i$. Define $v := y' - (y \wedge y')$, with $v_i = 0$, by construction. By monotonicity of f, we have

$f(t, y \wedge y') \leq f(t, y) < q' \leq f(t, y')$. Continuity of $f(t, \cdot)$ implies that there is some $\lambda \geq 0$ such that $f(t, y \wedge y' + \lambda v) = q$. Since $f(t, \cdot)$ is concave and supermodular, for all $t \in T$, by Proposition 2 in Quah (2007), it is $\mathcal{C}_i$-supermodular. Therefore,

$$q' - q \;\leq\; f(t, y') - f(t, y \wedge y' + \lambda v) \;\leq\; f(t', y') - f(t', y \wedge y' + \lambda v)$$
$$\leq\; f(t', y \vee y' - \lambda v) - f(t', y) \;\leq\; f(t', y \vee y' - \lambda v) - q,$$

where the second and the third inequality follow from supermodularity of f and $\mathcal{C}_i$-supermodularity of $f(t', \cdot)$, respectively. Hence, $f(t', y \vee y' - \lambda v) \geq q'$. Let $z := (y \wedge y') + \lambda v$ and $z' := y \vee y' - \lambda v$. Since $z + z' = (y \wedge y') + (y \vee y') = y + y'$, while $z \in \Gamma(t, q)$ and $z' \in \Gamma(t', q')$, the proof is complete. □

We conclude this section by completing the proof of Proposition 3.

Continuation of the proof of Proposition 3. In order to complete the proof of the proposition, we need to show that correspondence Γ is submodular.

To prove that it is upper submodular, take any $x' \geq x$ in X, $t' \geq t$ in T, as well as $y \in \Gamma(x, t')$ and $y' \in \Gamma(x', t)$. By definition of Γ, there is some $\lambda \in \Lambda(t)$ and $\lambda' \in \Lambda(t')$ such that $y_i = \delta_i(x)\lambda'(s_i)$ and $y'_i = \delta_i(x')\lambda(s_i)$, for $i = 1, \ldots, \ell$. Since correspondence Λ is increasing with respect to the strong set order, we have $(\lambda \wedge \lambda') \in \Lambda(t)$ and $(\lambda \vee \lambda') \in \Lambda(t')$, where $(\lambda \wedge \lambda')(s_i) = \max\{\lambda(s_i), \lambda'(s_i)\}$ and $(\lambda \vee \lambda')(s_i) = \min\{\lambda(s_i), \lambda'(s_i)\}$, for all $s_i \in S$. Hence, by the supermodularity of f, we have $\delta_i(x') \geq \delta_i(x)$ as well as

$$\delta_i(x')\big[(\lambda \vee \lambda')(s_i) \;-\; \lambda(s_i)\big] \;\leq\; \delta_i(x)\big[\lambda'(s_i) \;-\; (\lambda \wedge \lambda')(s_i)\big],$$

for all $i = 1, \ldots, \ell$. This, on the other hand, is equivalent to

$$\delta_i(x)(\lambda \wedge \lambda')(s_i) \;+\; \delta_i(x')(\lambda \vee \lambda')(s_i) \;\leq\; \delta_i(x)\lambda'(s_i) \;+\; \delta_i(x')\lambda(s_i),$$

for all $i = 1, \ldots, \ell$. Define vectors z, z' in $\mathbb{R}^\ell$ such that $z_i := \delta_i(x)(\lambda \wedge \lambda')(s_i)$ and $z'_i := \delta_i(x')(\lambda \vee \lambda')(s_i)$, for $i = 1, \ldots, \ell$. Since $z \in \Gamma(x, t)$ and $z' \in \Gamma(x', t')$, while $z + z' \leq y + y'$, we conclude that correspondence Γ is upper submodular.

To prove that Γ is lower submodular, take any $x' \geq x$ in X, $t' \geq t$ in T, as well as $z \in \Gamma(x, t)$ and $z' \in \Gamma(x', t')$. By definition, there are distributions $\lambda \in \Lambda(t)$ and

$\lambda' \in \Lambda(t')$ such that $z_i = \delta_i(x)\lambda(s_i)$ and $z_i' = \delta_i(x')\lambda'(s_i)$, for $i = 1, \ldots, \ell$. Since Λ is monotone, $(\lambda \wedge \lambda') \in \Lambda(t)$ and $(\lambda \vee \lambda') \in \Lambda(t')$. Given that $\delta_i(x') \geq \delta_i(x)$, we obtain

$$\delta_i(x')\big[(\lambda \wedge \lambda')(s_i) \;-\; \lambda'(s_i)\big] \;\geq\; \delta_i(x)\big[\lambda(s_i) \;-\; (\lambda \vee \lambda')(s_i)\big]$$

for all $i = 1, \ldots, \ell$. Equivalently, we have

$$\delta_i(x')(\lambda \wedge \lambda')(s_i) \;+\; \delta_i(x)(\lambda \vee \lambda')(s_i) \;\geq\; \delta_i(x)\lambda(s_i) \;+\; \delta_i(x')\lambda'(s_i)$$

for all $i = 1, \ldots, \ell$. Construct vectors y, y' in $\mathbb{R}^\ell$ such that $y_i := \delta_i(x)(\lambda \vee \lambda')(s_i)$ and $y_i' := \delta_i(x')(\lambda \wedge \lambda')(s_i)$, for $i = 1, \ldots, \ell$. Since $y \in \Gamma(x, t')$ and $y' \in \Gamma(x', t)$, we obtain $y + y' \geq z + z'$. We conclude that Γ is lower supermodular. □

References

Aliprantis, C. D., and K. C. Border (2006): *Infinite dimensional analysis: A hitchhiker's guide*. Berlin: Springer, 3rd edn.

Amir, R. (2002): "Complementarity and diagonal dominance in discounted stochastic games," *Annals of Operations Research*, 114, 39–56.

Balbus, Ł., K. Reffett, and Ł. Woźny (2014): "A constructive study of Markov equilibria in stochastic games with strategic complementarities," *Journal of Economic Theory*, 150(C), 815–840.

Debreu, G. (1975): *Theory of value. An axiomatic analysis of economic equilibrium.* Cowles Fundation.

Gilboa, I., and D. Schmeidler (1989): "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics*, 18(2), 141–153.

Milgrom, P., and J. Roberts (1990): "The economics of modern manufacturing: Technology, strategy, and organization," *American Economic Review*, 80(3), 511–28.

Milgrom, P., and C. Shannon (1994): "Monotone comparative statics," *Econometrica*, 62(1), 157–180.

Quah, J. K.-H. (2007): “The comparative statics of constrained optimization problems,” *Econometrica*, 75(2), 401–431.

Sharkey, W. (1981): “Convex games without side payments,” *International Journal of Game Theory*, 10, 101–106.

Topkis, D. M. (1968): “Ordered optimal solutions,” Ph.D. thesis, Stanford University.

——— (1978): “Minimazing a submodular function on a lattice,” *Operations Research*, 26(2), 305–321.

——— (1998): *Supermodularity and complementarity*, Frontiers of economic research. Princeton University Press.