On Choquet pricing for financial markets with frictions

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Abstract

The fundamental theory of asset pricing has been developed under the two main assumptions that markets are frictionless and the absence of arbitrage opportunities. In this case the market enforces that replicable assets are valued by a linear function of their payoffs, or as the discounted expectation with respect to the so-called risk-neutral probability. Important evidence of the presence of frictions in financial markets has led to study market pricing rules in such a framework. Recently, Cerreia-Vioglio, Maccheroni and Marinacci [3] have extended the Fundamental Theorem of Finance by showing that, in financial markets with frictions, requiring the Put-Call Parity to hold, is equivalent to the market pricing rule being represented as a discounted (Choquet) expectation with respect to a nonadditive risk-neutral probability.

This paper continues the study of market pricing rules f and its nonadditive risk-neutral probabilities when f is not assumed to be subadditive. First, Choquet pricing rules for which there exists a strictly positive linear functional below f or a strictly positive element in the core of f are characterized by a notion of arbitrage-free, stronger than the one in [3], that eliminates explicitly all the arbitrage opportunities that can be obtained by splitting payoffs in parts. Thus normalizing f, allows to get, below the nonadditive risk-neutral probability associated with f, a strictly positive additive risk-neutral probability, as in the standard formulation of the Fundamental Theorem of Finance in markets without frictions. Second, we show that the observed violation of the Call-Put Parity, a condition considered by Chateauneuf, Kast, and Lapied [4] similar to, but different from, the Put-Call Parity in [3], is consistent with the existence of bid-ask spreads.

Keywords: Market frictions, Absence of arbitrage opportunities, Nonadditive risk-neutral probability, Choquet pricing
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1. Introduction

The fundamental theory of asset pricing has been developed under the two main assumptions that markets are frictionless and the absence of arbitrage opportunities. In this case the market enforces that replicable assets are valued by a linear function of their payoffs, or as the discounted

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expectation with respect to the so-called risk-neutral probability. Important evidence of the presence of frictions in financial markets has led to study market pricing rules in such a framework. Recently, Cerreia-Vioglio, Maccheroni and Marinacci [3] have extended the Fundamental Theorem of Finance by showing that, in financial markets with frictions, requiring the Put-Call Parity to hold, is equivalent to the market pricing rule being represented as a discounted (Choquet) expectation with respect to a nonadditive risk-neutral probability.

This paper continues the study of market pricing rules f and its nonadditive risk-neutral probabilities when f is not assumed to be subadditive. First, Choquet pricing rules for which there exists a strictly positive linear functional below f or a strictly positive element in the core of f are characterized by a notion of arbitrage-free, stronger than the one in [3], that eliminates explicitly all the arbitrage opportunities that can be obtained by splitting payoffs in parts. Thus normalizing f, allows to get, below the nonadditive risk-neutral probability associated with f, a strictly positive additive risk-neutral probability μ, as in the standard formulation of the Fundamental Theorem of Finance in markets without frictions. Second, we show that the observed violation of the Call-Put Parity, a condition considered by Chateauneuf, Kast, and Lapied [4] similar to, but different from, the Put-Call Parity in [3], is consistent with the existence of bid-ask spreads.

The paper is organized as follows. After presenting in Section 2.1 the two-date stochastic model that will be considered throughout the paper, the next Section 2.2 introduces the main arbitrage-free notion, AF_{++}, together with two weaker notions, AF_{+} and AF. In Section 3.4, Theorem 3 characterizes Choquet pricing rules that are arbitrage-free, i.e., that satisfy AF_{++}, by the existence of a strictly positive linear functional below f or by a strictly positive element in the core of f. Thus normalizing f so that $f(\mathbf{1}_\Omega) = 1$ allows to get, below the nonadditive risk-neutral probability associated with f, a strictly positive additive risk-neutral probability μ, as in the standard formulation of the Fundamental Theorem of Finance in markets without frictions. For the sake of completeness, Theorem 3 will also characterize the two weaker notions AF_{+} and AF, by the existence of μ in the core of f that only satisfies $\mu \in \mathbb{R}^\Omega_+$, and $\mu \in \mathbb{R}^\Omega$, respectively.

Finally, Theorem 1 in Section 2.3 shows that the Call-Put Parity considered by Chateauneuf, Kast, and Lapied [4], together with the Arbitrage-free Condition AF, positive homogeneity, and translation invariance (which are standard assumptions for pricing rules and are satisfied if f is a Choquet pricing rule) implies that f is linear. Thus, even if the Call-Put and Put-Call Parity notions considered by [4] and [3] look similar, they are very different in nature, in view of the previous result [Theorem 1] and the Fundamental Theorem of Finance proved by [3]. Moreover, for Choquet pricing rules, Theorem 2 shows that a weaker version of the Call-Put Parity, denoted CPP_{+}, (in which the equality is replaced by an inequality) is shown to be equivalent to the absence of buy and sell arbitrage opportunities (of order 2). It is worth noticing that the condition CPP_{+} has been confirmed by empirical research, see e.g. Gould and Galai [7], Klemkosky and Resnick [8], and Sternberg [14].

2. The main results

2.1. The model

This paper considers the simple framework of a stochastic two-date model: $t = 0$ (today) is known, $t = 1$ (tomorrow) is uncertain, and the uncertainty is represented by a set Ω of states of nature (or simply states) that can prevail tomorrow, one (and only one) of which will be disclosed tomorrow and will be known (by everybody). A (stream of) payoff is a random variable $x : \Omega \to \mathbb{R}$ or a vector $x := (x(\omega))_\omega \in \mathbb{R}^\Omega$ with $x(\omega)$ representing the payoff (money) at $t = 1$ if states ω prevails. We adopt the convention that if $x(\omega) < 0$ then $|x(\omega)|$ is paid by the agent and if $x(\omega) > 0$ then $|x(\omega)|$ is received by the agent. A total (stream of) payoff is a vector $x := (x(0), x(1)) \in \mathbb{R} \times \mathbb{R}^\Omega$ specifying the payoff $x(0) \in \mathbb{R}$ known for sure at $t = 0$ and the random payoff $x(1) \in \mathbb{R}^\Omega$ at $t = 1$.

A financial market M is defined as a family of securities that allows any investor to generate total (streams of) payoffs by choosing adequately the quantities of the securities that are bought or sold. The hedging price - also called replication cost - associated with this market is then a function $f : \mathbb{R}^\Omega \to \mathbb{R}$ that associates to every payoff $x \in \mathbb{R}^\Omega$ the price/cost $f(x)$ to be paid today for the delivery of the random payoff $x : \mathbb{R}^\Omega \to \mathbb{R}$ at $t = 1$. In other words the total stream of payoffs associated with x is $(-f(x), x)$. We refer to [5] and [3] for the detailed presentation of a financial market and the way to derive from it the hedging pricing rule and the super-hedging pricing rule. This paper will take the pricing rule $f : \mathbb{R}^\Omega \to \mathbb{R}$ as the primitive concept, without assuming f to be increasing to be able to cover the two cases of a hedging pricing rule and a super-hedging pricing rule. In markets without frictions, the arbitrage-free condition enforces the hedging price f to be linear. In markets with frictions the hedging pricing rule may no longer be linear. Formally, a pricing rule is a (finite) real-valued function $f : \mathbb{R}^\Omega \to \mathbb{R}$ satisfying:

[Positive Homogeneity] $f(tx) = tf(x)$ for all $x \in \mathbb{R}^\Omega$, all $t \geq 0$;

[Translation Invariance] $f(x + t\mathbf{1}_\Omega) = f(x) + tf(\mathbf{1}_\Omega)$ for all $x \in \mathbb{R}^\Omega$, all $t \in \mathbb{R}$.

We will also say that (the bond) $\mathbf{1}_\Omega$ is frictionless if $f(-\mathbf{1}_\Omega) = -f(\mathbf{1}_\Omega)$, a condition that holds whenever f is translation invariant. The function f is said to be a Choquet pricing rule whenever, for all $x \in \mathbb{R}^\Omega$, $f(x) = \int_\Omega^C x(\omega)\, dv(\omega)$, the Choquet integral with respect to a set-function $v : 2^\Omega \to \mathbb{R}$, satisfying $v(\emptyset) = 0$. And indeed a Choquet pricing rule is a pricing rule, that is, it is both positively homogeneous and translation invariant. For every $A \subset \Omega$ we denote by $\mathbf{1}_A$ the vector in $\mathbb{R}^\Omega$ defined by $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and $\mathbf{1}_A(\omega) = 0$ otherwise, and we adopt the convention that $\mathbf{1}_\emptyset = 0$. We recall that, for a Choquet pricing rule, v is uniquely defined and equal to the set-function v_f defined by $v_f(A) := f(\mathbf{1}_A)$ for all $A \subset \Omega$. Hereafter, are also listed some additional standard properties that will be met in the paper.

[Monotonicity] $f(x) \leq f(x')$ for all $x \leq x'$.

[Subadditivity] $f(x + x') \leq f(x) + f(x')$ for all x, x'.

[Convexity] $f(tx + (1-t)x') \leq tf(x) + (1-t)f(x')$ for all x, x' and all $t \in [0, 1]$.

[Linearity] $f(\lambda x + \mu x') = \lambda f(x) + \mu f(x')$ for all x, x' and all λ, μ in $\mathbb{R}$.

We point out that if the pricing rule f is monotone and normalized, i.e., $f(\mathbf{1}_\Omega) = 1$, then v_f is a non-additive probability, i.e., it satisfies $v_f(\emptyset) = 0$, $v_f(\Omega) = 1$, and $v_f(A) \leq v_f(B)$ for all

$A \subset B \subset \Omega$. A nonadditive probability v is said to be submodular (or concave) if $v(A\cap B)+v(A\cup B) \leq v(A)+v(B)$ for all A, B. An additive probability is then a set-function $\mu : 2^\Omega \to \mathbb{R}_+$ that is additive and satisfies $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$. Hereafter, without any risk of confusion, we will use indifferently the same notation μ to represent a vector in $\mathbb{R}^\Omega$, or the associated linear function $x \to x \cdot \mu$, or the associated set-function $A \to \mu(A) := \mathbf{1}_A \cdot \mu$ for all $A \subset \Omega$. When markets are frictionless, the pricing rule f is additionally assumed to be linear, or equivalently v_f is assumed to be an additive probability. The study of markets with frictions has led to consider Choquet pricing rules that are subadditive (which is equivalent to being convex under positive homogeneity) or equivalently such that v_f is submodular/concave as in as in [4] and [5]. In this paper, we will follow [3] and study Choquet pricing rules that are not assumed to be subadditive. Moreover, the pricing rule will not be assumed to be monotone to allow the interpretation of f, either as the hedging price (replication cost) of the market, or as the super-hedging price (super-replication cost) of the market, the latter one being monotone when the first one is not in general.

2.2. Absence of arbitrage opportunities

This section introduces the main arbitrage-free notion, AF_{++}, together with two weaker notions, AF_+ and AF. Heuristically, Condition AF_{++} eliminates standard arbitrage opportunities, i.e., payoffs $x \in \mathbb{R}^\Omega$, such that $x > 0$ and $f(x) \leq 0$ (no payment is due at $t = 0$ for a positive payoff), but also arbitrage opportunities that may be generated by splitting the payoff $x > 0$ in parts, i.e., $x = x_1 + \cdots + x_k > 0$ and $f(x_1) + \cdots + f(x_k) \leq 0$ (no aggregate payment is due at $t = 0$ for a positive payoff). In particular, AF_{++} rules out "buy and sell" (or order 2) arbitrage opportunities, that is, payoffs $x \in \mathbb{R}^\Omega$ such that $f(x) + f(-x) < 0$.[3] Typically, the arbitrage opportunities of order $k > 1$ are not ruled out in general, unless f is subadditive (see Proposition 1 below). We now formally define the main arbitrage-free notion:

AF_{++} $\forall k \in \mathbb{N},\ \forall (x_1,\ldots,x_k) \in (\mathbb{R}^\Omega)^k,\ x_1 + \cdots + x_k > 0 \Rightarrow f(x_1) + \cdots + f(x_k) > 0,$

together with two weaker notions that will also be used hereafter:

AF_{+} $\forall k \in \mathbb{N},\ \forall (x_1,\ldots,x_k) \in (\mathbb{R}^\Omega)^k,\ x_1 + \cdots + x_k \geq 0 \Rightarrow f(x_1) + \cdots + f(x_k) \geq 0,$

AF $\forall k \in \mathbb{N},\ \forall (x_1,\ldots,x_k) \in (\mathbb{R}^\Omega)^k,\ x_1 + \cdots + x_k = 0 \Rightarrow f(x_1) + \cdots + f(x_k) \geq 0.$

The next proposition summarizes the main properties of the three arbitrage-free notions. First there is no need to consider arbitrage opportunities of order $k > 1$ when f is subadditive. Moreover, AF and AF_+ are equivalent whenever f is monotone and satisfies $f(0) = 0$.

Proposition 1. *Let $f : \mathbb{R}^\Omega \to \mathbb{R}$. (a) $AF_{++} \Rightarrow AF_+ \Rightarrow AF$ and*
AF (resp. AF_+, resp. AF_{++}) $\Rightarrow AF^1$ (resp. AF^1_+, resp. AF^1_{++}), where

AF^1 $\quad f(0) \geq 0,$ $\qquad$ *AF^1_+* $\quad f(x) \geq 0$ *for all* $x \geq 0,$

AF^1_{++} $\quad \forall x \in \mathbb{R}^\Omega,\ [\ x \geq 0 \Rightarrow f(x) \geq 0\ ]$ *and* $[\ x > 0 \Rightarrow f(x) > 0\ ].$

(b) If f is subadditive, then $AF \iff AF^1$, $AF_+ \iff AF^1_+$, and $AF_{++} \iff AF^1_{++}$.
(c) $AF_+ \iff AF$ if f is monotonic and $f(0) \leq 0$.

[3]Indeed, for every integer n, $nx + n(-x) + \mathbf{1}_\Omega > 0$, thus $nf(x) + nf(-x) + f(\mathbf{1}_\Omega) > 0$. Dividing by n, and taking the limit when $n \to +\infty$ we get $f(x) + f(-x) \geq 0$.

Proof. [$\mathrm{AF}_{++} \Rightarrow \mathrm{AF}_{+}$] Let $k \in \mathbb{N}$, let $(x_1, \ldots, x_k) \in (\mathbb{R}^\Omega)^k$ such that $x_1 + \cdots + x_k \geq 0$. Then, $n(x_1 + \cdots + x_k) + \mathbf{1}_\Omega > 0$ for all $n \in \mathbb{N}$. Hence, $n(f(x_1) + \cdots + f(x_k)) + f(\mathbf{1}_\Omega) > 0$, from AF_{++}. Dividing by n, and letting $n \to +\infty$, at the limit we get $f(x_1) + \cdots + f(x_k) \geq 0$.

The proofs of the remaining parts are straightforward. □

2.3. Characterizing order 2 arbitrage-free Choquet pricing rules

We first define the notion of Call-Put Parity considered by Chateauneuf, Kast, and Lapied [4]:

CPP [Call-Put Parity] $f([x - k\mathbf{1}_\Omega]_+) - f([-x + k\mathbf{1}_\Omega]_+) = -f(-x + k\mathbf{1}_\Omega)$ for all $x \in \mathbb{R}^\Omega$, $k \geq 0$,

and we notice that, whenever f is linear it is equivalent to the notion of Put-Call Parity introduced by Cerreia-Vioglio, Maccheroni and Marinacci [3]:

PCP [Put-Call Parity] $f([x - k\mathbf{1}_\Omega]_+) + f(-[-x + k\mathbf{1}_\Omega]_+) = f(x) - kf(\mathbf{1}_\Omega)]$ for all $x \in \mathbb{R}^\Omega$, $k \geq 0$.

Even if the notions of Call-Put Parity and of Put-Call Parity look similar, it is worth pointing out they are very different in nature in view of the Fundamental Theorem of Finance by Cerreia-Vioglio, Maccheroni and Marinacci [3] proved under the Put-Call Parity (together with other conditions) and the following result that states that a pricing rule f satisfying the Call-Put Parity and the Arbitrage-free Condition AF is in fact linear.

Theorem 1. *If the pricing rule $f : \mathbb{R}^\Omega \to \mathbb{R}$ satisfies the Arbitrage-free Condition AF, together with the Call-Put Parity CPP, then f is linear.*

The proof of the Theorem 1 is given in Section 3.1. We end this section with a result that shows that, for Choquet pricing rules, a weaker version of the Call-Put Parity, denoted CPP_+, (in which the equality is replaced by an inequality) is equivalent to the absence of buy and sell arbitrage opportunities (i.e., of order 2). It is worth noticing that the condition CPP_+ has been confirmed by empirical research, see e.g. Gould and Galai [7], Klemkosky and Resnick [8], and Sternberg [14].

Theorem 2. *Let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be a Choquet pricing rule, then the following assertions are equivalent*

CPP_+ $f([x - k\mathbf{1}_\Omega]_+) - f([-x + k\mathbf{1}_\Omega]_+) \geq -f(-x + k\mathbf{1}_\Omega)$ *for all* $x \in \mathbb{R}^\Omega$, $k \geq 0$*;*

$AF2$: $f(-x) + f(x) \geq 0$ *for all* $x \in \mathbb{R}^\Omega$*;*

(*) $v_f(A) + v_f(\bar{A}) \geq v_f(\Omega)$ *for all* $A \subset \Omega$*.*

The proof of Theorem 2 is given in Section 3.1.

2.4. Characterizing arbitrage-free Choquet pricing rules

The main result of this section characterizes Choquet pricing rules that are arbitrage-free, i.e., that satisfy AF_{++}, by the existence of a strictly positive linear functional below f or by a strictly positive element in the core of f. For the sake of completeness, the theorem will also characterize the two weaker notions AF_+ and AF, by weakening the strictly positive condition on μ and only assume that $\mu \in \mathbb{R}^\Omega_+$, and $\mu \in \mathbb{R}^\Omega$, respectively. Before stating the result we recall some definitions. Let $f : \mathbb{R}^\Omega \to \mathbb{R}$, the core of f, denoted $\mathrm{core}(f)$,[4] is defined by:

[4]The usual notion of the core of a cooperative game defined by its characteristic function, i.e., a set-function $v : 2^\Omega \to \mathbb{R}$ is defined by: $\mathrm{core}(v) := \{\mu \in \mathbb{R}^\Omega : \forall A \subset \Omega, \mu(A) \leq v(A) \text{ and } \mu(\Omega) = v(\Omega)\}$, and one checks that $\mathrm{core}(f) = \mathrm{core}(v_f)$. Note that $f(0) = 0$ whenever f is positively homogeneous, thus $v_f(\emptyset) := f(0) = 0$.

$$\text{core}(f) := \{\mu \in \mathbb{R}^\Omega : \forall A \subset \Omega,\ \mu(A) \le f(\mathbf{1}_A) \text{ and } \mu(\Omega) = f(\mathbf{1}_\Omega)\},$$

with the convention that $\mathbf{1}_\emptyset = 0$, and we let

$$\text{core}_+(f) = \text{core}(f) \cap \mathbb{R}^\Omega_+ \text{ and } \text{core}_{++}(f) = \text{core}(f) \cap \mathbb{R}^\Omega_{++}.$$

We can now state the characterization result of arbitrage-free Choquet pricing rules.

Theorem 3. *Let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be a Choquet pricing rule, the following assertions are equivalent:*
(*i*) *f satisfies AF_{++} (resp. f satisfies AF_+, f satisfies AF);*
(*ii*) *$\exists \mu \in \mathbb{R}^\Omega_{++}$, (resp. $\exists \mu \in \mathbb{R}^\Omega_+$, $\exists \mu \in \mathbb{R}^\Omega$) $\forall x \in \mathbb{R}^\Omega,\ x \cdot \mu \le f(x)$;*
(*iii*) *$\text{core}_{++}(f) \ne \emptyset$ (resp. $\text{core}_+(f) \ne \emptyset$, $\text{core}(f) \ne \emptyset$).*

Thus normalizing f so that $f(\mathbf{1}_\Omega) = 1$ allows to get, below the nonadditive risk-neutral probability v_f, a strictly positive additive risk-neutral probability μ, as in the standard formulation of the Fundamental Theorem of Finance in markets without frictions.

The proof of Theorem 3 is given in Section 3.2

We end the section with a remark that shows the relationship between the Arbitrage-free Condition AF on f and the Balancedness Condition of the game v_f.

Remark 1 (Arbitrage-free and balancedness conditions)**.** Since, for a Choquet pricing rule f, the Arbitrage-free Condition AF is equivalent to the nonvacuity of its core, from Bondareva [2], and Shapley [13]), it is also equivalent to the balancedness of v_f:

[Balancedness] $\forall \theta_A \ge 0$, $\sum_{A\subset\Omega} \theta_A \mathbf{1}_A = \mathbf{1}_\Omega \Rightarrow \sum_{A\subset\Omega} \theta_A v_f(A) \ge v_f(\Omega)$,

or equivalently $\sum_{A\subset\Omega} f(\theta_A \mathbf{1}_A) \ge f(\mathbf{1}_\Omega)$. In other words, if the bond $\mathbf{1}_\Omega$ is split in parts as event payoffs, $\theta_A \mathbf{1}_A$, then the aggregate cost of buying the parts cannot be smaller than the cost of buying the whole $\mathbf{1}_\Omega$. The balancedness assumption can be strengthened by replacing the event payoffs by arbitrary **nonnegative** payoffs as follows:

[Strong Balancedness] $\forall k \in \mathbb{N}, \forall (x_1, \dots, x_k) \in (\mathbb{R}^\Omega_+)^k$,

$$x_1 + \cdots + x_k = \mathbf{1}_\Omega \Rightarrow f(x_1) + \cdots + f(x_k) \ge f(\mathbf{1}_\Omega),$$

and the same interpretation applies. Moreover, for Choquet pricing rules, the three conditions: Arbitrage-free AF, Balancedness, and Strong Balancedness are equivalent. The proof of this assertion is straightforward and relies on the fact that the bond $\mathbf{1}_\Omega$ is frictionless, i.e., $-f(-\mathbf{1}_\Omega) = f(\mathbf{1}_\Omega)$, a standard property of the Choquet integral. □

3. Proofs

3.1. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. The proof is given, with CPP replaced by the weaker assumption:

CPP_- $f([x - k\mathbf{1}_\Omega]_+) - f([-x + k\mathbf{1}_\Omega]_+) \le -f(-x + k\mathbf{1}_\Omega)$ for all $x \in \mathbb{R}^\Omega$, all $k \ge 0$.

If f satisfies CPP_-, we first claim that f satisfies:

AF2_- $f(-x) + f(x) \le 0$ for all $x \in \mathbb{R}^\Omega$.

Indeed, choose $k \in \mathbb{R}$ such that $x \ge k\mathbf{1}_\Omega$, then $[x - k\mathbf{1}_\Omega]_+ = x - k\mathbf{1}_\Omega$ and $[-x + k\mathbf{1}_\Omega]_+ = 0$. Consequently,

$$
\begin{aligned}
0 &\geq f([x-k\mathbf{1}_\Omega]_+) - f([-x+k\mathbf{1}_\Omega]_+) + f(-x+k\mathbf{1}_\Omega) \text{ by } (CPP_-)\\
&= f(x-k\mathbf{1}_\Omega) - f(0) + f(-x+k\mathbf{1}_\Omega)\\
&= f(x) - kf(\mathbf{1}_\Omega) + f(-x) + kf(\mathbf{1}_\Omega) \text{ since } f \text{ is translation invariant and } f(0)=0\\
&= f(x) + f(-x).
\end{aligned}
$$

□

Since f satisfies the Arbitrage-free Condition AF, from Theorem 3, there exists $\mu \in \mathbb{R}^\Omega$ such that $x \cdot \mu \leq f(x)$ for all $x \in \mathbb{R}^\Omega$. Thus we also have $(-x) \cdot \mu \leq f(-x)$. Consequently, using the inequality in AF2$_-$, we get $x \cdot \mu \leq f(x) \leq -f(-x) \leq -(-x) \cdot \mu = x \cdot \mu$. Hence all inequalities are equalities. This proves that $f(x) = x \cdot \mu$ for all x, that is, f is linear. □

Proof of Theorem 2. $[(CPP_+) \Rightarrow (AF^2)]$ Choose $k \in \mathbb{R}$ such that $x \geq k\mathbf{1}_\Omega$, then $[x-k\mathbf{1}_\Omega]_+ = x - k\mathbf{1}_\Omega$ and $[-x+k\mathbf{1}_\Omega]_+ = 0$. Consequently,

$$
\begin{aligned}
0 &\leq f([x-k\mathbf{1}_\Omega]_+) - f([-x+k\mathbf{1}_\Omega]_+) + f(-x+k\mathbf{1}_\Omega) \text{ by } (CPP_+)\\
&= f(x-k\mathbf{1}_\Omega) - f(0) + f(-x+k\mathbf{1}_\Omega)\\
&= f(x) - kf(\mathbf{1}_\Omega) + f(-x) + kf(\mathbf{1}_\Omega) \text{ since } f \text{ is translation invariant and } f(0)=0\\
&= f(x) + f(-x).
\end{aligned}
$$

$[(AF^2) \Rightarrow (*)]$ Indeed, let $x := \mathbf{1}_A$, then $-x = \mathbf{1}_{\bar{A}} - \mathbf{1}_\Omega$. Thus:

$$
\begin{aligned}
0 &\leq f(-x) + f(x) = f(\mathbf{1}_{\bar{A}} - \mathbf{1}_\Omega) + f(\mathbf{1}_A) \text{ by } (AF^2)\\
&= f(\mathbf{1}_{\bar{A}}) - f(\mathbf{1}_\Omega) + f(\mathbf{1}_A) \text{ since } f \text{ is translation invariant}\\
&= v_f(\bar{A}) - v_f(\Omega) + v_f(A) \text{ since, by definition, } v_f(B) := f(\mathbf{1}_B) \text{ for all } B \subset \Omega.
\end{aligned}
$$

$[(*) \Rightarrow (AF^2)]$ Let $x \in \mathbb{R}^\Omega$, whose set of values, $\{x(\omega) : \omega \in \Omega\} = \{x_1, \ldots x_K\}$, is ordered decreasingly as $x_1 > \cdots > x_k > \cdots > x_K$. Since f is a Choquet pricing rule, we have

$$
\begin{aligned}
f(x) := {} & (x_1 - x_2)v_f(A_1) + \cdots + (x_k - x_{k+1})v_f(A_1 \cup \cdots \cup A_k) + \ldots\\
& + (x_{K-1} - x_K)v_f(A_1 \cup \cdots \cup A_{K-1}) + x_K v_f(\Omega),
\end{aligned}
$$

where $A_k := \{\omega \in \Omega : x(\omega) = x_k\}$.

Similarly

$$
\begin{aligned}
f(-x) := {} & (-x_K + x_{K-1})v_f(B_K) + \cdots + (-x_k + x_{k+1})v_f(B_K \cup \cdots \cup B_k) + \ldots\\
& + (-x_2 + x_1)v_f(B_K \cup \cdots \cup B_2) + (-x_1)v_f(\Omega),
\end{aligned}
$$

where $B_k := \{\omega \in \Omega : -x(\omega) = -x_k\} = A_k$.

Hence, from $(*)$ we deduce that:

$$
\begin{aligned}
f(x) + f(-x) = {} & (x_1 - x_2)[v_f(A_1) + v_f(A_K \cup \cdots \cup A_2] + \ldots\\
& + (x_{K-1} - x_K)[v_f(A_1 \cup \cdots \cup A_{K-1}) + v_f(A_K)] + x_K v_f(\Omega) - x_1 v_f(\Omega)\\
\geq {} & (x_1 - x_2 + \cdots + x_{K-1} - x_K + x_K - x_1)v_f(\Omega) = 0 \,.\, v_f(\Omega) = 0.
\end{aligned}
$$

$[AF^2 \Rightarrow CPP_+]$ Let $x \in \mathbb{R}^\Omega$ and let $k \geq 0$. Using the equality $y = [y]_+ - [-y]_+$ and the fact that $[y]_+$ and $-[-y]_+$ are comonotonic, we get

$$f(y) = f([y]_+) + f(-[-y]_+) \text{ since } f \text{ is a Choquet pricing rule [see Schmeidler [12]].}$$

Thus $\quad f(y) - f([y]_+) + f([-y]_+) = f(-[-y]_+) + f([-y]_+) \geq 0$ by (AF^2).

Taking $y := -x + k\mathbf{1}_\Omega$ we get:

$$f(-x + k\mathbf{1}_\Omega) - f([-x + k\mathbf{1}_\Omega]_+) + f([x - k\mathbf{1}_\Omega]_+) \geq 0 \text{ that is } CPP_+ \text{ holds.}$$

□

3.2. Proof of Theorem 3

3.2.1. Proof of Theorem 3 with AF_{++}

We prepare the proof with several claims:

Claim 3.1. *Let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be a Choquet pricing rule on $\mathbb{R}^\Omega$. Then one has:*
$\{\mu \in \mathbb{R}^\Omega : x \cdot \mu \leq f(x)\ \forall x \in \mathbb{R}^\Omega\} = \mathrm{core}(f)$.

Proof. We prove the following inclusions:

$\mathrm{core}(f) \subset \{\mu \in \mathbb{R}^\Omega : x \cdot \mu \leq f(x)\ \forall x\} \subset \{\mu \in \mathbb{R}^\Omega : x \cdot \mu \leq f(x)\ \forall x \text{ and } \mu(\Omega) = f(\mathbf{1}_\Omega)\} \subset \mathrm{core}(f)$.

We prove the first inclusion. Indeed, let $\mu \in \mathrm{core}(f)$, then, for all $A \subset \Omega$, $\mu(A) \leq v_f(A) := f(\mathbf{1}_A)$, and $\mu(\Omega) = v_f(\Omega)$. Assume that the values $\{x(\omega) : \omega \in \Omega\} = \{x_1, \ldots, x_m\}$ of x are ranked in decreasing order, that is, $x_1 > x_2 > \cdots > x_i > \cdots > x_m$, we let $A_i := \{\omega \in \Omega\ :\ x(\omega) = x_i\}$, and we recall that:

$$\begin{aligned} f(x) &= \textstyle\int_\Omega^C x\, dv_f := \sum_{i=1}^{m-1}(x_i - x_{i+1}) v_f(A_1 \cup \cdots \cup A_i) + x_m v_f(\Omega) \\ &\geq \textstyle\sum_{i=1}^{m-1}(x_i - x_{i+1}) \mu(A_1 \cup \cdots \cup A_i) + x_m \mu(\Omega) \text{ [since } \mu \in \mathrm{core}(f)] \\ &= x \cdot \mu \text{ [since } \mu \text{ is additive]}. \end{aligned}$$

The second inclusion is proved hereafter. Let $\mu \in \mathbb{R}^\Omega$ such that $x \cdot \mu \leq f(x)$ for all x. Then, $\mathbf{1}_\Omega \cdot \mu \leq f(\mathbf{1}_\Omega)$ and $(-\mathbf{1}_\Omega) \cdot \mu \leq f(-\mathbf{1}_\Omega) = -f(\mathbf{1}_\Omega)$ since $\mathbf{1}_\Omega$ is frictionless. Consequently, $\mathbf{1}_\Omega \cdot \mu = f(\mathbf{1}_\Omega)$. The proof of the third inclusion is immediate by taking $x := \mathbf{1}_A$ for all $A \subset \Omega$. □

Claim 3.2. *f satisfies AF (resp. AF_+, resp. AF_{++}) if*
there exists $\mu \in \mathbb{R}^\Omega$, (resp. $\mathbb{R}^\Omega_+$, resp. $\mathbb{R}^\Omega_{++}$) such that $f(x) \geq x \cdot \mu$ for all $x \in \mathbb{R}^\Omega$.

Proof. Assume that there exists $\mu \in \mathbb{R}^\Omega$ [resp. $\mathbb{R}^\Omega_+$, resp. $\mathbb{R}^\Omega_{++}$] such that $f(x) \geq x \cdot \mu$ for all x. Let $k \in \mathbb{N}$, let $(x_1, \ldots, x_k) \in (\mathbb{R}^\Omega)^k$ such that $x_1 + \cdots + x_k = 0$ [resp. ≥ 0, resp. > 0]. Then $f(x_i) \geq x_i \cdot \mu$ for all i and summing up we get:

$$f(x_1) + \cdots + f(x_k) \geq (x_1 + \cdots + x_k) \cdot \mu = 0 \cdot \mu = 0$$
$$[\text{resp. } \geq 0 \cdot \mu = 0 \text{ since } \mu \geq 0].$$
$$[\text{resp. } > 0 \text{ since } \mu \gg 0].$$
□

We define for all $x \in \mathbb{R}^\Omega$

$c_+(x) := \sup\{x \cdot \mu : \mu \in \mathrm{core}_+(f)\} \in [-\infty, +\infty]$,

with the convention that $c_+(x) := -\infty$ if $\mathrm{core}_+(f) = \emptyset$.

Claim 3.3. *Let f be a Choquet pricing rule. (a) If f satisfies AF_+, then*

$0 \leq c_+(x) < +\infty$ *for all $x \geq 0$ [hence $\mathrm{core}_+(f) \neq \emptyset$].*

(b) If f satisfies AF_{++}, then $c_+(x) > 0$ for all $x > 0$.

Proof. (a) For all $x \in \mathbb{R}^\Omega$, we notice that:

$$c_+(x) := \sup\{x \cdot \mu : \mu \geq 0, \mathbf{1}_A \cdot \mu \leq v_f(A)\, \forall A \subsetneq \Omega \text{ and } \mathbf{1}_\Omega \cdot \mu = v_f(\Omega)\}$$

is the value of a (primal) linear programming problem whose dual is defined by:

$$c_+^*(x) := \inf\{\textstyle\sum_{A \subset \Omega} \theta_A v_f(A) : \theta_A \geq 0 \text{ if } A \neq \Omega, \theta_\Omega \in \mathbb{R}, \sum_{A \subset \Omega} \theta_A \mathbf{1}_A \geq x\}$$

$= \inf\{\sum_{A\subset\Omega} f(\theta_A \mathbf{1}_A) : \theta_A \geq 0 \text{ if } A \neq \Omega, \theta_\Omega \in \mathbb{R}, \sum_{A\subset\Omega} \theta_A \mathbf{1}_A \geq x\}$

noticing that, for all $A \neq \Omega$, $\theta_A f(\mathbf{1}_A) = f(\theta_A \mathbf{1}_A)$ since $\theta_A \geq 0$ and f is positively homogeneous, and $\theta_\Omega f(\mathbf{1}_\Omega) = f(\theta_\Omega \mathbf{1}_\Omega)$ for $\theta_\Omega \in \mathbb{R}$, since f is positively homogeneous and $f(-\mathbf{1}_\Omega) = -f(\mathbf{1}_\Omega)$ since f is a Choquet pricing rule. But, for $x \geq 0$, we have

$c_+^*(x) \leq \theta_\Omega f(\mathbf{1}_\Omega) < +\infty$ with $\theta_\Omega := \max\{x(\omega) : \omega \in \Omega\}$ since $x \leq \theta_\Omega \mathbf{1}_\Omega$.

$c_+^*(x) \geq \inf\{f(x_1) + \cdots + f(x_k) : x_1 + \cdots + x_k \geq x, \ (x_1, \ldots, x_k) \in (\mathbb{R}^\Omega)^k, k \in \mathbb{N}\}$

≥ 0 since $x \geq 0$ and f satisfies AF_+.

We have thus proved that the value, $c_+^*(x)$, of the dual linear programming problem is finite. Thus by the strong duality theorem of linear programming, the values of the primal and the dual are equal. Hence $c_+(x) = c_+^*(x) \geq 0$. In particular $c_+(\mathbf{1}_\Omega)$ is finite, thus $\mathrm{core}_+(f) \neq \emptyset$.

(b) Let $x > 0$. From Part (a), the value, $c_+^*(x)$, of the dual linear programming problem is finite, thus by the strong duality theorem of linear programming, the dual problem has a solution. Hence

$c_+(x) = c_+^*(x) = \sum_{A\subset\Omega} f(\theta_A \mathbf{1}_A)$, for some $\theta_A \in \mathbb{R}$ ($A \subset \Omega$) such that $\sum_{A\subset\Omega} \theta_A \mathbf{1}_A \geq x > 0$,

> 0 since f satisfies AF_{++}. □

Claim 3.4. *If f is a Choquet pricing rule satisfying AF_{++}, then* $\mathrm{core}_{++}(f) \neq \emptyset$.

Proof. For all $\omega \in \Omega$, $c_+(\mathbf{1}_\omega) > 0$ by Claim 3.3, since $\mathbf{1}_\omega > 0$. Thus, there exists $\mu^\omega \in \mathrm{core}_+(f)$

$0 < \frac{1}{2} c_+(\mathbf{1}_\omega) < \mathbf{1}_\omega \cdot \mu^\omega \leq c_+(\mathbf{1}_\omega) := \sup\{x \cdot \mu : \mu \in \mathrm{core}_+(f)\}$.

Define $\mu := (1/\#\Omega) \sum_{\omega\in\Omega} \mu^\omega$, then $\mu \gg 0$. Indeed, for all ω, $\mathbf{1}_\omega \cdot \mu \geq \mathbf{1}_\omega \cdot \mu^\omega$ since each $\mu^{\omega'} \geq 0$ as an element of $\mathrm{core}_+(f) \subset \mathbb{R}_+^\Omega$, hence $\mathbf{1}_\omega \cdot \mu \geq \mathbf{1}_\omega \cdot \mu^\omega > 0$. Moreover $\mu \in \mathrm{core}_+(f)$ since $\mathrm{core}_+(f)$ is convex. Thus, $\mu \in \mathrm{core}_+(f) \cap \mathbb{R}_{++}^\Omega = \mathrm{core}_{++}(f)$ that is nonempty. □

We can now give the proof of Theorem 3 with AF_{++}. Let f be a Choquet pricing rule, we have the following implications:

[$\mathrm{AF}_{++} \Rightarrow \mathrm{core}_{++}(f) \neq \emptyset$] by Claim 3.4,

[$\mathrm{core}_{++}(f) \neq \emptyset \iff \exists \mu \in \mathbb{R}_{++}^\Omega, \ \mu \leq f$] by Claim 3.1 since $\mathrm{core}_{++}(f) := \mathrm{core}(f) \cap \mathbb{R}_{++}^\Omega$,

[$\exists \mu \in \mathbb{R}_{++}^\Omega, \ \mu \leq f \Rightarrow \mathrm{AF}_{++}$] by Claim 3.2. □

3.2.2. Proof of Theorem 3 with AF_+

Let f be a Choquet pricing rule, we have the following implications:

[$\mathrm{AF}_{+} \Rightarrow \mathrm{core}_{+}(f) \neq \emptyset$] by Claim 3.3,

[$\mathrm{core}_{+}(f) \neq \emptyset \iff \exists \mu \in \mathbb{R}_{+}^\Omega, \ \mu \leq f$] by Claim 3.1 since $\mathrm{core}_{+}(f) := \mathrm{core}(f) \cap \mathbb{R}_{+}^\Omega$,

[$\exists \mu \in \mathbb{R}_{+}^\Omega, \ \mu \leq f \Rightarrow \mathrm{AF}_{+}$] by Claim 3.2. □

3.2.3. Proof of Theorem 3 with AF

Let f be a Choquet pricing rule, we have the following implications:

[AF $\Rightarrow$ Balancedness $\Rightarrow \mathrm{core}\,(f) \neq \emptyset$] by Remark 1 and Bondareva-Shapley Theorem ([2], [13])

[$\mathrm{core}\,(f) \neq \emptyset \iff \exists \mu \in \mathbb{R}^\Omega, \ \mu \leq f$] by Claim 3.1,

[$\exists \mu \in \mathbb{R}^\Omega, \ \mu \leq f \Rightarrow$ AF] by Claim 3.2. □

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