Arch. Math. 99 (2012), 493–500

0003-889X/12/050493-8
published online November 16, 2012
DOI 10.1007/s00013-012-0437-4

Archiv der Mathematik

Stability index jump for constant mean curvature hypersurfaces of spheres

Oscar Perdomo and Aldir Brasil Jr.

To Myriam Santamaria

Abstract. It is known that the totally umbilical hypersurfaces in the $(n+1)$-dimensional spheres are characterized as the only hypersurfaces with weak stability index 0. That is, a compact hypersurface with constant mean curvature, cmc, in S^{n+1}, different from an Euclidean sphere, must have stability index greater than or equal to 1. In this paper we prove that the weak stability index of any non-totally umbilical compact hypersurface $M \subset S^{n+1}$ with cmc cannot take the values $1, 2, 3 \ldots, n$.

Mathematics Subject Classification (2010). Primary 53C42; Secondary 53C10.

Keywords. Constant mean curvature, Stability index, Spheres.

1. Introduction and preliminaries. Let us denote by $S^{n+1} \subset \mathbf{R}^{n+2}$ the $(n+1)$-dimensional sphere, and let us assume that $M \subset S^{n+1}$ is an n-dimensional oriented manifold immersed in the unit sphere. For any pair of vectors v, w in $\mathbf{R}^{n+2}$, $\langle v, w\rangle$ denotes the Euclidean inner product between them. Let us denote by $\nu(x)$ the Gauss map of the immersion, that is, $\nu(x)$ is a unit vector field perpendicular to x and to the tangent space of M at x. We will denote by $\bar{\nabla}$ the Levi–Cita connection in $\mathbf{R}^{n+2}$. We denote by A the shape operator of M, that is, $A_x : T_xM \to T_xM$ is given by $A_x(v) = -\bar{\nabla}_v \nu(x)$. We will denote by $\kappa_1(x), \ldots, \kappa_n(x)$ the principal curvatures at x, that is, we have that

$$A_x(v_i) = \kappa_i v_i \quad \text{for some orthonormal basis } \{v_1, \ldots, v_n\} \text{ of } T_xM.$$

We will denote the mean curvature by H. Recall that H is the average of the principal curvatures. The square of the norm of A, denoted by $|A|^2$, is the sum of the squares of the principal curvatures. A manifold M is said to be totally umbilical if for every $x \in M$ all the n principal curvatures are equal.

An inequality that we will be using in the main theorem of this paper states that

$$|A|^2 \geq nH^2 \quad \text{with equality if and only if } M \text{ is totally umbilical.} \tag{1.1}$$

The proof follows easily from the Cauchy–Schwarz inequality as follows

$$n^2H^2 = (\kappa_1 + \cdots + \kappa_n)^2 \leq n(\kappa_1^2 + \cdots + \kappa_n^2) = n|A|^2.$$

We will be studying hypersurfaces with constant mean curvature, here are some examples.

Example 1.1. Let $M = \{x = (x_1, \ldots, x_{n+2}) \in S^{n+1} : x_{n+2} = c\}$ with $-1 < c < 1$. If $e_{n+2} = (0, \ldots, 0, 1) \in \mathbf{R}^{n+2}$, then a direct computation shows that

$$\nu(x) = \frac{1}{\sqrt{1-c^2}}(e_{n+2} - cx).$$

If $v \in T_xM$ and α is a curve in M such that $\alpha(0) = x$ and $\alpha'(0) = v$, then

$$\bar{\nabla}_v\nu = \frac{1}{\sqrt{1-c^2}} \frac{d(e_{n+2} - c\alpha(t))}{dt}\Big|_{t=0} = -\frac{c}{\sqrt{1-c^2}}v.$$

Therefore, $A(v) = -\bar{\nabla}_v\nu = \frac{c}{\sqrt{1-c^2}}v$, and the principal curvatures are

$$\kappa_1(x) = \cdots = \kappa_n(x) = \frac{c}{\sqrt{1-c^2}} \quad \text{for all } x \in M.$$

Therefore, M is totally umbilical.

Example 1.2. For any pair of positive integers k and l such that $l + k = n$ and any real number $r \in (0, 1)$, let us define

$$M = \{x \in \mathbf{R}^{n+2} : x_1^2 + \cdots + x_{k+1}^2 = r^2 \quad \text{and} \quad x_{k+2}^2 + \cdots + x_{n+2}^2 = 1 - r^2\}.$$

Clearly we have that $M \subset S^{n+1}$. Let us denote any $x \in M$ as $x = y + z$ where $y = (x_1, \ldots, x_{k+1}, 0, \ldots, 0)$ and $z = (0, \ldots, 0, x_{k+2}, \ldots, x_{n+2})$. A direct computation shows that the tangent space of M at x can be written as the direct sum of the vectors spaces V_1 and V_2 where

$$V_1 = \{v = (v_1, \ldots, v_{k+1}, 0 \ldots, 0) : \langle v, y\rangle = 0\} \quad \text{and}$$
$$V_2 = \{v = (0, \ldots, 0, v_{k+2}, \ldots, v_{n+2}) : \langle v, z\rangle = 0\}.$$

Notice that V_1 has dimension k and V_2 has dimension l. A direct verification shows that we can define the Gauss map as

$$\nu(x) = \frac{\sqrt{1-r^2}}{r} \quad y - \frac{r}{\sqrt{1-r^2}}z$$

Therefore, if v is a tangent vector of M at x and $v \in V_1$, then $\bar{\nabla}_v\nu = \frac{\sqrt{1-r^2}}{r}v$, and if $v \in V_2$, we have that $\bar{\nabla}_v\nu = -\frac{r}{\sqrt{1-r^2}}v$. Therefore, the principal curvatures of M at x are $-\frac{\sqrt{1-r^2}}{r}$ with multiplicity k and $\frac{r}{\sqrt{1-r^2}}$ with multiplicity l. Moreover, the mean curvature H and the square of the norm of the shape operator are given by

$$H = \frac{lr}{n\sqrt{1-r^2}} - \frac{k\sqrt{1-r^2}}{nr} \quad \text{and} \quad |A|^2 = \frac{k(1-r^2)}{r^2} + \frac{lr^2}{1-r^2}.$$

These examples are called Clifford hypersurfaces.

In general, the principal curvatures change for different points in the manifold, the reason they are constant in the previous two examples is because of the big amount of symmetries that these examples have.

The main reason we study minimal and constant mean curvature hypersurfaces is because they are critical points of the area functional in the minimal case and, in the cmc case, they are critical points of the area functional among those hypersurfaces that preserve the algebraic volume enclosed by them. If A denotes the shape operator of M, we have that the stability operator J is given by $J(f) = -\Delta f - |A|^2 f - nf$. We will also refer to the operator J as the Jacobi operator. We define the weak stability index, denoted by $\text{ind}_T(M)$, as the maximum dimension of a vector space $V \subset C^\infty(M)$ satisfying

$$\int_M f = 0 \quad \text{and} \quad \int_M fJ(f) < 0 \quad \text{for all non-zero} \quad f \in V.$$

For a compact minimal hypersurface of the sphere, the Jacobi operator is defined in the same way, and the stability index, denoted by $\text{ind}(M)$, is defined as the number of negative eigenvalues of this operator. The reason for this slight difference lies in the fact that minimal surfaces locally minimize area while cmc surfaces locally minimize area among those variations that preserve the algebraic volume of M. In this way, the condition $\int_M f = 0$, for the cmc case, assures that we are comparing the area among hypersurfaces that locally preserve enclosed volume. In both cases, minimal and cmc, the surfaces are critical values of a functional. Viewed as critical points, we can prove that up to finite dimensional space, they are local minima. The dimension of this final dimensional space is the stability index of the surface. In other words, the stability index of a minimal or cmc surface, measures the number of essential directions for which the surface fails to minimize area. Here is exactly how this happens. Let us assume that $J(f) = \lambda f$ for some smooth function $f : M \to \mathbf{R}$; we can think of f locing the velocity vector at $t = 0$ of the following curve of surfaces

$$t \longrightarrow \alpha(t) = \{\cos(tf(p))p + \sin(tf(p))\nu(p) : p \in M\}$$

where ν is the Gauss map. Notice that $\alpha(t) \subset S^{n+1}$ and $\alpha(0)$ is the surface M. We have that

$$\frac{d(\text{Area}(\alpha(t))}{dt}\Big|_{t=0} = -n \int_M Hf = 0.$$

As it was expected, because M is a critical point of the area functional. We also have that

$$\frac{d^2(\text{Area}(\alpha(t))}{dt^2}\Big|_{t=0} = \int_M fJ(f) = \lambda \int_M f^2.$$

Notice that when $\lambda < 0$, we have that, along the curve of surfaces $\alpha(t)$, the area functional has a local maximum, that is, in the direction f, M fails to minimize area.

It turns out that very important results come out from understanding the spectrum of the stability operator. For example, the final step in the proof that the only complete area minimizing hypersurfaces in $\mathbf{R}^n$ are hyperplanes, for $n \leq 7$; and, a part of the proof of the regularity of minimal hypersurfaces in *any* riemannian manifold with dimension less than 8, comes from the fact that Simons, [15], showed that the first eigenvalue of the stability operator of a compact minimal hypersurface in the $(n+1)$-dimensional sphere is smaller than or equal to $-2n$. The fact that the regularity statement is true in any riemannian manifold shows why the study of the spectrum of the stability operator for hypersurfaces of spheres is, somehow, more important than the study of the spectrum of the stability operator of hypersurfaces in other spaces. Another result that uses the spectrum of the Jacobi operator of minimal surfaces of spheres is the classification of all cmc surfaces in S^3 and R^3 given by Sterling–Pinkall in [14].

We have that $\text{ind}(M) \geq n+3$ for any compact minimal hypersurface different from an equator; the reason is that when $H = 0$, the projection of the Gauss map to a fix vector defines an eigenfunction of J, [15]. It is known that all minimal Clifford hypersurfaces have stability index $n+3$. A natural problem that remains open is the question whether they are the only minimal examples with stability index $n+3$. For $n = 2$ Urbano, [16], gave an affirmative answer and, for any dimension n, Perdomo [12] gave an affirmative answer among those hypersurfaces with a special kind of symmetries, in particular he proved that the conjecture is true among those hypersurfaces with antipodal symmetry. Other results among hypersurfaces that satisfy additional conditions on $|A|^2$ can be found in [7,8,13].

For the constant mean curvature case, the motivation for the study of the stability operator is similar to the minimal case. For example in [1], Alias and Piccione showed the existence of embedded examples using the spectrum of the Jacobi operator in a part of their arguments. It looks like a few of their embedded examples agree with those found by Perdomo in [11].

To obtain information for the stability index for the non-minimal case is more difficult due to the fact that, besides the isoparametric examples, no explicit formula for any of the eigenfunctions of J is known. For this reason, so far, the only general result in this direction was proven by Barbosa and Do Carmo [5], and it states that $ind_T(M) = 0$ if and only if M is totally umbilical. Generalizations to other ambient spaces and alternative proofs of Barbosa and Do Carmo's result can be found in [6,10,17]. All other results have additional assumptions; for example, in [2] and [3], some estimates are found for the $\text{ind}_T(M)$ under the fairly strong additional condition that $|A|^2$ is constant. Also in [4], some estimates are found for hypersurfaces with antipodal symmetry. In [9] some estimates are found by adding additional conditions on the value of H and conditions on the coordinate functions.

2. Main Theorem. Let us start this section with the notation introduced in [12]. We will denote $l_v : M \to \mathbf{R}$ the function given by $l_v(x) = \langle x, v\rangle$ and by

$f_v : M \to \mathbf{R}$ the function given by $f_v(x) = \langle \nu(x), v\rangle$, where $\nu : M \to S^n$ is the Gauss map. The following relations are well known [2]

$$\Delta l_v = -nl_v + nHf_v, \quad \Delta f_v = -|A|^2 f_v + nHl_v.$$

Before proving the main theorem, let us prove the following small lemma which is just a direct application of the divergency theorem.

Lemma 2.1. *Let $M \subset S^{n+1}$ be a compact hypersurface with constant mean curvature H. If l_v and f_v are defined as in the beginning of this section, then for all $u \in \mathbf{R}^{n+2}$, we have that*

$$\int_M |A|^2 f_u l_u = n \int_M f_u l_u - nH \int_M f_u^2 + nH \int_M l_u^2.$$

Proof. The lemma follows directly from the divergency theorem as follows,

$$\begin{aligned}\int_M |A|^2 f_u l_u &= \int_M l_u(-\Delta f_u + nHl_u)\\ &= -\int_M f_u \Delta l_u + \int_M nHl_u^2\\ &= \int_M f_u(nl_u - nHf_u) + \int_M nHl_u^2.\end{aligned}$$

□

We also need the following result shown in [3].

Lemma 2.2. *If M is a compact hypersurface with constant mean curvature in S^{n+1} and M is neither Clifford nor totally umbilical, then for all non zero $v \in \mathbf{R}^{n+2}$, no function l_v is a multiple of the function f_v.*

Theorem 2.3. *Let $M \subset S^{n+1}$ be a compact hypersurface with constant mean curvature H. If M is not totally umbilical, then* $\text{ind}_T(M) \geq n+1$.

Proof. We will assume without loss of generality that M is neither Clifford nor minimal. The reason we may assume this is because for both of these cases it is already known that $\text{ind}_T(M) \geq n+2$, [2].

Let us consider the following subspace of $C^\infty(M)$.

$$V = \left\{ h_u = f_u + \frac{\sqrt{1+H^2}-1}{H} l_u \ : \ u \in \mathbf{R}^{n+2} \quad \text{and} \quad \int_M h_u = 0 \right\}$$

Let us check that the dimension of V is at least $n+1$. Let us denote $e_1 = (1,0,\ldots,0), e_2 = (0,1,\ldots,0)$, and $e_{n+2} = (0,\ldots,0,1)$, and let us prove that the functions $h_{e_1},\ldots,h_{e_{n+2}}$ are linearly independent. If for some real numbers $c_1,\ldots,c_{n+2}$ which are not all zero we have that

$$(c_1 h_{e_1} + \cdots + c_{n+2} h_{e_{n+2}})(x) = 0 \quad \text{for all } x \in M,$$

then making $v = (c_1, \ldots, c_{n+2})$, the equation above states that the function $h_v = c_1 h_{e_1} + \cdots + c_{n+2} h_{e_{n+2}}$ vanishes. Therefore

$$l_v = \frac{H}{1 - \sqrt{1+H^2}} f_v.$$

By Lemma 2.2, M has to be Clifford or totally umbilical, which contradicts our assumptions for M. We now can say that the vector space W spanned by the functions $h_{e_1}, \ldots, h_{e_{n+2}}$ has dimension $n+2$. Since the map $T : W \to \mathbf{R}$ given by $T(h_u) = \int_M h_u$ is linear and V is the kernel of T, we can say that either the dimension of V is $n+2$ when T vanishes on all W, or it is $n+1$ otherwise. Let us continue the proof of the Theorem. A direct computation shows that

$$\Delta h_u = -|A|^2 f_u + nHl_u - n\frac{\sqrt{1+H^2}-1}{H}\, l_u + nH\frac{\sqrt{1+H^2}-1}{H}\, f_u.$$

Therefore,

$$J(h_u) = -|A|^2 \frac{\sqrt{1+H^2}-1}{H}\, l_u - nf_u - nHl_u - nH\frac{\sqrt{1+H^2}-1}{H}\, f_u$$

and

$$\begin{aligned}
&\int_M h_u J(h_u) \\
&= \int_M \left(-|A|^2 \frac{\sqrt{1+H^2}-1}{H}\, l_u f_u - nf_u^2 - nHl_u f_u - nH\frac{\sqrt{1+H^2}-1}{H}\, f_u^2 \right) \\
&\quad + \int_M \left(-|A|^2 \Big(\frac{\sqrt{1+H^2}-1}{H}\Big)^2 l_u^2 - n\frac{\sqrt{1+H^2}-1}{H}\, f_u l_u \right) \\
&\quad + \int_M \left(-nH\frac{\sqrt{1+H^2}-1}{H}\, l_u^2 - nH\Big(\frac{\sqrt{1+H^2}-1}{H}\Big)^2 f_u l_u \right).
\end{aligned}$$

Using Lemma 2.1 to change the first term on the right hand side of the equation above and also using the inequality $|A|^2 \geq nH^2$, we get

$$\begin{aligned}
&\int_M h_u J(h_u) \\
&\quad \leq -\int_M \left(2n\frac{\sqrt{1+H^2}-1}{H} + nH + nH\left(\frac{\sqrt{1+H^2}-1}{H}\right)^2 \right) f_u l_u \\
&\qquad - \int_M nf_u^2 - \int_M \left(nH^2 \left(\frac{\sqrt{1+H^2}-1}{H}\right)^2 + 2nH\frac{\sqrt{1+H^2}-1}{H} \right) l_u^2
\end{aligned}$$

$$= -n \int_M (2H f_u l_u + f_u^2 + H^2 l_u^2)$$
$$= -n \int_M \left(f_u + H l_u\right)^2$$
$$< 0.$$

The strict inequality at the end of the computations above follows again by the main result in [2] since we know that $f_u + Hl_u$ cannot vanish identically because we are assuming that M is neither totally umbilical nor Clifford. Since the dimension of V is at least $n+1$ and $\int_M fJ(f) < 0$ for all $f \in V$, we conclude that $\mathrm{ind}_T(M) \geq n+1$. □

Acknowledgements. This work was started while the first author was visiting the Department of Mathematics of the Universidade Federal do Ceara in Fortaleza Brazil. He would like to thank that institution and the members of the Department for their hospitality. A. Brasil Jr. was partially supported by CNPq, Brazil.

References

[1] L. Alias, P. Piccione, Bifurcation of constant mean curvature tori in Euclidean spheres, arXiv:0905.2128v2

[2] L. Alias, A. Brasil, and O. Perdomo, On the stability index of hypersurfaces with constant mean curvature in spheres, PAMS **135** (2007), 3685–3693.

[3] L. Alias, A. Brasil, and O. Perdomo A characterization of quadric constant mean curvature hypersurfaces of spheres, J. Geom. Anal. **18** (2008), 687–703.

[4] L. Alias, A. Brasil and O. Perdomo, Stable constant mean curvature hypersurfaces in the real projective space, Manuscripta Math. **121** (2006), 329–338.

[5] J.L. Barbosa and M. do Carmo, Stability of hypersurfaces with constant mean curvature, Math Z. **185** (1984), 339–353.

[6] J.L. Barbosa, M. do Carmo, and J. Eschenburg, Stability of hypersurfaces with constant mean curvature in Riemannian manifolds, Math Z. **197** (1988), 123–138.

[7] A. Barros and P. Sousa, Estimate for index of closed minimal hypersurfaces in spheres, Kodai Mathematical Journal **32** (2009), 442–449.

[8] A. Brasil, J. A. Delgado and I. Guadalupe, A characterization of the Clifford torus, Rend. Circ. Ma. Palermo. **48** (1999) 537–540.

[9] E. Colberg, A.M. de Jesus and K. Kinneberg, On the index of Constant Mean Curvature Hypersurfaces, arXiv:0901.4398

[10] S. Montiel, Stable constant mean curvature hypersurfaces in some Riemannian manifolds, Comment. Math. Helv. **73**(1998), 584–602.

[11] O. Perdomo, Embedded constant mean curvature hypersurfaces of spheres, Asian J. Math. **14** (2010), 73–108.

[12] O. Perdomo, Low index minimal hypersurfaces of spheres, Asian J. Math. **5** (2001), 741–749.

[13] O. Perdomo, On the average of the scalar curvature for minimal hypersurfaces of spheres with low index, Illinois J. Math. **48** (2004), 559–565.

[14] U. Pinkall and I. Sterling, On the classification of constant mean curvature tori, Ann. of Math. **130** (1989), 407–451

[15] J. Simons, Minimal varieties in Riemannian manifolds, Ann. of Math. **88** (1968) 62–105.

[16] F. Urbano, Minimal surfaces with low index in the three-dimensional sphere, Proc. Amer. Math. Soc. **108** (1990), 989–992.

[17] A. Veeravalli, Stability of constant mean curvature hypersurfaces in a wide class of Riemannian manifolds, Geom. Dedicata, doi:10.1007/s10711-011-9638-4

Oscar Perdomo
Department of Mathematics,
Central Connecticut State University,
New Britain, CT 06050,
USA
e-mail: perdomoosm@ccsu.edu

Aldir Brasil Jr.
Departamento de Matematicas,
Universidade Federal do Ceara,
Fortaleza, Ceara,
Brazil
e-mail: aldir@mat.ufc.br

Received: 16 February 2012