GENERALIZED QUASI-EINSTEIN MANIFOLDS WITH HARMONIC ANTI-SELF DUAL WEYL TENSOR

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ABSTRACT. We prove that a 4-dimensional generalized m-quasi-Einstein manifold with harmonic anti-self dual Weyl tensor is locally a warped product with 3-dimensional Einstein fibers provided an additional condition holds.

1. INTRODUCTION

In [10], Catino introduced a class of special Riemannian (cf. [5, 9] and [20]). According to [10], a complete Riemannian manifold (M^n, g), $n \geq 3$, is a *generalized quasi-Einstein manifold*, if there exist three smooth functions f, μ and β on M such that

$$Ric + \nabla^2 f - \mu df \otimes df = \beta g, \tag{1.1}$$

where Ric and $\nabla^2 f$, denotes, respectively, the Ricci tensor and Hessian of the metric g. When f is a constant function, we say that (M^n, g) is a trivial generalized quasi-Einstein manifold.

Catino [10] proved that a complete generalized quasi-Einstein manifold with harmonic Weyl tensor and vanishing radial Weyl curvature ($W(\nabla f, \cdot, \cdot, \cdot) = 0$) is locally a special warped product. In [4] and [13] the authors considered a special case of (1.1). Following the terminology used in [4] and [13], we consider a special case of (1.1). More precisely, in this paper we consider the following definition.

Definition 1. *A Riemannian manifold $(M^n,\ g)$ will be called a generalized m-quasi-Einstein manifold, or simply generalized quasi-Einstein manifold, if there exists a constant m with $0 < m \leq +\infty$, as well as two smooth functions, f and β, on M^n satisfying*

$$Ric + \nabla^2 f - \frac{1}{m} df \otimes df = \beta g.$$

In a system of local coordinates, we have

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{m} \nabla_i f \nabla_j f = \beta g_{ij}. \tag{1.2}$$

Let us point out that if $m = \infty$ and β is constant, (1.2) becomes the fundamental equation of gradient Ricci soliton, see, for instance, [7] and [12]. Further, if $m = \infty$ and β is a smooth function, (1.2) reduces to Ricci almost soliton equation (cf. [3] and [19]).

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A generalized m-quasi-Einstein manifold will be called *trivial* if the potential function f is constant. Otherwise, it will be called *nontrivial.* Let us point out that the triviality definition implies that M^n is an Einstein manifold. On the other hand, the converse statement is not true (cf. [4]).

Barros and Ribeiro [4] obtained explicitly examples of nontrivial generalized m-quasi-Einstein manifolds. Moreover, by assuming that a generalized m-quasi-Einstein manifold is also an Einstein manifold, they were able to show that such manifold must be a space form. They also obtained a formula for the Laplacian of its scalar curvature which provided some integral formulae for such a class of compact manifolds that permit to obtain some rigidity results. Barros and Gomes [2] showed that a compact gradient generalized quasi-Einstein manifold with constant scalar curvature must be isometric to a standard Euclidean sphere $\mathbb{S}^n$. At the same time, some rigidity results on compact gradient generalized quasi-Einstein manifolds were obtained in [18]. Moreover, Huang and Zeng [18] as well as Guangyue and Wei [17] provided some classifications for generalized quasi-Einstein manifolds under the assumption that the Bach tensor is flat. Ghosh [16] proved that if a complete K-contact manifold $M^{(2n+1)}$ of dimension $(2n+1)$ admits a generalized quasi-Einstein structure with $m \neq 1$, then $M^{(2n+1)}$ is compact, Einstein and Sasakian. In particular, $M^{(2n+1)}$ is isometric to a standard Euclidean sphere $\mathbb{S}^{2n+1}$. For more details on this subject, we indicate, for instance, [4, 10, 13, 14, 16, 17] and [18].

In order to proceed, we recall that 4-dimensional Riemannian manifolds are special. In such manifolds, the bundle of 2-forms can be invariantly decomposed as a direct sum; some relevant facts may be found in [1, 6, 15]. For instance, on an oriented Riemannian manifold $(M^4,\, g)$, the Weyl curvature tensor W is an endomorphism of the bundle of 2-forms $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ such that

$$W = W^+ \oplus W^-,$$

where $W^\pm : \Lambda^2_\pm \longrightarrow \Lambda^2_\pm$ are called the self-dual and anti-self dual parts of W. Half conformally flat metrics are also known as self-dual or anti-self dual if $W^- = 0$ or $W^+ = 0$, respectively.

Recently, Deng [13] proved that a 4-dimensional half conformally flat generalized quasi-Einstein manifold satisfying (1.2) must be either Einstein or locally conformally flat, given that its potential function f is a real analytic function.

For what follows, remember that viewing W^+ as a tensor of type $(0,4)$, the tensor W^+ is harmonic if $\delta W^+ = 0$, where δ is the formal divergence defined for any $(0,4)$-tensor T by

$$\delta T(X_1, X_2, X_3) = trace_g\{(Y, Z) \mapsto \nabla_Y T(Z, X_1, X_2, X_3)\},$$

where g is the metric of M^4. Furthermore, it should be emphasized that every 4-dimensional Einstein manifold has harmonic tensor W^+ (cf. 16.65 in [6], see also Lemma 6.14 in [15]). Therefore, it is natural to ask which geometric implications has the assumption of the harmonicity of the tensor W^+ on generalized quasi-Einstein manifolds.

In this article, inspired by the results obtained in [1, 10, 13], we study harmonic anti-self dual Weyl tensor (i.e., W^+ is harmonic) on 4-dimensional generalized m-quasi-Einstein manifolds satisfying (1.2). In this sense, we have established the following results.

Theorem 1.1. *Let $(M^4,\, g,\, f)$ be a nontrivial generalized m-quasi-Einstein manifold with harmonic anti-self dual Weyl tensor and $W(\nabla f, \cdot, \cdot, \cdot) = 0$. Then, around any regular point of f, (M^4, g) is locally a warped product with 3-dimensional Einstein fibers.*

As it was pointed out by Catino [10], the condition $W(\nabla f, \cdot, \cdot, \cdot) = 0$ can not be removed when a generalized quasi-Einstein manifold satisfying (1.1) has harmonic Weyl tensor. Nonetheless, our next theorem shows that when a 4-dimensional generalized m-quasi-Einstein manifold satisfying (1.2) has harmonic anti-self dual Weyl tensor, the condition $W(\nabla f, \cdot, \cdot, \cdot) = 0$ can be replaced.

Theorem 1.2. *Let $(M^4, g,\, f)$ be a nontrivial generalized m-quasi-Einstein manifold with harmonic anti-self dual Weyl tensor. Then, around any regular point of f, we have*

$$3|Ric|^2 \geq 3R_{44}^2 + (R - R_{44})^2.$$

In addition, if equality holds, (M^4, g) is locally a warped product with 3-dimensional Einstein fibers.

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