SOLUTIONS TO THE SINGULAR σ_2-YAMABE PROBLEM WITH ISOLATED SINGULARITIES

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Abstract. Given (M, g_0) a closed Riemannian manifold and a nonempty closed subset X in M, the singular σ_k-Yamabe problem asks for a complete metric g on $M\backslash X$ conformal to g_0 with constant σ_k-curvature. The σ_k-curvature is defined as the k-th elementary symmetric function of the eigenvalues of the Schouten tensor of a Riemannian metric. The main goal of this paper is to find solutions to the singular σ_2-Yamabe problem with isolated singularities in any nondegenerate closed Riemannian manifold such that the Weyl tensor vanishing to sufficiently high order at the singular points. We will use perturbation techniques and gluing methods.

1. Introduction

Since the complete resolution of the Yamabe problem by Yamabe [35], Trudinger [33], Aubin [1] and Schoen [30], much attention has been given to the study of conformal geometry. To understand the problem we are interested in this work, first let us recall some background definition from Riemannian Geometry. Given a Riemannian manifold (M, g), there exists an orthogonal decomposition of the curvature tensor Rm_g which is given by

$$Rm_g = W_g + A_g \odot g,$$

where $\odot$ is the Kulkarni-Nomizu produt, A_g is the Shouten Tensor defined as

$$A_g = \frac{1}{n-2}\left(Ric_g - \frac{1}{2(n-1)}R_g g\right), \tag{1}$$

Ric_g and R_g are respectively the Ricci tensor and the scalar curvature of the metric g, see [10] for instance. Since the Weyl tensor W_g is conformally invariant in the sense that $W_{e^f g} = e^f W_g$, then to understand the conformal class of the metric g it is natural to study the Schouten tensor A_g. For $k \in \{1, \dots, n\}$, the σ_k-curvature is defined as

$$\sigma_k(A_g) := \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1}\cdot \ldots \cdot \lambda_{i_k},$$

that is, $\sigma_k(A_g)$ is the k-th elementary symmetric function of the eigenvalue (λ_1, $\dots$, λ_n) of A_g.

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The σ_k-Yamabe problem asks for a conformal metric in a given closed Riemannian manifold (M,g) with constant σ_k-curvature. Note that since $\sigma_1(A_g) = \frac{1}{2(n-1)}R_g$, then the case $k=1$ is the classical Yamabe problem. The σ_k-Yamabe problem has been extensively studied in the past years. We direct the reader to the papers [6], [7], [14], [15], [23], [31] and the references contained therein.

It is then natural to ask whether every noncompact Riemannian manifold is conformally equivalent to a complete manifold with constant σ_k-curvature. When the noncompact manifold has a simple structure at infinity, this question may be studied by solving the singular σ_k-Yamabe problem:

> Given (M,g_0) a closed Riemannian manifold and a nonempty closed set X in M, find a complete metric g on $M\backslash X$ conformal to g_0 with constant σ_k-curvature.

For $k=1$ this problem has been extensively studied in recent years, and many existence results as well as obstructions to existence are known. See [32] and the references contained therein (See also [2]).

The equation $\sigma_k(A_g) =$ constant is always elliptic for $k=1$, while for $k\geq 2$ we need some additional hypothesis, for example, a sufficient condition for this is that g is k-admissible. By definition a metric g on M is said to be k-admissible if it belongs to the k-th positive cone Γ_k^+, this means that

$$g\in\Gamma_k^+ \Longleftrightarrow \sigma_1(A_g),\ldots,\sigma_k(A_g)>0.$$

We will produce conformal complete metric in a given closed manifold (M,g_0) with nonremovable isolated singularity with positive constant σ_2-curvature. Before write precisely our main result let us remember some well known facts about the σ_k-curvature.

For $4\leq 2k<n$ it was proved in [9] and [11] that if $\mathbb{S}^n\backslash X$ admits a complete Riemannian metric g conformal to the round metric $g_{\mathbb{S}^n}$ with $\sigma_1(A_g)\geq c>0$ and $\sigma_2(A_g),\ldots,\sigma_k(A_g)\geq 0$, then the Hausdorff dimension of X is less then or equal to $(n-2k)/2$. On the other hand, using the estimate, obtained in [13], namely,

$$Ric_g \geq \frac{(2k-n)(n-1)}{(k-1)}\left(\begin{array}{c} n\\ k\end{array}\right)^{-1/k}\sigma_k^{1/k}(A_g)g,$$

for locally conformally flat manifold and the Bonnet-Myers's Theorem, Gonzalez [11] observed that there is no singular metric in $\mathbb{S}^n$ with positive constant σ_k-curvature and $n<2k$. In [8] the authors proved that there is no complete metric with positive constant $\sigma_{n/2}$-curvature conformally related with the canonical metric in $\mathbb{R}^n\backslash\{0\}$ and with radial conformal factor. For $2\leq k\leq n$, Han, Li and Teixeira [16] proved, as in the case $k=1$ (see [4], [21] and [24]), that any complete metric in $\mathbb{S}^n$ with nonremovable isolated singularity, positive constant σ_k-curvature and conformal to the round metric is asymptotic to some rotationally symmetric metric near the singular set. Although these results are for locally conformally flat manifold, they motivate us to consider the singular σ_k-Yamabe problem with $2\leq 2k<n$.

In [24], Marques has proved that given a closed manifold (M,g), not necessarilly locally conformally flat, with dimension $3\leq n\leq 5$ then every complete metric with positive constant scalar curvature and with nonremovable isolated singularity is asymptotic to a radial metric near the singular set. It should be an interesting question ask whether there is an analogous result for singular metrics with positive constant σ_k-curvature. Another interested problem is related with the Hausdorff dimension estimate $(n-2k)/2$. For $k=1$ this estimate is sharp. In [25], the authors have constructed metrics with positive constant scalar curvature that are singular at any given disjoint union of smooth submanifolds of $\mathbb{S}^n$ of dimensions $0 < k_i \leq (n-2)/2$. In fact, a model to the positive singular Yamabe problem is the manifold $\mathbb{S}^{n-l-1} \times \mathbb{H}^{l+1}$ which is conformal to $\mathbb{S}^n \backslash \mathbb{S}^l$ and has positive constant scalar curvature equal to $(n-2l-2)(n-1)$ for all $l < (n-2)/2$, see [2] and [11]. Up to our knowledge, it is not known if the correspond estimate for $k > 1$ is sharp. Gonzalez [11] has showed that

$$l_k := \sup\{l \geq 0; P_1(l), \dots, P_k(l) > 0\} \to \frac{n-2}{2} - O(\sqrt{n}), \quad \text{as} \quad n \to \infty,$$

where $P_r(l)$ is the σ_r-curvature of $\mathbb{S}^{n-l-1} \times \mathbb{H}^{l+1}$. See [12] for more details about this subject.

Only few results are known about the singular σ_k-Yamabe problem. Using a similar method as Mazzeo and Pacard [25] used to construct singular metrics in the sphere $\mathbb{S}^n$ with positive constant scalar curvature, Mazzieri and Ndiaye [27] have proved the following existence result:

Theorem 1.1 (Mazzieri-Ndiaye [27]). *Suposse $\Lambda \subset \mathbb{S}^n$ is a finite set which is symmetrically balanced, that is, there exists an orthogonal transformation $T \in O(n+1)$ of $\mathbb{R}^{n+1}$ such that $T(\Lambda) = \Lambda$ and 1 is not an eigenvalue of T. Assuming $2 \leq 2k < n$, then there exists a family of complete Riemannian metric on $\mathbb{S}^n \backslash \Lambda$ with positive constant σ_k-curvature, which are conformal to the standard metric in $\mathbb{S}^n$.*

We notice that by a result in [8] there is no complete metric in $\mathbb{S}^n \backslash \{p\}$ with positive constant σ_k-curvature which is radially simmetric and conformal to the standard round metric. If $\Lambda = \{p_1, \dots, p_m\} \subset \mathbb{S}^n$ is a finite set which is symmetrically balanced, then $T(p) = p$, where $p = \sum p_i$ and $T \in O(n+1)$ is a linear orthogonal transformation such that 1 is not an eigenvalue of T and Λ is T-invariant. This implies that the only possibility is that $p = 0$, and so $m \geq 2$.

Inspired by the construction presented in [26], Mazzieri and Segatti [28] have constructed complete noncompact locally conformally flat metrics with positive constant σ_k-curvature with $2 \leq k < n$. The method consists in performing the connected sum of a finite number of Delaunay-type metrics. For connected sum in the compact case see [5].

Our main result is concerned with the positive singular σ_2-Yamabe problem in the case where X is a finite set, which can be a single point. We will construct solutions to this problem under a condition on the Weyl tensor. The method applied in the proof is based on perturbation techniques and gluing procedure. Basically, we will construct a family of complete constant σ_2-curvature metrics on a punctured ball with prescribed Dirichlet boudary data. Also we will construct a family of constant σ_2-curvature metrics on the complement of a geodesic ball with prescribed Dirichlet boundary data. Both families of metrics depend on $n+2$ parameters, where n is the dimension of the manifold. We construct these families of conformal metrics using perturbation techniques. In this way, both families of metrics comes with some estimates on the parameters and in the Dirichlet boundary data, which in order to apply the gluing procedure, they have to be similar. We restrict ourselves to the case $k = 2$ since by the identity $2\sigma_2(A_g) = (\mathrm{tr}_g A_g)^2 - |A_g|_g^2$ we find explicitly an expression to the equation $\sigma_2(A_g)$ =constant and then we are able to find the estimates that it will be needed in the final step of our proof.

This method was applied in [32] to solve the problem in the case $k = 1$. It is an interesting problem to solve the singular σ_k-Yamabe problem for $3 \leq k < n/2$.

The main result of this paper reads as follows.

Main Theorem: *Let (M^n, g_0) be a compact Riemannian manifold nondegenerate with dimension $n \geq 5$, g_0 conformal to some 2-admissible metric and the σ_2-curvature equal to $n(n-1)/8$. Let $\{p_1, \dots, p_m\}$ be a set of points in M such that $\nabla^j_{g_0} W_{g_0}(p_i) = 0$ for $j = 0, 1, \dots, \left[\frac{n-4}{2}\right]$ and $i = 1, \dots, m$, where W_{g_0} is the Weyl tensor of the metric g_0. Then, there exist a constant $\varepsilon_0 > 0$ and a one-parameter family of complete metrics g_ε on $M \backslash \{p_1, \dots, p_m\}$ defined for all $\varepsilon \in (0, \varepsilon_0)$, conformal to g_0, with constant σ_2-curvature equal to $n(n-1)/8$, obtained by attaching Delaunay-type ends to the points $p_1, \dots, p_m$. Moreover, $g_\varepsilon \to g_0$ uniformly on compact sets in $M \backslash \{p_1, \dots, p_m\}$ as $\varepsilon \to 0$.*

We notice here that the condition on the σ_2-curvature in the Main Theorem is not restrictive. By the results of Sheng-Trudinger-Wang [31] and Viaclovsky [34] we know that in the conformal class of a 2-admissible Riemannian metric on a compact manifold M there exists a positive constant σ_2-curvature metric. Since the hypothesis on the Weyl tensor is conformally invariant, we can work with any metric in the conformal class of the metric, in particular with the metric with positive constant σ_2-curvature. In fact in [31] the authors have showed that for all $4 \leq 2k \leq n$ the positive σ_k-Yamabe problem always has a solution, since the problem is variational and the initial metric is conformal to another k-admissible metric, while in [34] Viaclovsky has showed that the σ_k-Yamabe problem is always variational in the case $k = 2$. In the other side, in [3] Branson and Gover have showed that the σ_k-Yamabe problem for $k \in \{3, \dots, n\}$ is variational if and only if the manifold is locally conformally flat. Thus, to solve the singular σ_k-Yamabe problem with isolated singularities for $k \geq 3$ it will be necessary to overcome two main problems. The first problem is the lack of an explicit expression to the equation $\sigma_k =$ constant to get the right estimates for the gluing procedure. And the second problem is that in our proof we need of a positive constant σ_k-curvature metric to construct the family of metrics on the complement of the geodesic ball. This we can only ensure if the manifold is locally conformally flat where the Weyl tensor vanishing.

The nondegeneracy is defined as follows

Definition 1.2. *A metric g with constant σ_2-curvature equal to $n(n-1)/8$ is nondegenerate if the operator $L^1_g : C^{2,\alpha}(M) \to C^{0,\alpha}(M)$ is surjective for some $\alpha \in (0,1)$, where L^1_g is defined in (11). Here $C^{k,\alpha}(M)$ is the standard Hölder spaces on M.*

When the operator L^1_g is elliptic, we need only check the injectivity. For example, it is clear that the round sphere $\mathbb{S}^n$ is degenerate because $L^1_{g_0} = c_n(\Delta_{g_0} + n)$ annihilates the restrictions of linear functions on $\mathbb{R}^{n+1}$ to $\mathbb{S}^n$, where c_n is a constant which depends only on n.

Mazzieri and Ndiaye have proved their theorem in the sphere, which is locally conformally flat. With this assumption, in the neighborhood of p_i the metric is essentially the standard metric on $\mathbb{R}^n$, and in this case it is possible to transfer the metric to cylindrical coordinates, where there is a family of well known Delaunay-type solutions. In our case we only have that the Weyl tensor vanishing to sufficiently high order at each point p_i. Since the singular σ_k-Yamabe problem is conformally invariant, it is more convenient to work in conformal normal coordinates. As indicated in [20] in such coordinates we get some simplifications. The order $\left[\frac{n-4}{2}\right]$ comes up naturally in our method and it will be fundamental to solve the problem locally, although we do not know if it is the optimal one.

The organization of this paper is as follows.

All the analysis in the paper were done considering $m = 1$. In Section 2 we record some notation that it will be used throughout the work. We review some results concerning the Delaunay-type solutions for the constant σ_k-curvature equation and using the right inverse found in [27] and a perturbation argument we construct a right inverse for the linearized operator about such solution. In Section 3 we work with conformal normal coordinates in a neighborhood of p, since in these coordinates we get some simplifications. We use the assumption on the Weyl tensor to find a family of complete constant σ_2-curvature metrics in a small punctured ball, which depends on $n+2$ parameters and it has prescribed Dirichlet boundary data. In Section 4 we work with a metric with constant σ_2-curvature and we find a family of constant σ_2-curvature metrics, which also depends on $n+2$ parameters and it has prescribed Dirichlet boundary data. Finally, the fact that the metric is conformal to some 2-admissible metric allows us to use elliptic regularity. In Section 5 we put all results obtained in the previous sections together to prove the Main Theorem for the case $m = 1$. For the general case we briefly explain the minor changes that need to be made in order to deal with more than one singular point.

2. Preliminaries

In this section we record some notation and results that it will be used frequently, throughout the rest of the work and sometimes without comment.

We use the symbols c, C, with or without subscript, to denote various positive constants. We write $f = O'(Cr^k)$ to mean $f = O(Cr^k)$ and $\nabla f = O(Cr^{k-1})$, where C is a fixed constant. O'' is defined similarly.

2.1. **Notation.** Let us denote by e_j, for $j \in \mathbb{N}$, the eigenfunction of the Laplace operator on $\mathbb{S}^{n-1}$ with corresponding eigenvalue λ_j, where $\lambda_0 = 0$, $\lambda_1 = \cdots = \lambda_n = n-1$, $\lambda_{n+1} = 2n, \ldots$ and $\lambda_j \le \lambda_{j+1}$ with unit L^2-norm. That is,

$$\Delta_{\mathbb{S}^{n-1}} e_j + \lambda_j e_j = 0 \quad \text{and} \quad \|e_j\|_2^2 = \int_{\mathbb{S}^{n-1}} e_j^2 = 1.$$

Remember that $\{e_j\}$ is an orthonormal basis of $L^2(\mathbb{S}^{n-1})$. These eigenfunctions are restrictions to $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ of homogeneous harmonic polynomials in $\mathbb{R}^n$. The i-th eigenvalue counted without multiplicity is $i(i+n-2)$.

Let $\mathbb{S}^{n-1}_r$ be the sphere with radius $r > 0$. If the eigenfunction decomposition of the function $\phi \in L^2(\mathbb{S}^{n-1}_r)$ is given by

$$\phi(r\theta) = \sum_{j=0}^{\infty} \phi_j(r) e_j(\theta) \quad \text{where} \quad \phi_j(r) = \int_{\mathbb{S}^{n-1}} \phi(r\cdot) e_j,$$

then we define the projection π_r onto the *high frequencies space* by the formula

$$\pi_r(\phi)(r\theta) := \sum_{j=n+1}^{\infty} \phi_j(r) e_j(\theta).$$

The *low frequencies space* on $\mathbb{S}_r^{n-1}$ is spanned by the constant functions and the restrictions to $\mathbb{S}_r^{n-1}$ of linear functions on $\mathbb{R}^n$.

2.2. **The constant σ_k-curvature equation.** Let (M, g_0) be a closed Riemannian manifold of dimension $n \geq 3$. Let A_{g_0} be the Schouten tensor of the metric g_0 defined in (1).

The so called σ_k-curvature of (M, g_0), which is a smooth function denoted by $\sigma_k(A_{g_0})$, is defined pointwise for each $p \in M$ as the k-th symmetric elementary function of the eigenvalues of the tensor $A_{g_0}(p)$. Since

$$\sigma_1(A_{g_0}) = \mathrm{tr}_{g_0}(A_{g_0}) \quad \text{and} \quad \sigma_2(A_{g_0}) = \frac{1}{2}\left((\mathrm{tr}_{g_0}(A_{g_0}))^2 - |A_{g_0}|^2_{g_0}\right),$$

then note that

$$\sigma_1(A_{g_0}) = \frac{R_{g_0}}{2(n-1)} \qquad \text{and} \qquad \sigma_2(A_{g_0}) = \frac{n}{8(n-1)(n-2)^2}R^2_{g_0} - \frac{|Ric_{g_0}|^2_{g_0}}{2(n-2)^2}. \tag{2}$$

The Euclidean space $\mathbb{R}^n$ with its standard metric is σ_k-flat for any $1 \leq k \leq n$, whereas the standard sphere $\mathbb{S}^n$ has $A_{\mathbb{S}^n} = \frac{1}{2}g_{\mathbb{S}^n}$ and thus

$$\sigma_k(A_{\mathbb{S}^n}) = \frac{1}{2^k}\left(\begin{array}{c} n \\ k \end{array}\right) \quad \text{for} \quad 1 \leq k \leq n.$$

For a given nonempty closed set $X \subset M$, the positive singular σ_k-Yamabe problem amounts to find a conformal factor $u \in C^\infty(M\backslash X)$ such that the metric $g = u^{\frac{4k}{n-2k}}g_0$ is complete on $M\backslash X$ and verifies

$$\sigma_k(A_g) = \frac{1}{2^k}\left(\begin{array}{c} n \\ k \end{array}\right) \quad \text{in} \quad M\backslash X. \tag{3}$$

Now we define the nonlinear operator

$$H_{g_0}(u) = \left(\frac{n-2k}{4k}\right)^k u^{\frac{2kn}{n-2k}}\sigma_k(A_g) - 2^{-k}\left(\begin{array}{c} n \\ k \end{array}\right)\left(\frac{n-2k}{4k}\right)^k u^{\frac{2kn}{n-2k}}. \tag{4}$$

The equation (3) is equivalent to

$$H_{g_0}(u) = 0 \quad \text{in} \quad M\backslash X, \tag{5}$$

with a suitable condition in the singular set, for instance, the function u goes to infinity with a sufficiently fast rate. This equation is fully nonlinear for $k > 1$.

The operator H_{g_0} obeys the following relation concerning conformal changes of the metric

$$H_{v^{4k/(n-2k)}g}(u) = v^{-\frac{2kn}{n-2k}}H_g(vu) \tag{6}$$

and the Schouten tensor obeys the following well-known transformation law

$$A_{v^{4k/(n-2k)}g} = A_g - \frac{2k}{n-2k}u^{-1}\nabla^2_g u + \frac{2kn}{(n-2k)^2}u^{-2}du \otimes du - \frac{2k^2}{(n-2k)^2}u^{-2}|du|^2_g g$$

In this work we are interested in the case $k = 2$. So, using the second formula in (2) we obtain the expression for the nonlinear operator H_{g_0} in this case

$$
\begin{aligned}
H_{g_0}(u) &= \left(\frac{n-4}{4}\right)^2 u^4\sigma_2(A_{g_0}) + \frac{u^2}{2}(\Delta_{g_0}u)^2 - \frac{n-4}{8(n-2)}R_{g_0}u^2|\nabla_{g_0}u|^2_{g_0} \\
&- \frac{n-4}{8(n-2)}R_{g_0}u^3\Delta_{g_0}u + \frac{n-2}{n-4}u|\nabla_{g_0}u|^2_{g_0}\Delta_g u - \frac{u^2}{2}|\nabla^2_{g_0}u|^2_{g_0} \\
&+ \left\langle Ric_{g_0}, \frac{n-4}{4(n-2)}u^3\nabla^2_{g_0}u - \frac{n}{4(n-2)}u^2\nabla_{g_0}u\otimes\nabla_{g_0}u\right\rangle_{g_0} \\
&+ \frac{n}{n-4}u\langle\nabla_{g_0}u\otimes\nabla_{g_0}u, \nabla^2_{g_0}u\rangle_{g_0} - \frac{n(n-1)(n-4)^2}{128}|u|^{\frac{3n+4}{n-4}}u.
\end{aligned} \tag{7}
$$

We seek a positive function which solves (5). We will use perturbation techniques and gluing methods to finding this solution. Expanding H_g about a function u, not necessarilly a solution, gives

$$H_g(u+v) = H_g(u) + L^u_g(v) + Q^u_g(v),$$

where

$$
\begin{aligned}
L^u_g(v) &= \left.\frac{d}{dt}\right|_{t=0} H_g(u+tv) \\
&= \left(u^2\Delta u - \frac{n-4}{8(n-2)}R_g u^3 + \frac{n-2}{n-4}u|\nabla u|^2\right)\Delta v \\
&+ \left(\frac{(n-4)^2}{4}u^3\sigma_2(A_g) + u(\Delta u)^2 - \frac{n-4}{4(n-2)}R_g u|\nabla u|^2\right. \\
&+ \frac{n-2}{n-4}|\nabla u|^2\Delta u - \frac{3(n-4)}{8(n-2)}R_g u^2\Delta u - u|\nabla^2 u|^2 \\
&+ \frac{3(n-4)}{4(n-2)}u^2\langle Ric_g, \nabla^2 u\rangle - \frac{n}{2(n-2)}u\langle Ric_g, \nabla u\otimes\nabla u\rangle \\
&+ \left.\frac{n}{n-4}\langle\nabla u\otimes\nabla u, \nabla^2 u\rangle - \frac{n^2(n-1)(n-4)}{32}|u|^{\frac{3n+4}{n-4}}\right) v \\
&+ \left\langle \frac{2(n-2)}{n-4}u\nabla u\Delta u - \frac{n-4}{4(n-2)}R_g u^2\nabla u, \nabla v\right\rangle \\
&+ \left\langle \frac{n-4}{4(n-2)}u^3 Ric_g - u^2\nabla^2 u + \frac{n}{n-4}u\nabla u\otimes\nabla u, \nabla^2 v\right\rangle \\
&+ \left\langle \frac{2n}{n-4}u\nabla^2 u - \frac{n}{2(n-2)}u^2 Ric_g, \nabla u\otimes\nabla v\right\rangle
\end{aligned} \tag{8}
$$

and

$$Q_g^u(v) = \int_0^1 \int_0^1 \frac{d}{ds} L_g^{u+tsv}(v) ds dt. \tag{9}$$

Note that, by the property (6), we obtain

$$L^v_{u^{4k/(n-2k)}g}(w) = u^{-\frac{2kn}{n-2k}} L_g^{uv}(uw). \tag{10}$$

It is important to emphasize here that in this work (M, g_0) always will be a compact Riemannian manifold of dimension $n \geq 5$ with constant σ_2-curvature equal to $n(n-1)/8$ and nondegenerate, see Definition 1.2. This implies that the operator $L^1_{g_0} : C^{2,\alpha}(M) \to C^{0,\alpha}(M)$ is surjective for some $\alpha \in (0,1)$, where

$$L^1_{g_0} u = -\frac{n-4}{8(n-2)} R_{g_0} \Delta_{g_0} u - \frac{n(n-1)(n-4)}{8} u + \frac{n-4}{4(n-2)} \langle Ric_{g_0}, \nabla^2_{g_0} u \rangle_{g_0}. \tag{11}$$

In the round sphere $\mathbb{S}^n$ we have that the scalar curvature is equal to $n(n-1)$ and the Ricci tensor equal to $(n-1)g_{\mathbb{S}^n}$. Thus

$$L^1_{g_{\mathbb{S}^n}} u = -\frac{(n-1)(n-4)}{8} (\Delta_{\mathbb{S}^n} + n) u,$$

which implies that the round sphere is degenerate, since n is a eigenvalue of $\Delta_{\mathbb{S}^n}$.

2.3. **Delaunay-type solutions.** In this section we recall some facts about the Delaunay-type solutions in the σ_k-curvature setting. Our solution to the singular σ_k-Yamabe problem will be asymptotic to some Delaunay-type solution.

If $g = u^{\frac{4k}{n-2k}} g_{eucl}$ is a complete metric in $\mathbb{R}^n \backslash \{0\}$ conformal to the Euclidean standard metric g_{eucl} on $\mathbb{R}^n$ with constant σ_k-curvature equal to $2^{-k} \begin{pmatrix} n \\ k \end{pmatrix}$, then u is a solution of the equation

$$H_{g_{eucl}}(u) = 0 \quad \text{in} \quad \mathbb{R}^n \backslash \{0\}. \tag{12}$$

Let us consider that u is rotationally invariant, and thus the equation it satisfies may be reduced to an ordinary differential equation. These metrics has been studied in [8], see also [27].

Since $\mathbb{R}^n \backslash \{0\}$ is conformally diffeomorphic to a cylinder, it will be convenient to use the cylindrical background. In other words, consider the conformal diffeomorphism $\Phi : (\mathbb{S}^{n-1} \times \mathbb{R}, g_{cyl}) \to (\mathbb{R}^n \backslash \{0\}, g_{eucl})$ defined by $\Phi(\theta, t) = e^{-t}\theta$ and where $g_{cyl} := d\theta^2 + dt^2$. Then $\Phi^* g_{eucl} = e^{-2t} g_{cyl}$. Define $v(t) := e^{\frac{2k-n}{2k}t} u(e^{-t}\theta) = |x|^{\frac{n-2k}{2k}} u(x)$, where $t = -\log |x|$ and $\theta = x|x|^{-1}$. Note that v is defined in the whole cylinder and $\Phi^* g = v^{\frac{4k}{n-2k}} g_{cyl}$.

Therefore, the conformal factor v satisfies the following ODE

$$\left(v^2 - \left(\frac{2k}{n-2k} \right)^2 (v')^2 \right)^{k-1} \left(v - \left(\frac{2k}{n-2k} \right)^2 v'' \right) = \frac{n}{n-2k} v^{\frac{2kn}{n-2k} - 1}. \tag{13}$$

The Hamiltonian energy, given by

$$H(v, w) = \left(v^2 - \left(\frac{2k}{n-2k} \right)^2 w^2 \right)^k - v^{\frac{2kn}{n-2k}}, \tag{14}$$

is constant along solutions of (13). We summarize the basic properties of this solutions in the next proposition (see Propositon 2.1 in [27] and Proposition 3.1 in [28], see also [8]).

Proposition 2.1. *Suppose* $H(v,v') = H_0 \in \left[0, \frac{2k}{n-2k}\left(\frac{n-2k}{n}\right)^{\frac{n}{2k}}\right]$, *then we have three cases:*

a) *If* $H_0 = 0$, *then either we have the trivial solution* $v \equiv 0$ *or* $v(t) = \cosh^{-\frac{n-2k}{2k}}(t-c)$, *for some* $c \in \mathbb{R}$. *The latter conformal factor gives rise to a metric on* $\mathbb{S}^{n-1} \times \mathbb{R}$ *which is non complete and which corresponds in fact to the standard metric* $g_{\mathbb{S}^n}$ *on* $\mathbb{S}^n \backslash \{p,-p\}$.

b) *If* $0 < H_0 < \frac{2k}{n-2k}\left(\frac{n-2k}{n}\right)^{\frac{n}{2k}}$, *then, in correspondence of each* H_0, *there exists a unique solution* v *of (13) satisfying the conditions* $v'(0) = 0$ *and* $v''(0) > 0$. *This solution is periodic and it is such that* $0 < v(t) < 1$ *for all* $t \in \mathbb{R}$. *This family of solutions gives rise to a family of complete and periodic metrics on* $\mathbb{R} \times \mathbb{S}^{n-1}$.

c) *If* $H_0 = \frac{2k}{n-2k}\left(\frac{n-2k}{n}\right)^{\frac{n}{2k}}$, *then there exists a unique solution to (13) given by* $v(t) = \left(\frac{n-2k}{n}\right)^{\frac{n-2k}{4k^2}}$, *for* $t \in \mathbb{R}$. *This solution give rise to a complete metric on* $\mathbb{S}^{n-1} \times \mathbb{R}$ *and it is in fact a constant multiple of the cylindrical metric* g_{cyl}.

We will write the solution of (13) given by the Proposition 2.1 when $H_0 > 0$ as v_ε, where $v_\varepsilon(0) = \min v_\varepsilon = \varepsilon^{(n-2k)/2k}$, for $\varepsilon \in (0, ((n-2k)/n)^{\frac{1}{2k}})$ and the corresponding solution of (12) as $u_\varepsilon(x) = |x|^{\frac{2k-n}{2k}} v_\varepsilon(-\log|x|)$. For our purposes, the next proposition gives sufficient information about their behavior as ε tends to zero. Its proof can be found in [27], but we include it here for the sake of the reader.

Proposition 2.2. *For* $0 < \varepsilon < \left(\frac{n-2k}{n}\right)^{\frac{1}{2k}}$. *Then we have that there exists a positive constant* $c_{n,k} > 0$ *depending only on* n *and* k *such that for all* $t \in \mathbb{R}$ *we have*

$$
\begin{aligned}
\left| v_\varepsilon(t) - \varepsilon^{\frac{n-2k}{2k}} \cosh\left(\frac{n-2k}{2k} t\right) \right| &\leq c_{n,k} \varepsilon^{\frac{n+2k}{2k}} e^{\frac{n+2k}{2k}|t|}, \\
\left| v_\varepsilon'(t) - \frac{n-2k}{2k} \varepsilon^{\frac{n-2k}{2k}} \sinh\left(\frac{n-2k}{2k} t\right) \right| &\leq c_{n,k} \varepsilon^{\frac{n+2k}{2k}} e^{\frac{n+2k}{2k}|t|}, \\
\left| v_\varepsilon''(t) - \left(\frac{n-2k}{2k}\right)^2 \varepsilon^{\frac{n-2k}{2k}} \cosh\left(\frac{n-2k}{2k} t\right) \right| &\leq c_{n,k} \varepsilon^{\frac{n+2k}{2k}} e^{\frac{n+2k}{2k}|t|}.
\end{aligned}
$$

Proof. Since the Hamiltonian energy (14) is constant along solutions of (13) and $v_\varepsilon(0) = \varepsilon^{\frac{n-2k}{2k}}$ is the minimum of v_ε, then $H(v_\varepsilon, v_\varepsilon') = \varepsilon^{n-2k} - \varepsilon^n > 0$. From [8] we have that

$$
h_\varepsilon := v_\varepsilon^2 - \left(\frac{2k}{n-2k}\right)^2 (v_\varepsilon')^2 > 0. \tag{15}
$$

Thus

$$
(v_\varepsilon')^2 = \left(\frac{n-2k}{2k}\right)^2 \left(v_\varepsilon^2 - \left(v_\varepsilon^{\frac{2kn}{n-2k}} + \varepsilon^{n-2k} - \varepsilon^n \right)^{1/k} \right)
$$

and $\varepsilon^{\frac{n-2k}{2k}} \leq v_\varepsilon(t)$, for all $t \in \mathbb{R}$, implies that

$$(v_\varepsilon')^2 \leq \left(\frac{n-2k}{2k}\right)^2 \left(v_\varepsilon^2 - \varepsilon^{\frac{n-2k}{k}}\right).$$

Therefore, using that $\cosh t \leq e^{|t|}$ for all $t \in \mathbb{R}$, we get that

$$v_\varepsilon \leq \varepsilon^{\frac{n-2k}{2k}} \cosh\left(\frac{n-2k}{2k}t\right) \leq \varepsilon^{\frac{n-2k}{2k}} e^{\frac{n-2k}{2k}|t|}. \tag{16}$$

Next, writing the equation (13) for v_ε as

$$v_\varepsilon'' - \left(\frac{n-2k}{2k}\right)^2 v_\varepsilon = -\frac{n(n-2k)}{4k^2} v_\varepsilon^{\frac{2kn}{n-2k}-1} h_\varepsilon^{1-k}, \tag{17}$$

and noting that $\cosh\left(\frac{n-2k}{2k}t\right)$ satisfies the equation

$$\left(\cosh\left(\frac{n-2k}{2k}t\right)\right)'' - \left(\frac{n-2k}{2k}\right)^2 \cosh\left(\frac{n-2k}{2k}t\right) = 0,$$

we can represent v_ε as

$$\begin{aligned} v_\varepsilon(t) &= \varepsilon^{\frac{n-2k}{2k}} \cosh\left(\frac{n-2k}{2k}t\right) \\ &- \frac{n(n-2k)}{4k^2} e^{\frac{n-2k}{2k}t} \int_0^t e^{\frac{2k-n}{k}s} \int_0^s e^{\frac{n-2k}{2k}z} v_\varepsilon^{\frac{2kn}{n-2k}-1}(z) h_\varepsilon(z)^{1-k} dz ds. \end{aligned} \tag{18}$$

Now, since $H(v_\varepsilon, v_\varepsilon') > 0$, we get by (14) and (16) that

$$v_\varepsilon^{\frac{2kn}{n-2k}-1} h_\varepsilon^{1-k} = \left(\frac{v_\varepsilon^{\frac{2kn}{n-2k}}}{H(v_\varepsilon, v_\varepsilon') + v_\varepsilon^{\frac{2kn}{n-2k}}}\right)^{\frac{k-1}{k}} v_\varepsilon^{\frac{n+2k}{n-2k}} \leq \varepsilon^{\frac{n+2k}{2k}} e^{\frac{n+2k}{2k}|t|}. \tag{19}$$

By (16), (18) and (19), for all $t > 0$, we get that

$$0 \leq \varepsilon^{\frac{n-2k}{2k}} \cosh\left(\frac{n-2k}{2k}t\right) - v_\varepsilon(t) \leq c_{n,k} \varepsilon^{\frac{n+2k}{2k}} e^{\frac{n+2k}{2k}t},$$

for some constant $c_{n,k} > 0$ which depends only on n and k

Differentiating the identity (18), we get

$$v_\varepsilon'(t) = \frac{n-2k}{2k} \varepsilon^{\frac{n-2k}{2k}} \sinh\left(\frac{n-2k}{2k}t\right) - I_1(t) - I_2(t), \tag{20}$$

where

$$I_1(t) = \frac{n(n-2k)^2}{(2k)^3} e^{\frac{n-2k}{2k}t} \int_0^t e^{\frac{2k-n}{k}s} \int_0^s e^{\frac{n-2k}{2k}z} v_\varepsilon^{\frac{2kn}{n-2k}-1}(z)\, h_\varepsilon(z)^{1-k} dz ds$$

and

$$I_2(t) = \frac{n(n-2k)^2}{(2k)^2} e^{-\frac{n-2k}{2k}t} \int_0^t e^{\frac{n-2k}{2k}z} v_\varepsilon^{\frac{2kn}{n-2k}-1}(z)\, h_\varepsilon(z)^{1-k} dz\ .$$

Using (16) and (19), for all $t > 0$, we get that $I_1(t) \leq c_{n,k}\varepsilon^{\frac{n+2k}{2k}} e^{\frac{n+2k}{2k}t}$ and $I_1(t) \leq c_{n,k}\varepsilon^{\frac{n+2k}{2k}} e^{\frac{n+2k}{2k}t}$. From this and (20) we obtain the second inequality. The third inequality we obtain in analogous way. □

Proposition 2.3. *For any $\varepsilon \in (0, ((n-2k)/n)^{1/2k})$ and any $x \in \mathbb{R}^n \backslash \{0\}$ with $|x| \leq 1$, the Delaunay-type solution $u_\varepsilon(x)$ satisfies the estimates*

$$\begin{aligned}
\left| u_\varepsilon(x) - \frac{\varepsilon^{\frac{n-2k}{2k}}}{2}(1+|x|^{\frac{2k-n}{k}}) \right| &\leq c_{n,k}\varepsilon^{\frac{n+2k}{2k}}|x|^{-\frac{n}{k}}, \\
\left| |x|\partial_r u_\varepsilon(x) + \frac{n-2k}{2k}\varepsilon^{\frac{n-2k}{2k}}|x|^{\frac{2k-n}{k}} \right| &\leq c_{n,k}\varepsilon^{\frac{n+2k}{2k}}|x|^{-\frac{n}{k}} \\
\left| |x|^2\partial_r^2 u_\varepsilon(x) - \frac{(n-2k)^2}{2k^2}\varepsilon^{\frac{n-2k}{2k}}|x|^{\frac{2k-n}{k}} \right| &\leq c_{n,k}\varepsilon^{\frac{n+2k}{2k}}|x|^{-\frac{n}{k}},
\end{aligned}$$

for some positive constant $c_{n,k}$ that depends only on n and k.

Proof. The first inequality follows from the first one in the Proposition 2.2 and the fact that for $t = -\log|x| \geq 0$ with $0 < |x| < 1$ we have $|x|^{\frac{2k-n}{2k}}e^{\frac{n+2k}{2k}|t|} = |x|^{-\frac{n}{k}}$, $u_\varepsilon(x) = |x|^{\frac{2k-n}{2k}}v_\varepsilon(-\log|x|)$ and $\varepsilon^{\frac{n-2k}{2k}}|x|^{\frac{2k-n}{2k}}\cosh\left(\frac{n-2k}{2k}t\right) = \frac{\varepsilon^{\frac{n-2k}{2k}}}{2}\left(1+|x|^{\frac{2k-n}{k}}\right)$.

For the second inequality, note that

$$|x|\partial_r u_\varepsilon(x) = \frac{2k-n}{2k}u_\varepsilon(x) - |x|^{\frac{2k-n}{2k}}v_\varepsilon'(-\log|x|),$$

and

$$\frac{n-2k}{2k}\varepsilon^{\frac{n-2k}{2k}}|x|^{\frac{2k-n}{2k}}\sinh\left(\frac{n-2k}{2k}t\right) = \frac{n-2k}{2k}\varepsilon^{\frac{n-2k}{2k}}\frac{|x|^{\frac{2k-n}{k}}-1}{2}.$$

Therefore, again by Proposition 2.2 we obtain

$$\begin{aligned}
\left| |x|\partial_r u_\varepsilon(x) + \frac{n-2k}{2k}\varepsilon^{\frac{n-2k}{2k}}|x|^{\frac{2k-n}{k}} \right| &\leq \left|\frac{2k-n}{2k}\right| \left| u_\varepsilon(x) - \frac{\varepsilon^{\frac{n-2k}{2k}}}{2}(1+|x|^{\frac{2k-n}{k}}) \right| \\
+ \left| |x|^{\frac{2k-n}{2k}}v_\varepsilon'(-\log|x|) - \frac{n-2k}{2k}\varepsilon^{\frac{n-2k}{2k}}|x|^{\frac{2k-n}{2k}}\sinh\left(\frac{n-2k}{2k}t\right) \right| &\leq c_{n,k}\varepsilon^{\frac{n+2k}{2k}}|x|^{-\frac{n}{k}}.
\end{aligned}$$

In analogous way we get the third inequality. □

For our purposes, it is convenient to consider the following $(n+2)$-dimensional family of solution to (12) in a small punctured ball centered at the origin

$$u_{\varepsilon,R,a}(x) := |x-a|x|^2|^{\frac{2k-n}{2k}}v_\varepsilon(-2\log|x| + \log|x-a|x|^2| + \log R), \tag{21}$$

where only translations along the Delaunay axis and of the "point at infinity" are allowed (see [27]). This family of solutions comes from the fact that if u_ε is a solution then the functions $R^{\frac{2-n}{2}}u_\varepsilon(R^{-1}x)$, $u_\varepsilon(x+b)$ and $|x|^{\frac{2k-n}{k}}u_\varepsilon(x|x|^{-2})$ are still solutions in a small punctured ball centered at the origin for any $R > 0$ and $b \in \mathbb{R}^n$. The last function is related with the inversion $I(x) = x|x|^{-2}$ of the $\mathbb{R}^n\backslash\{0\}$.

In order to simplify the notation we will define $u_{\varepsilon,R} := u_{\varepsilon,R,0}$ and $u_\varepsilon := u_{\varepsilon,1}$.

Corollary 2.1. *For any $\varepsilon \in (0, ((n-2k)/n)^{1/2k})$ and any x in $\mathbb{R}^n$ with $|x| \leq 1$, the function $u_{\varepsilon,R}$ satisfies the estimates*

$$u_{\varepsilon,R}(x) = \frac{\varepsilon^{\frac{n-2k}{2k}}}{2}\left(R^{\frac{2k-n}{2k}} + R^{\frac{n-2k}{2k}}|x|^{\frac{2k-n}{k}}\right) + O''(R^{\frac{n+2k}{2k}}\varepsilon^{\frac{n+2k}{2k}}|x|^{-\frac{n}{k}}),$$

$$|x|\partial_r u_{\varepsilon,R}(x) = \frac{2k-n}{2k}\varepsilon^{\frac{n-2k}{2k}}R^{\frac{n-2k}{2k}}|x|^{\frac{2k-n}{k}} + O'(R^{\frac{n+2k}{2k}}\varepsilon^{\frac{n+2k}{2k}}|x|^{-\frac{n}{k}})$$

and

$$|x|^2\partial_r^2 u_{\varepsilon,R}(x) = \frac{(n-2k)^2}{2k^2}\varepsilon^{\frac{n-2k}{2k}}R^{\frac{n-2k}{2k}}|x|^{\frac{2k-n}{k}} + O(R^{\frac{n+2k}{2k}}\varepsilon^{\frac{n+2k}{2k}}|x|^{-\frac{n}{k}}).$$

Proof. Directly by the Proposition 2.3. □

Corollary 2.2. *There exists a constant $r_0 \in (0,1)$, such that for any x and a in $\mathbb{R}^n$ with $|x| \le 1$, $|a||x| < r_0$, $R \in \mathbb{R}^+$, and $\varepsilon \in (0, ((n-2k)/n)^{1/2k})$ the solution $u_{\varepsilon,R,a}$ satisfies the estimate*

$$u_{\varepsilon,R,a}(x) = u_{\varepsilon,R}(x) + \left(\frac{n-2k}{k}u_{\varepsilon,R}(x) + |x|\partial_r u_{\varepsilon,R}(x)\right) a\cdot x + O''\left(|a|^2|x|^{\frac{6k-n}{2k}}\right) \tag{22}$$

and if $R \le |x|$, the estimate

$$\begin{aligned} u_{\varepsilon,R,a}(x) &= u_{\varepsilon,R}(x) + \left(\frac{n-2k}{k}u_{\varepsilon,R}(x) + |x|\partial_r u_{\varepsilon,R}(x)\right) a\cdot x \\ &\quad + O''\left(|a|^2\varepsilon^{\frac{n-2k}{2k}}R^{\frac{2k-n}{2k}}|x|^2\right). \end{aligned} \tag{23}$$

Proof. First note that

$$|x - a|x|^2|^{\frac{2k-n}{2k}} = |x|^{\frac{2k-n}{2k}} + \frac{n-2k}{2k}a\cdot x|x|^{\frac{2k-n}{2k}} + O''(|a|^2|x|^{\frac{6k-n}{2k}}) \tag{24}$$

and

$$\log\left|\frac{x}{|x|} - a|x|\right| = -a\cdot x + O''(|a|^2|x|^2),$$

for $|a||x| < r_0$ and some $r_0 \in (0,1)$. Using the Taylor's expansion we obtain that

$$\begin{aligned} v_\varepsilon\left(-\log|x| + \log\left|\frac{x}{|x|} - a|x|\right| + \log R\right) &= v_\varepsilon(-\log|x| + \log R) \\ &\quad - v_\varepsilon'(-\log|x| + \log R)a\cdot x + v_\varepsilon'(-\log|x| + \log R)O''(|a|^2|x|^2) \\ &\quad + v_\varepsilon''(-\log|x| + \log R + t_{a,x})O''(|a|^2|x|^2) \end{aligned} \tag{25}$$

for some $t_{a,x} \in \mathbb{R}$ with $0 < |t_{a,x}| < \left|\log\left|\frac{x}{|x|} - a|x|\right|\right|$. Observe that $t_{a,x} \to 0$ when $|a||x| \to 0$. By (15) and (17) we obtain $|v_\varepsilon'| \le c_{n,k}v_\varepsilon$ and $|v_\varepsilon''| \le c_{n,k}v_\varepsilon$. Then, multiplying (24) by (25), we get (22).

For the second equality, note that if $R \le |x|$, then $-\log|x| + \log R \le 0$. Therefore, the result follows by (16) and

$$|x|\partial_r u_{\varepsilon,R}(x) = \frac{2k-n}{2k}u_{\varepsilon,R}(x) - |x|^{\frac{2k-n}{2k}}v_\varepsilon'(-\log|x| + \log R).$$

□

In the Section 5 we put together all the analysis done in the Section 3 and 4 to perform the gluing procedure. To do that we divided the analysis in the high and low frequencies spaces in the boundary of the geodesic ball. The high frequencies space is controlled by the Dirichlet boundary data, while we use the Corollaries 2.1 and 2.2 to control the low frequencies space, since the first two terms in the expansions (22) and (23) belongs to this space. Observe that by Corollary 2.1 the parameter R will be used to control the space spanned by the constant functions and by the Corollary 2.2 the parameter a will be used to control the space spanned by the coordinate functions.

2.4. **Function spaces.** In this section we define some function spaces that we use in this work. This spaces has appeared in [17], [18], [25], [27] and [32]. See these works for more details.

Definition 2.1. *For each $k \in \mathbb{N}$, $r > 0$, $\alpha \in (0,1)$, $\sigma \in (0, r/2)$ and $u \in C^k_{loc}(B_r(0)\backslash\{0\})$, we define*

$$\begin{aligned}\|u\|_{(k,\alpha),[\sigma,2\sigma]} &= \sup_{|x|\in[\sigma,2\sigma]}\left(\sum_{j=0}^{k}\sigma^j|\nabla^j u(x)|\right)\\ &+ \sigma^{k+\alpha}\sup_{|x|,|y|\in[\sigma,2\sigma]}\frac{|\nabla^k u(x)-\nabla^k u(y)|}{|x-y|^\alpha}.\end{aligned}$$

Then, for any $\mu \in \mathbb{R}$, the weighted Hölder space $C^{k,\alpha}_\mu(B_r(0)\backslash\{0\})$ is the collection of functions $u \in C^{k,\alpha}_{loc}(B_r(0)\backslash\{0\})$ which the weighted Hölder norm

$$\|u\|_{C^{k,\alpha}_\mu(B_r(0)\backslash\{0\})} = \sup_{0<\sigma\leq\frac{r}{2}}\sigma^{-\mu}\|u\|_{(k,\alpha),[\sigma,2\sigma]}$$

is finite.

Note that if $\mu \geq \delta$ and $k \geq l$, then $C^{k,\alpha}_\mu(B_r(0)\backslash\{0\}) \subseteq C^{l,\alpha}_\delta(B_r(0)\backslash\{0\})$ and

$$\|u\|_{C^{l,\alpha}_\delta(B_r(0)\backslash\{0\})} \leq C\|u\|_{C^{k,\alpha}_\mu(B_r(0)\backslash\{0\})}$$

for all $u \in C^{k,\alpha}_\mu(B_r(0)\backslash\{0\})$.

Definition 2.2. *For each $k \in \mathbb{N}$, $0 < \alpha < 1$ and $r > 0$. The space $C^{k,\alpha}(\mathbb{S}^{n-1}_r)$ is the collection of functions $\phi \in C^k(\mathbb{S}^{n-1}_r)$ for which the norm*

$$\|\phi\|_{C^{k,\alpha}(\mathbb{S}^{n-1}_r)} := \|\phi(r\cdot)\|_{C^{k,\alpha}(\mathbb{S}^{n-1})}.$$

is finite.

We often will write

$$C^{k,\alpha}(\mathbb{S}^{n-1}_r)^\perp := \left\{\phi \in C^{k,\alpha}(\mathbb{S}^{n-1}_r);\ \pi_r(\phi) = \phi\right\},$$

$$C^{k,\alpha}_\mu(B_r(0)\backslash\{0\})^\perp := \left\{u \in C^{k,\alpha}_\mu(B_r(0)\backslash\{0\});\ \pi_s(u(s\cdot)) = u(s\cdot), \forall s \in (0,r)\right\}$$

and

$$C^{k,\alpha}_\mu(B_r(0)\backslash\{0\})^\top := \left\{u \in C^{k,\alpha}_\mu(B_r(0)\backslash\{0\}); \pi_s(u(s\cdot)) = 0, \forall s \in (0,r)\right\}.$$

Next, consider (M,g) an n-dimensional compact Riemannian manifold and $\Psi : B_{r_1}(0) \to M$ some coordinate system on M centered at some point $p \in M$, where $B_{r_1}(0) \subset \mathbb{R}^n$ is the ball of radius $r_1 > 0$ centered in the origin. For $0 < r < s \leq r_1$ define $M_r := M\backslash\Psi(B_r(0))$ and $\Omega_{r,s} := \Psi(A_{r,s})$, where $A_{r,s} := \{x \in \mathbb{R}^n; r \leq |x| \leq s\}$.

Definition 2.3. *For all $k \in \mathbb{N}$, $\alpha \in (0,1)$, $0 < r < s \leq r_1$ and $\mu \in \mathbb{R}$, the spaces $C^{k,\alpha}_\mu(\Omega_{r,s})$ and $C^{k,\alpha}_\mu(M_r)$ are the spaces of functions $v \in C^{k,\alpha}_{loc}(M\backslash\{p\})$ for which the following norms*

$$\|v\|_{C^{k,\alpha}_\mu(\Omega_{r,s})} := \sup_{r\leq\sigma\leq\frac{s}{2}}\sigma^{-\mu}\|v\circ\Psi\|_{(k,\alpha),[\sigma,2\sigma]}$$

and

$$\|v\|_{C^{k,\alpha}_\mu(M_r)} := \|v\|_{C^{k,\alpha}(M_{\frac{1}{2}r_1})} + \|v\|_{C^{k,\alpha}_\mu(\Omega_{r,r_1})},$$

respectively, are finite.

2.5. **The linearized operator about the Delaunay-type solutions.** Since we will need of the inverse of the linearized operator, we start this section recalling the expression for the linearized operator about the Delaunay-type solution v_ε and a proposition from [27].

Lemma 2.4 (Mazzieri-Ndiaye [27])**.** *In the cylinder $\mathbb{S}^{n-1}\times\mathbb{R}$ the linearization of the operator defined in (4) about the Delaunay-type solution v_ε is given by*

$$\begin{aligned} L^{v_\varepsilon}_{g_{cyl}}(w) &= C_{n,k}v_\varepsilon h_\varepsilon^{k-1}\left(\frac{\partial^2}{\partial t^2}+a_\varepsilon\Delta_{\mathbb{S}^{n-1}}+b_\varepsilon\frac{\partial}{\partial t}+c_\varepsilon\right)w \\ &= C_{n,k}v_\varepsilon h_\varepsilon^{\frac{k-1}{2}}\mathcal{L}_\varepsilon(h_\varepsilon^{\frac{k-1}{2}}w), \end{aligned}$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the Laplace-Beltrami operator for standard round metric on the unit sphere,

$$\begin{aligned} \mathcal{L}_\varepsilon &= \frac{\partial^2}{\partial t^2}+a_\varepsilon\Delta_{\mathbb{S}^{n-1}}+c_\varepsilon+d_\varepsilon, \\ a_\varepsilon &= 1-\frac{n(k-1)}{k(n-1)}\frac{H_\varepsilon}{h_\varepsilon^k}=\frac{n-k}{k(n-1)}+\frac{v_\varepsilon^{\frac{2kn}{n-2k}}}{h_\varepsilon^k}, \\ b_\varepsilon &= -\left(\frac{2(n-k)}{n-2k}-\frac{2k(n-1)}{n-2k}a_\varepsilon\right)\frac{v_\varepsilon'}{v_\varepsilon}=(k-1)\frac{h_\varepsilon'}{h_\varepsilon}, \\ c_\varepsilon &= -\frac{(n-1)(n-2k)}{2k}a_\varepsilon+\frac{n-2k}{2k}+\frac{v_\varepsilon''}{v_\varepsilon}+\frac{n^2}{2k}h^{1-k}v_\varepsilon^{\frac{2kn}{n-2k}-2}, \\ d_\varepsilon &= -\frac{k-1}{2}\frac{\partial^2}{\partial t^2}\log h_\varepsilon-\left(\frac{k-1}{2}\right)^2\left(\frac{\partial}{\partial t}\log h_\varepsilon\right)^2, \end{aligned}$$

$H_\varepsilon:=H(v_\varepsilon,v_\varepsilon')$ is the Hamiltonian energy, h_ε is defined in (15) and the constant $C_{n,k}$ depends only on n and k.

Before we proceed let us recall the notion of Jacobi fields.

First, note that by (21) we can write

$$u_{\varepsilon,R,a}=|x|^{\frac{2k-n}{2k}}v_{\varepsilon,R,a}(-\log|x|,x|x|^{-1}),$$

where $v_{\varepsilon,R,a}$ is a function defined in $(0,+\infty)\times\mathbb{S}^{n-1}\backslash\{(\log|a|,a|a|^{-1})\}$ given by

$$v_{\varepsilon,R,a}(t,\theta)=|\theta-ae^{-t}|^{\frac{2k-n}{2k}}v_\varepsilon(t+\log|\theta-ae^{-t}|+\log R)$$

and v_ε is defined right after the Proposition 2.1. Now, it is easy to see that if $s\mapsto v_s$ is a variation of a solution v such that for each fixed s the function v_s is a solution of

$$H_{g_{cyl}}(v_s)=0, \quad \text{with} \quad v_0=v,$$

then deriving, we get that

$$\left.\frac{\partial}{\partial s}\right|_{s=0}H_{g_{cyl}}(v_s)=L^v_{g_{cyl}}\left(\left.\frac{\partial}{\partial s}\right|_{s=0}v_s\right)=0.$$

Therefore, the function $\left.\frac{\partial}{\partial s}\right|_{s=0}v_s$ belongs to the kernel of $L^v_{g_{cyl}}$ and it is the so called Jacobi fields.

Applying this to the family of solutions $R \mapsto v_{\varepsilon,R} := v_{\varepsilon,R,0}$ and $a \mapsto v_{\varepsilon,1,a}$ we define the following Jacobi fields

$$\Psi_\varepsilon^{0,+}(t,\theta) := \frac{\partial}{\partial R}\bigg|_{R=1} v_{\varepsilon,R}(t,\theta) = v'_\varepsilon(t)$$

and for $j = 1, \dots, n$

$$\Psi_\varepsilon^{j,+}(t,\theta) := \frac{\partial}{\partial a_j}\bigg|_{a=0} v_{\varepsilon,1,a}(t,\theta) = \left(\frac{n-2k}{2k}v_\varepsilon(t) - v'_\varepsilon(t)\right)e^{-t}e_j(\theta),$$

where the functions $\{e_j\}_{j=1}^n$ are the n eigenfunctions of the Laplacian on $\mathbb{S}^{n-1}$ with eigenvalue $n-1$ as defined in Section 2.1. Therefore, the Jacobi fields $\{\Psi_\varepsilon^{j,+}\}_{j=0}^n$ belongs to the low frequencies space. In the same way we can define more Jacobi fields $\{\Psi_\varepsilon^{j,-}\}_{j=0}^n$, see [27] for more details about the subject.

We can also linearize the operator $H_{g_{cyl}}$ about the solution $v_{\varepsilon,R}$. The expression is the same as in the Lemma 2.4 with the function $v_{\varepsilon,R}$ instead of v_ε. Then, in this case we will denote with the letters ε and R the quantities in the Lemma 2.4, for instance, $\mathcal{L}_{\varepsilon,R}$, $h_{\varepsilon,R}$, $a_{\varepsilon,R}$ and so on.

Let $\mathcal{C}_+$ be the half cylinder $(0,+\infty)\times\mathbb{S}^n$ with the canonical metric. In [27] the authors define the weighted Hölder space $C_\gamma^{k,\alpha}(\mathcal{C}_+)$ analogous to the Definition 2.1. We notice that if $u \in C_{loc}^{k,\alpha}(B_1(0)\backslash\{0\})$ and $v \in C_{loc}^{k,\alpha}(\mathcal{C}_+)$ are functions such that $v(t,\theta) = u(e^{-t}\theta)$, then we can show that $u \in C_\mu^{k,\alpha}(B_1(0)\backslash\{0\})$ if only if $v \in C_\mu^{k,\alpha}(\mathcal{C}_+)$. Besides, the norms are equal. We define the spaces $C_\mu^{k,\alpha}(\mathcal{C}_+)^\perp$ and $C_\mu^{k,\alpha}(\mathcal{C}_+)^\top$ analogously to those right after Definition 2.2 on page 13.

Now, define

$$\delta_{n,k} := \sqrt{\frac{2n(n-k)}{k(n-1)} + \left(\frac{n-2k}{2k}\right)^2}. \tag{26}$$

Note that $\delta_{n,k} + 1 - n/2k < 2$ for all $k > 1$.

Proposition 2.4 (Mazzieri-Ndiaye [27]). *Let $R > 0$, $\gamma \in (-\delta_{n,k}, \delta_{n,k})$, $\overline{\gamma} > n/2$ and $\alpha \in (0,1)$. There exists a positive real number $\varepsilon_0 = \varepsilon_0(\gamma,\overline{\gamma},n,k,\alpha) > 0$ such that for every $\varepsilon \in (0,\varepsilon_0]$, the bounded linear operator*

$$\mathcal{L}_{\varepsilon,R} : [C_\gamma^{2,\alpha}(\mathcal{C}_+)^\perp \oplus C_{\overline{\gamma}}^{2,\alpha}(\mathcal{C}_+)^\top \oplus \mathcal{W}_\varepsilon(\mathcal{C}_+)]_0 \to C_\gamma^{0,\alpha}(\mathcal{C}_+)^\perp \oplus C_{\overline{\gamma}}^{0,\alpha}(\mathcal{C}_+)^\top$$

is an isomorphism, where $\mathcal{W}_\varepsilon(\mathcal{C}_+)$ is a finite vetorial space called the deficiency space, which is generated by the Jacobi fields $\Psi_\varepsilon^{j,+}$. Moreover, if $w \in C_\gamma^{2,\alpha}(\mathcal{C}_+)^\perp \oplus C_{\overline{\gamma}}^{2,\alpha}(\mathcal{C}_+)^\top \oplus \mathcal{W}_\varepsilon(\mathcal{C}_+)$ and $f \in C_\gamma^{0,\alpha}(\mathcal{C}_+)^\perp \oplus C_{\overline{\gamma}}^{0,\alpha}(\mathcal{C}_+)^\top$ verify $\mathcal{L}_{\varepsilon,R}w = f$, and, with the notations introduced above, we decompose w and f as

$$w = w^\perp + w^\top + h_\varepsilon^{\frac{k-1}{2}}\sum_{j=0}^n a_j\Psi_\varepsilon^{j,+} \quad \textit{and} \quad f = f^\perp + f^\top,$$

then we have that there exists a positive constant $C = C(\gamma,\overline{\gamma},n,k,\alpha) > 0$ such that, for every $\varepsilon \in (0,\varepsilon_0]$,

$$\|w^\perp\|_{C_\gamma^{2,\alpha}(\mathcal{C}_+)} \le C\|f^\perp\|_{C_\gamma^{0,\alpha}(\mathcal{C}_+)},$$

$$\|w^\top\|_{C_{\overline{\gamma}}^{2,\alpha}(\mathcal{C}_+)} \le C\|f^\top\|_{C_{\overline{\gamma}}^{0,\alpha}(\mathcal{C}_+)}$$

and

$$\varepsilon^{\frac{n-2k}{2}}\sum_{j=0}^{n}|a_j|\leq C\|f^{\top}\|_{C^{0,\alpha}_{\overline{\gamma}}(\mathcal{C}_+)}.$$

Proof. A carefully reading of the proof in [27] we see that the constant C does not depend on R. □

From this proposition we get the following proposition.

Proposition 2.5. *Let $R>0$, $\gamma\in(-\delta_{n,k}+1-n/2k,\delta_{n,k}+1-n/2k)$, $\overline{\gamma}>n/2+1-n/2k$ and $\alpha\in(0,1)$. There exists a positive real number $\varepsilon_0=\varepsilon_0(\gamma,\overline{\gamma},n,k,\alpha)>0$ such that, for every $\varepsilon\in(0,\varepsilon_0]$, there is an operator*

$$G_{\varepsilon,R}:C^{0,\alpha}_{\gamma-1-n+\frac{n}{2k}}(B_1(0)\backslash\{0\})^{\perp}\oplus C^{0,\alpha}_{\overline{\gamma}-1-n+\frac{n}{2k}}(B_1(0)\backslash\{0\})^{\top}$$
$$\longrightarrow C^{2,\alpha}_{\gamma}(B_1(0)\backslash\{0\})^{\perp}\oplus C^{2,\alpha}_{\overline{\gamma}}(B_1(0)\backslash\{0\})^{\top}$$

such that for $f=f^{\perp}+f^{\top}\in C^{0,\alpha}_{\gamma}(B_1(0)\backslash\{0\})^{\perp}\oplus C^{0,\alpha}_{\overline{\gamma}}(B_1(0)\backslash\{0\})^{\top}$, the function $w:=G_{\varepsilon,R}(f)=w^{\perp}+w^{\top}$ solves the equation

$$\begin{cases} L^{u_{\varepsilon,R}}_{g_{eucl}}(w)=f & \text{in} \quad B_1(0)\backslash\{0\} \\ \pi_1(w|_{\mathbb{S}^{n-1}})=0 & \text{on} \quad \partial B_1(0)\end{cases},$$

and the norm satisfies

$$\|w^{\perp}\|_{C^{2,\alpha}_{\gamma}(B_1(0)\backslash\{0\})}\leq C\|f^{\perp}\|_{C^{0,\alpha}_{\gamma-1-n+\frac{n}{2k}}(B_1(0)\backslash\{0\})}$$

and

$$\|w^{\top}\|_{C^{2,\alpha}_{\overline{\gamma}}(B_1(0)\backslash\{0\})}\leq C\|f^{\top}\|_{C^{0,\alpha}_{\overline{\gamma}-1-n+\frac{n}{2k}}(B_1(0)\backslash\{0\})},$$

where $C=C(\gamma,\overline{\gamma},n,k,\alpha)>0$ is a constant.

Proof. Since $\Phi^*g_{eucl}=(e^{\frac{2k-n}{2k}t})^{\frac{4k}{n-2k}}g_{cyl}$, then using the conformal equivariance (10) we obtain

$$L^{u_{\varepsilon,R}}_{g_{eucl}}(w)\circ\Phi=e^{nt}L^{v_{\varepsilon,R}}_{g_{cyl}}(e^{\frac{2k-n}{2k}t}w\circ\Phi). \tag{27}$$

From this and Proposition 2.4 the result follows. □

To construct a family of complete constant σ_2-curvature metrics with Dalaunay-type ends on a punctured ball $B_r(p)\backslash\{p\}$, we need to linearize the operator $H_{g_{eucl}}$ about the Delaunay-type solution $u_{\varepsilon,R,a}$ and to find a (right) inverse. To do that, first let f be a function defined in $B_1(0)\backslash\{0\}$ and let v be a solution of

$$L_{\varepsilon,R}(v)=f \text{ in } B_1(0)\backslash\{0\}.$$

Here we are setting $L_{\varepsilon,R}:=L^{u_{\varepsilon,R}}_{g_{eucl}}$.

Note that, for $r>0$, if $v_r(x)=v(r^{-1}x)$, then by Proposition 2.4 and (27) we get

$$L_{\varepsilon,rR}(v_r)(x)=r^{-1-n+\frac{n}{2k}}L_{\varepsilon,R}(v)(r^{-1}x).$$

So, if we define $g(x)=r^{-1-n+\frac{n}{2k}}f(r^{-1}x)$, then

$$L_{\varepsilon,rR}(v_r)=g \text{ in } B_r(0)\backslash\{0\}.$$

Besides, the norm satisfies

$$\|v_r\|_{C^{2,\alpha}_{\mu}(B_r(0)\backslash\{0\})}=r^{-\mu}\|v\|_{C^{2,\alpha}_{\mu}(B_1(0)\backslash\{0\})}$$

and

$$\|g\|_{C^{2,\alpha}_{\mu}(B_r(0)\backslash\{0\})} = r^{-\mu}\|f\|_{C^{2,\alpha}_{\mu}(B_1(0)\backslash\{0\})}.$$

Therefore we obtain the next result.

Proposition 2.6. *Let $R>0$, $\gamma\in(-\delta_{n,k}+1-n/2k,\delta_{n,k}+1-n/2k)$, $\overline{\gamma}>n/2+1-n/2k$ and $\alpha\in(0,1)$. There exists a positive real number $\varepsilon_0=\varepsilon_0(\gamma,\overline{\gamma},n,k,\alpha)>0$ such that, for every $\varepsilon\in(0,\varepsilon_0]$, there is an operator*

$$\begin{aligned} G_{\varepsilon,R,r}: C^{0,\alpha}_{\gamma-1-n+\frac{n}{2k}}(B_r(0)\backslash\{0\})^{\perp}\oplus C^{0,\alpha}_{\overline{\gamma}-1-n+\frac{n}{2k}}(B_r(0)\backslash\{0\})^{\top}\\ \longrightarrow C^{2,\alpha}_{\gamma}(B_r(0)\backslash\{0\})^{\perp}\oplus C^{2,\alpha}_{\overline{\gamma}}(B_r(0)\backslash\{0\})^{\top}\end{aligned}$$

such that for $f=f^{\perp}+f^{\top}\in C^{0,\alpha}_{\gamma}(B_r(0)\backslash\{0\})^{\perp}\oplus C^{0,\alpha}_{\overline{\gamma}}(B_r(0)\backslash\{0\})^{\top}$, the function $w:=G_{\varepsilon,R}(f)=w^{\perp}+w^{\top}$ solves the equation

$$\begin{cases} L_{\varepsilon,R}(w)=f & in \quad B_r(0)\backslash\{0\}\\ \pi_r(w|_{\mathbb{S}^{n-1}_r})=0 & on \quad \partial B_r(0)\end{cases},$$

and the norm satisfies

$$\|w^{\perp}\|_{C^{2,\alpha}_{\gamma}(B_r(0)\backslash\{0\})}\leq C\|f^{\perp}\|_{C^{0,\alpha}_{\gamma-1-n+\frac{n}{2k}}(B_r(0)\backslash\{0\})}$$

and

$$\|w^{\top}\|_{C^{2,\alpha}_{\overline{\gamma}}(B_r(0)\backslash\{0\})}\leq C\|f^{\top}\|_{C^{0,\alpha}_{\overline{\gamma}-1-n+\frac{n}{2k}}(B_r(0)\backslash\{0\})},$$

where $C=C(\gamma,\overline{\gamma},n,k,\alpha)>0$ is a constant.

Now, since the weight in the high and low frequencies space in the Proposition 2.6 are different, then we need to define the following space

$$C^{k,\alpha}_{(\mu,\nu)}(B_r(0)\backslash\{0\}) := C^{k,\alpha}_{\mu}(B_r(0)\backslash\{0\})^{\perp}\oplus C^{k,\alpha}_{\nu}(B_r(0)\backslash\{0\})^{\top},$$

with the norm

$$\|u\|_{C^{k,\alpha}_{(\mu,\nu)}(B_r(0)\backslash\{0\})} := r^{\gamma-\overline{\gamma}}\|u^{\perp}\|_{C^{k,\alpha}_{\mu}(B_r(0)\backslash\{0\})}+\|u^{\top}\|_{C^{k,\alpha}_{\nu}(B_r(0)\backslash\{0\})}, \tag{28}$$

where $u=u^{\perp}+u^{\top}$ with $u^{\perp}\in C^{k,\alpha}_{\mu}(B_r(0)\backslash\{0\})^{\perp}$ and $u^{\top}\in C^{k,\alpha}_{\nu}(B_r(0)\backslash\{0\})^{\top}$ and γ and $\overline{\gamma}$ are given by Proposition 2.6.

From now on we will write

$$L_{\varepsilon,R,a} := L^{u_{\varepsilon,R,a}}_{g_{eucl}}.$$

By a perturbation argument we obtain the next corollary.

Corollary 2.3. *Let $R>0$, $\gamma\in(-\delta_{n,k}+1-n/2k,\delta_{n,k}+1-n/2k)$, $\overline{\gamma}>n/2+1-n/2k$ and $\alpha\in(0,1)$. There exist a positive real numbers $\varepsilon_0=\varepsilon_0(\mu,\delta,n,\alpha)>0$ and $r_0>0$, such that, for every $\varepsilon\in(0,\varepsilon_0]$, $a\in\mathbb{R}^n$ and $r\in(0,1]$ with $|a|r\leq r_0$, there is an operator*

$$G_{\varepsilon,R,r,a}: C^{0,\alpha}_{(\gamma-1-n+\frac{n}{2k},\overline{\gamma}-1-n+\frac{n}{2k})}(B_r(0)\backslash\{0\})\to C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_r(0)\backslash\{0\})$$

with the norm bounded independently of ε and R, such that for every function $f\in C^{0,\alpha}_{(\gamma-1-n+\frac{n}{2k},\overline{\gamma}-1-n+\frac{n}{2k})}(B_r(0)\backslash\{0\})$, the function $w:=G_{\varepsilon,R,r,a}(f)$ solves the equation

$$\begin{cases} L_{\varepsilon,R,a}(w)=f & in \quad B_r(0)\backslash\{0\}\\ \pi_r(w|_{\mathbb{S}^{n-1}})=0 & on \quad \partial B_r(0)\end{cases}$$

and the norm satisfies

$$\|w\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_r(0)\backslash\{0\})} \leq C\|f\|_{C^{0,\alpha}_{(\gamma-1-n+\frac{n}{2k},\overline{\gamma}-1-n+\frac{n}{2k})}(B_r(0)\backslash\{0\})}, \tag{29}$$

where $C = C(\gamma,\overline{\gamma},n,k,\alpha) > 0$ is a constant.

Proof. By Lemma 2.4 and (27) we obtain

$$\|L^{u_{\varepsilon,R,a}}_{g_{eucl}}(v) - L^{u_{\varepsilon,R}}_{g_{eucl}}(v)\|_{(0,\alpha),[\sigma,2\sigma]} \leq c_{n,k}|a|\sigma^{-n+\frac{n}{2k}}\|v\|_{(2,\alpha),[\sigma,2\sigma]}.$$

This inequality holds for the low and high frequencies spaces. Thus by definition of the norm (28) we get that

$$\|L^{u_{\varepsilon,R,a}}_{g_{eucl}}(v) - L^{u_{\varepsilon,R}}_{g_{eucl}}(v)\|_{C^{0,\alpha}_{(\gamma-1-n+\frac{n}{2k},\overline{\gamma}-1-n+\frac{n}{2k})}(B_r(0)\backslash\{0\})} \leq c_{n,k}|a|r\|v\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_r(0)\backslash\{0\})}.$$

Therefore if we choose $r_0 \leq (2c_{n,k}\|G_{\varepsilon,R,r}\|)^{-1}$, where $G_{\varepsilon,R,r}$ is the operator given by Proposition 2.6, then

$$\|L_{\varepsilon,R,a}\circ G_{\varepsilon,R,r} - I\| \leq \|L_{\varepsilon,R,a} - L_{\varepsilon,R}\|\|G_{\varepsilon,R,r}\| \leq \frac{1}{2}.$$

This implies that the operator $L_{\varepsilon,R,a}\circ G_{\varepsilon,R,r}$ has a bounded right inverse given by

$$(L_{\varepsilon,R,a}\circ G_{\varepsilon,R,a})^{-1} = \sum_{i=0}^{\infty}(I - L_{\varepsilon,R,a}\circ G_{\varepsilon,R,r})^i$$

and it has norm bounded independently of ε, R, a and r,

$$\|\left(\pi'\circ L_{\varepsilon,R,a}\circ G_{\varepsilon,R,a}\right)^{-1}\| \leq \sum_{i=0}^{\infty}\|L_{\varepsilon,R,a}\circ G_{\varepsilon,R,a} - I\|^i \leq 1.$$

Therefore we define a right inverse for $L_{\varepsilon,R,a}$ by

$$G_{\varepsilon,R,r,a} := G_{\varepsilon,R,r}\circ(L_{\varepsilon,R,a}\circ G_{\varepsilon,R,r})^{-1}.$$

□

The right inverse $G_{\varepsilon,R,r,a}$ found in the Corollary 2.3 will be play a important role in the next section. Given a Riemannian metric g in the ball $B_r(0)$, we will be interested in to find a solution v to the equation $H_g(u_{\varepsilon,R,a}+v) = 0$ in the punctured ball $B_r(0)\backslash\{0\}$ such that the resulting conformal metric is complete. Since in normal coordinates the metric g is a perturbation of the euclidian metric and $H_{g_{eucl}}(u_{\varepsilon,R,a}) = 0$, then we can use the operator $G_{\varepsilon,R,r,a}$ to reduce the problem to a fixed point problem. Besides, any function in the domain of the operator $G_{\varepsilon,R,r,a}$ is dominated by $u_{\varepsilon,R,a}$ and this it is enough to ensure the completeness of the metric. In order to do that we use the assumption on the Weyl tensor as we will explain.

3. Interior Analysis

In this section we will use the assumption on the Weyl tensor to find a family of complete constant σ_2-curvature metrics with Delaunay-type ends on a punctured ball, with Dirichlet boundary data, which depends on $n+2$ parameters. First we use the operator given by Corollary 2.3 to reduce the problem to a fixed point problem and then we find a fixed point.

3.1. **Nonlinear analysis.** Throughout the rest of the paper $d = \left[\frac{n}{2}\right]$. Recall that (M, g_0) is a compact Riemannian manifold with dimension $n \geq 5$, σ_2-curvature equal to $n(n-1)/8$, and with the Weyl tensor W_{g_0} vanishing at $p \in M$ up to order $d-2$, that is,

$$\nabla^l W_{g_0}(p) = 0, \quad l = 0, 1, \dots, d-2. \tag{30}$$

Since our problem is conformally invariant, in this section it will be more convenient to work in conformal normal coordinates given by Theorem 2.7 in [22]. By the proof in [22] we see that there exists a positive smooth function $\mathcal{F} \in C^\infty(M)$ such that $g = \mathcal{F}^{\frac{8}{n-4}} g_0$ and $\mathcal{F}(x) = 1 + \overline{f}$, with $\overline{f} = O(|x|^2)$ in g-normal coordinates at p. Also, since the Weyl tensor is conformally invariant, it follows that the Weyl tensor of the metric g satisfies the condition (30).

In these coordinates it is convenient to consider the Taylor expansion of the metric. We will write $g_{ij} = \exp(h_{ij})$, where h_{ij} is a symmetric two-tensor satisfying $h_{ij} = O(|x|^2)$ and $\mathrm{tr} h_{ij}(x) = O(|x|^N)$, where N is as big as we want. In this case $\det(g_{ij}) = 1 + O(|x|^N)$. Using the assumption of the Weyl tensor (30), we obtain that $h_{ij} = O(|x|^{d+1})$. Therefore, we conclude that $g = g_{eucl} + O(|x|^{d+1})$, $R_g = O(|x|^{d-1})$ and $|Ric_g| = O(|x|^{d-1})$.

Next let us recall from [32] the following proposition, see also [17].

Proposition 3.1. *Let $\mu \leq 2$, $0 < r < 1$ and $\alpha \in (0,1)$. For each $\phi \in C^{2,\alpha}(\mathbb{S}_r^{n-1})^\perp$ there is a function $v_\phi \in C_2^{2,\alpha}(B_r(0)\backslash\{0\})^\perp$ so that*

$$\begin{cases} \Delta v_\phi = 0 & \text{in} \quad B_r(0)\backslash\{0\} \\ v_\phi = \phi & \text{on} \quad \partial B_r(0) \end{cases}$$

and

$$\|v_\phi\|_{C_\mu^{2,\alpha}(B_r(0)\backslash\{0\})} \leq C r^{-\mu} \|\phi\|_{C^{2,\alpha}(\mathbb{S}_r^{n-1})}, \tag{31}$$

where the constant $C > 0$ does not depend on r.

The main goal of this section is to solve the PDE

$$H_g(u_{\varepsilon,R,a} + r^{-\overline{\gamma}}|x|^{\overline{\gamma}} v_\phi + h + v) = 0 \tag{32}$$

in $B_r(0)\backslash\{0\} \subset \mathbb{R}^n$ for some $r > 0$, $\varepsilon > 0$, $R > 0$, $\phi \in C^{2,\alpha}(\mathbb{S}_r^{n-1})^\perp$, $a \in \mathbb{R}^n$ and $\overline{\gamma} > 1 + n/4$, with $u_{\varepsilon,R,a} + r^{-\overline{\gamma}}|x|^{\overline{\gamma}} v_\phi + h + v > 0$ and prescribed Dirichlet boundary data, where the operator H_g is defined in (7) and $u_{\varepsilon,R,a}$ in (21). Here, the function h is defined as

$$h = \frac{1}{2}\left((1-\overline{\gamma}) r^{-\overline{\gamma}-1}|x|^{\overline{\gamma}+1} + (\overline{\gamma}+1) r^{-\overline{\gamma}+1}|x|^{\overline{\gamma}-1}\right) f,$$

where $f = O(|x|^2)$ will be chosen later. We observe that $h = O(|x|^{\overline{\gamma}+1})$ and if $|x| = 1$ then $h(x) = f$ and $\partial_r h(x) = 0$. This function is needed to do the analysis in the Section 5, where we will clarify why it has to appear in (32). The term $r^{-\overline{\gamma}}|x|^{\overline{\gamma}} v_\phi$ in (32) is needed because we want to prescribe the Dirichlet boundary data. Only v_ϕ is not enough because it does not belong to the domain of the operator given by Corollary 2.3, which is used to reduce the equation (32) to a fixed point problem.

Using that $H_{g_{eucl}}(u_{\varepsilon,R,a}) = 0$, then by the Taylor's expansion we see that (32) is equivalent to

$$\begin{aligned} L_{\varepsilon,R,a}(v) &= L_{\varepsilon,R,a}(v) - L_g^{u_{\varepsilon,R,a}}(v) - Q_{\varepsilon,R,a}(r^{-\overline{\gamma}}|x|^{\overline{\gamma}} v_\phi + h + v) \\ &+ H_{g_{eucl}}(u_{\varepsilon,R,a}) - H_g(u_{\varepsilon,R,a}) - L_g^{u_{\varepsilon,R,a}}(r^{-\overline{\gamma}}|x|^{\overline{\gamma}} v_\phi + h), \end{aligned} \tag{33}$$

where $Q_{\varepsilon,R,a}(v) := Q^{u_{\varepsilon,R,a}}_{g_{eucl}}(v)$ is defined in (9).

Therefore, using the right inverse to the operator $L_{\varepsilon,R,a}$ given by Corollary 2.3, we reduce the equation to a fixed point problem

$$\begin{aligned} v &= G_{\varepsilon,R,a,r}\left(L_{\varepsilon,R,a}(v) - L_g^{u_{\varepsilon,R,a}}(v) - Q_{\varepsilon,R,a}(r^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi + h + v)\right. \\ &+ \left. H_{g_{eucl}}(u_{\varepsilon,R,a}) - H_g(u_{\varepsilon,R,a}) - L_g^{u_{\varepsilon,R,a}}(r^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi + h)\right). \end{aligned} \tag{34}$$

But, first we have to show that the right hand side of (34) is well defined, that is, all terms of the right hand side of the equation (33) belong to the right space, which is $C^{0,\alpha}_{(\gamma-1-\frac{3n}{4},\overline{\gamma}-1-\frac{3n}{4})}(B_r(0)\backslash\{0\})$, for some $\gamma\in(-\delta_{n,2}+1-n/4,\delta_{n,2}+1-n/4)$ and $\overline{\gamma}>1+n/4$.

Lemma 3.1. *Let $\gamma=\delta_{n,2}+1-n/4-\varepsilon_1$ and $\overline{\gamma}=n/4+1+\varepsilon_1$, where $\varepsilon_1>0$ is very small. For all $v\in C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_r(0)\backslash\{0\})$ and $\phi\in C^{2,\alpha}(\mathbb{S}^{n-1}_r)^\perp$, the right hand side of (33) belongs to $C^{0,\alpha}_{(\gamma-1-\frac{3n}{4},\overline{\gamma}-1-\frac{3n}{4})}(B_r(0)\backslash\{0\})$.*

Proof. If $v\in C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_r(0)\backslash\{0\})$, then $v=v^\perp+v^\top$ with $v^\perp=O(|x|^\gamma)$ and $v^\top=O(|x|^{\overline{\gamma}})$. Since $\gamma<2<\overline{\gamma}$, $v_\phi=O(|x|^2)$ and $h=O(|x|^{\overline{\gamma}+1})$, we have that $v=O(|x|^\gamma)$ and $r^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi+h+v=O(|x|^\gamma)$. Thus, using (7) we obtain that

$$H_{g_{eucl}}(u_{\varepsilon,R,a}) - H_g(u_{\varepsilon,R,a}) = O(|x|^{d+1-n}) = O(|x|^{\overline{\gamma}-1-\frac{3n}{4}}), \tag{35}$$

since $d\geq\overline{\gamma}+n/4-2$. Now, by (8) we get that

$$L_{\varepsilon,R,a}(v^\perp) - L_g^{u_{\varepsilon,R,a}}(v^\perp) = O(|x|^{d+\gamma-\frac{3n}{4}}) = O(|x|^{\overline{\gamma}-1-\frac{3n}{4}}),$$

since $d+\gamma\geq\overline{\gamma}-1$, and also

$$L_{\varepsilon,R,a}(v^\top) - L_g^{u_{\varepsilon,R,a}}(v^\top) = O(|x|^{d+\overline{\gamma}-\frac{3n}{4}}) = O(|x|^{\overline{\gamma}-1-\frac{3n}{4}}).$$

Using the definition (8) and (9) we obtain that

$$Q_{\varepsilon,R,a}(r^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi+h+v) = O(|x|^{2\gamma-2-\frac{n}{2}}) = O(|x|^{\overline{\gamma}-1-\frac{3n}{4}}),$$

since $2\gamma-1+n/4\geq\overline{\gamma}$. Finally, using again (8) and the fact that $r^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi+h=O(|x|^{\overline{\gamma}+1})$, we obtain that

$$L_g^{u_{\varepsilon,R,a}}(r^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi+h) = O(|x|^{\overline{\gamma}-\frac{3n}{4}}) = O(|x|^{\overline{\gamma}-1-\frac{3n}{4}}).$$

From these estimates and the definition of the norm (28), we obtain the result. □

Now, by the Lemma 3.1 it follows that the right hand side of (34) is well defined. Let $\gamma=\delta_{n,2}+1-n/4-\varepsilon_1$ and $\overline{\gamma}=n/4+1+\varepsilon_1$, where $\varepsilon_1>0$ is small. To solve the equation (32) we need to show that the map

$$N_\varepsilon(R,a,\phi,\cdot): C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_r(0)\backslash\{0\}) \to C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_r(0)\backslash\{0\})$$

has a fixed point for suitable parameters ε, R, a and ϕ. Here $N_\varepsilon(R,a,\phi,\cdot)$ is defined by

$$
\begin{aligned}
N_\varepsilon(R,a,\phi,v) &= G_{\varepsilon,R,r,a}\big(L_{\varepsilon,R,a}(v) - L_g^{u_{\varepsilon,R,a}}(v)\\
&- Q_{\varepsilon,R,a}(r^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi + h + v)\\
&+ H_{g_{eucl}}(u_{\varepsilon,R,a}) - H_g(u_{\varepsilon,R,a})\\
&- L_g^{u_{\varepsilon,R,a}}(r^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi + h)\big),
\end{aligned}
\tag{36}
$$

where $G_{\varepsilon,R,r,a}$ is the right inverse for $L_{\varepsilon,R,a}$ given by Corollary 2.3.

3.2. **Complete Delaunay-type ends with constant σ_2-curvature.** In this section we will show that the map (36) has a fixed point. First we need to do the following restriction to obtain some estimates that it will be necessary in the gluing procedure. From now on, we will consider $r_\varepsilon = \varepsilon^s$, for $\varepsilon > 0$ and $s > 0$ small, and

$$
R^{\frac{4-n}{4}} = 2(1+b)\varepsilon^{\frac{4-n}{4}},
\tag{37}
$$

with $|b| \leq 1/2$.

Next we will prove the main result of this section.

Proposition 3.2. *Let $\gamma = \delta_{n,2} + 1 - n/4 - \varepsilon_1$ and $\overline{\gamma} = n/4 + 1 + \varepsilon_1$, where $\varepsilon_1 > 0$ is a small constant. There exists a constant $\varepsilon_0 \in (0,1)$ such that for each $\varepsilon \in (0,\varepsilon_0)$, $\kappa > 0$, $\tau > 0$, $|b| < 1/2$, $a \in \mathbb{R}^n$, $\delta_1, \delta_2, l \in \mathbb{R}_+$ small and $\phi \in C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})^\perp$, with $3\delta_2 > \max\{\delta_1, l\}$, $|a| r_\varepsilon^{1-\delta_2} \leq 1$ and $\|\phi\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \leq \kappa r_\varepsilon^{2+l-\delta_1}$, there exists a fixed point $u \in C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})$ of the map $N_\varepsilon(R,a,\phi,\cdot)$ in the ball of radius $\tau r_\varepsilon^{2+l-\overline{\gamma}}$.*

Proof. First let us recall the norm in the space $C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})$,

$$
\|v\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})} = {r_\varepsilon}^{\gamma-\overline{\gamma}}\|v^\perp\|_{C^{2,\alpha}_{\gamma}(B_{r_\varepsilon}(0)\backslash\{0\})} + \|v^\top\|_{C^{2,\alpha}_{\overline{\gamma}}(B_{r_\varepsilon}(0)\backslash\{0\})}.
$$

Then, since $r_\varepsilon < 1$ and $\gamma < \overline{\gamma}$, for any $v \in C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})$ we obtain that

$$
\begin{aligned}
\|v\|_{(2,\alpha),[\sigma,2\sigma]} &\leq \sigma^\gamma\|v^\perp\|_{C^{2,\alpha}_{\gamma}(B_{r_\varepsilon}(0)\backslash\{0\})} + \sigma^{\overline{\gamma}}\|v^\top\|_{C^{2,\alpha}_{\overline{\gamma}}(B_{r_\varepsilon}(0)\backslash\{0\})}\\
&\leq \sigma^\gamma\|v\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})}.
\end{aligned}
\tag{38}
$$

In what follows, we use the letter μ to mean either γ or $\overline{\gamma}$.

Note that

$$
\begin{aligned}
N_\varepsilon(R,a,\phi,0) &= -G_{\varepsilon,R,r,a}\big(Q_{\varepsilon,R,a}({r_\varepsilon}^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi + h) + H_g(u_{\varepsilon,R,a})\\
&- H_{g_{eucl}}(u_{\varepsilon,R,a}) + L_g^{u_{\varepsilon,R,a}}({r_\varepsilon}^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi + h)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
N_\varepsilon(R,a,\phi,v_1) - N_\varepsilon(R,a,\phi,v_2) &= G_{\varepsilon,R,r,a}\Big(L_{\varepsilon,R,a}(v_1-v_2) - L_g^{u_{\varepsilon,R,a}}(v_1-v_2)\\
&- \int_0^1 \frac{d}{dt}Q_{\varepsilon,R,a}({r_\varepsilon}^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi + h + v_2 + t(v_1-v_2))dt\Big).
\end{aligned}
$$

By (35) we get that

$$
\sigma^{-\mu+1+\frac{3n}{4}}\|H_\delta(u_{\varepsilon,R,a}) - H_g(u_{\varepsilon,R,a})\|_{(0,\alpha),[\sigma,2\sigma]} \leq C\sigma^{2+d-\mu-\frac{n}{4}}.
$$

Since $d > n/4 > \gamma + n/4 - 2$ then for $\mu = \gamma$ we have that

$$\sigma^{-\gamma+1+\frac{3n}{4}} r_\varepsilon{}^{\gamma-\overline{\gamma}} \|H_\delta(u_{\varepsilon,R,a}) - H_g(u_{\varepsilon,R,a})\|_{(0,\alpha),[\sigma,2\sigma]} \leq C r_\varepsilon{}^{d-\frac{n}{4}-l} r_\varepsilon{}^{2+l-\overline{\gamma}} \tag{39}$$

and for $\mu = \overline{\gamma}$ we have that

$$\sigma^{-\overline{\gamma}+1+\frac{3n}{4}} \|H_\delta(u_{\varepsilon,R,a}) - H_g(u_{\varepsilon,R,a})\|_{(0,\alpha),[\sigma,2\sigma]} \leq C r_\varepsilon{}^{d-\frac{n}{4}-l} r_\varepsilon{}^{2+l-\overline{\gamma}}. \tag{40}$$

Now, by (9) we obtain that

$$\begin{aligned}&\left\|\int_0^1 \frac{d}{dt} Q_{\varepsilon,R,a}(v_2 + t(v_1 - v_2))dt\right\|_{(0,\alpha),[\sigma,2\sigma]} \\ &\leq C\sigma^{-2-\frac{n}{2}} \left(\|v_1\|_{(2,\alpha),[\sigma,2\sigma]} + \|v_2\|_{(2,\alpha),[\sigma,2\sigma]}\right) \|v_1 - v_2\|_{(2,\alpha),[\sigma,2\sigma]}.\end{aligned} \tag{41}$$

Thus, the estimate $|h| \leq C r_\varepsilon{}^{-\overline{\gamma}-1}|x|^{\overline{\gamma}+3} + C r_\varepsilon{}^{-\overline{\gamma}+1}|x|^{\overline{\gamma}+1}$, implies that

$$\begin{aligned}&\sigma^{-\mu+1+\frac{3n}{4}} \|Q_{\varepsilon,R,a}(r_\varepsilon{}^{-\overline{\gamma}}|x|^{\overline{\gamma}} v_\phi + h)\|_{(0,\alpha),[\sigma,2\sigma]} \\ &\leq C\left(r_\varepsilon{}^{3-\mu+\frac{n}{4}+2l-2\delta_1} + r_\varepsilon{}^{3-\mu+\frac{n}{4}}\right) \leq C r_\varepsilon{}^{1+\overline{\gamma}-\mu+\frac{n}{4}-l} r_\varepsilon{}^{2+l-\overline{\gamma}},\end{aligned}$$

and this implies that

$$\|Q_{\varepsilon,R,a}(r_\varepsilon{}^{-\overline{\gamma}}|x|^{\overline{\gamma}} v_\phi + h)\|_{C^{0,\alpha}_{(\gamma-1-\frac{3n}{4},\overline{\gamma}-1-\frac{3n}{4})}(B_{r_\varepsilon}(0)\backslash\{0\})} \leq C r_\varepsilon{}^{1+\frac{n}{4}-l} r_\varepsilon{}^{2+l-\overline{\gamma}}. \tag{42}$$

Now, by (8), we have that

$$\begin{aligned}&\|L_g^{u_{\varepsilon,R,a}}(r_\varepsilon{}^{-\overline{\gamma}}|x|^{\overline{\gamma}} v_\phi + h)\|_{(0,\alpha),[\sigma,2\sigma]} \\ &\leq C\left(\sigma^{-4}\|u_{\varepsilon,R,a}\|^3_{(2,\alpha),[\sigma,2\sigma]} + \|u_{\varepsilon,R,a}^{\frac{3n+4}{n-4}}\|_{(2,\alpha),[\sigma,2\sigma]}\right)\times \\ &\left(\sigma^{\overline{\gamma}+2} r_\varepsilon{}^{-\overline{\gamma}+l-\delta_1} + \sigma^{\overline{\gamma}+1} r_\varepsilon{}^{-\overline{\gamma}+1} + \sigma^{\overline{\gamma}+3} r_\varepsilon{}^{-\overline{\gamma}-1}\right).\end{aligned} \tag{43}$$

Note that, by Corollary 2.2, we obtain that

$$\|u_{\varepsilon,R,a}\|_{(2,\alpha),[\sigma,2\sigma]} \leq \|u_{\varepsilon,R}\|_{(2,\alpha),[\sigma,2\sigma]} + C|a|\sigma^{2-\frac{n}{4}}$$

If $r_\varepsilon^{1+\lambda} \leq |x| \leq r_\varepsilon$ with $\lambda > 0$, then by (37) we obtain that

$$(1-s)\log\varepsilon + \log(2+2b)^{\frac{4}{4-n}} \leq \log(|x|^{-1}R) \leq (1-s(1+\lambda))\log\varepsilon + \log(2+2b)^{\frac{4}{4-n}} < 0.$$

Thus, (16) implies that

$$v_\varepsilon(-\log|x| + \log R) \leq \varepsilon^{\frac{n-4}{4}s}(2+2b).$$

Therefore,

$$u_{\varepsilon,R}(x) \leq C|x|^{\frac{4-n}{4}} r_\varepsilon^{\frac{n-4}{4}}(2+2b)$$

and then

$$\|u_{\varepsilon,R,a}\|_{(2,\alpha),[\sigma,2\sigma]} \leq C\sigma^{1-\frac{n}{4}}(r_\varepsilon^{\frac{n}{4}-1} + |a|\sigma) \leq C\sigma^{1-\frac{n}{4}+\delta_2}.$$

This implies that

$$\begin{aligned}&\sigma^{-\mu-3+\frac{3n}{4}}\|u_{\varepsilon,R,a}\|^3_{(2,\alpha),[\sigma,2\sigma]}\left(\sigma^{\overline{\gamma}+2} r_\varepsilon{}^{-\overline{\gamma}+l-\delta_1} + \sigma^{\overline{\gamma}+1} r_\varepsilon{}^{-\overline{\gamma}+1} + \sigma^{\overline{\gamma}+3} r_\varepsilon{}^{-\overline{\gamma}-1}\right) \\ &\leq C\left(\sigma^{3\delta_2-\mu+\overline{\gamma}+2} r_\varepsilon{}^{-\overline{\gamma}+l-\delta_1} + \sigma^{3\delta_2-\mu+\overline{\gamma}+1} r_\varepsilon{}^{-\overline{\gamma}+1} + \sigma^{3\delta_2-\mu+\overline{\gamma}+3} r_\varepsilon{}^{-\overline{\gamma}-1}\right).\end{aligned}$$

For $\mu = \gamma$, we obtain

$$\sigma^{-\gamma-3+\frac{3n}{4}} {r_\varepsilon}^{\gamma-\overline{\gamma}} \|u_{\varepsilon,R,a}\|^3_{(2,\alpha),[\sigma,2\sigma]} \left(\sigma^{\overline{\gamma}+2} {r_\varepsilon}^{-\overline{\gamma}+l-\delta_1} + \sigma^{\overline{\gamma}+1} {r_\varepsilon}^{-\overline{\gamma}+1} + \sigma^{\overline{\gamma}+3} {r_\varepsilon}^{-\overline{\gamma}-1}\right)$$
$$\leq C({r_\varepsilon}^{3\delta_2-\delta_1} + {r_\varepsilon}^{3\delta_2-l}){r_\varepsilon}^{2+l-\overline{\gamma}}.$$

since $3\delta_2 > \max\{\delta_1, l\}$. For $\mu = \overline{\gamma}$, we have an analogous inequality.

If $0 \leq \sigma \leq {r_\varepsilon}^{1+\lambda}$, then

$$\sigma^{-\mu-3+\frac{3n}{4}} \|u_{\varepsilon,R,a}\|^3_{(2,\alpha),[\sigma,2\sigma]} (\sigma^{\overline{\gamma}+2} {r_\varepsilon}^{-\overline{\gamma}+l-\delta_1} + \sigma^{\overline{\gamma}+1} {r_\varepsilon}^{-\overline{\gamma}+1} + \sigma^{\overline{\gamma}+3} {r_\varepsilon}^{-\overline{\gamma}-1})$$
$$\leq C(\sigma^{\overline{\gamma}-\mu+2} {r_\varepsilon}^{-\overline{\gamma}+l-\delta_1} + \sigma^{\overline{\gamma}-\mu+1} {r_\varepsilon}^{-\overline{\gamma}+1} + \sigma^{\overline{\gamma}-\mu+3} {r_\varepsilon}^{-\overline{\gamma}-1}),$$

which implies that

$$\sigma^{-\gamma-3+\frac{3n}{4}} {r_\varepsilon}^{\gamma-\overline{\gamma}} \|u_{\varepsilon,R,a}\|^3_{(2,\alpha),[\sigma,2\sigma]} (\sigma^{\overline{\gamma}+2} {r_\varepsilon}^{-\overline{\gamma}+l-\delta_1} + \sigma^{\overline{\gamma}+1} {r_\varepsilon}^{-\overline{\gamma}+1} + \sigma^{\overline{\gamma}+3} {r_\varepsilon}^{-\overline{\gamma}-1})$$
$$\leq C({r_\varepsilon}^{(\overline{\gamma}-\gamma+2)\lambda-\delta_1} + {r_\varepsilon}^{(\overline{\gamma}-\gamma+1)\lambda-l} + {r_\varepsilon}^{(\overline{\gamma}-\mu+3)\lambda-l}){r_\varepsilon}^{2+l-\overline{\gamma}}.$$

and an analogous inequality for $\mu = \overline{\gamma}$.

Analogously we estimate the term with $u_{\varepsilon,R,a}^{\frac{3n+4}{n-4}}$ in the inequality (43).

Therefore,

$$\|L_g^{u_{\varepsilon,R,a}}({r_\varepsilon}^{-\overline{\gamma}}|x|^{\overline{\gamma}} v_\phi + h)\|_{C^{0,\alpha}_{(\gamma-1-\frac{3n}{4},\overline{\gamma}-1-\frac{3n}{4})}(B_{r_\varepsilon}(0)\backslash\{0\})} \leq C{r_\varepsilon}^c {r_\varepsilon}^{2+l-\overline{\gamma}}, \tag{44}$$

for some constant $c > 0$. By (39), (40), (42), (44) and using the inequality (29) given by Corollary 2.3, we get that

$$\|N_\varepsilon(R,a,\phi,0)\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})} \leq \frac{1}{2}\tau r_\varepsilon^{2+l-\overline{\gamma}}, \tag{45}$$

for $\varepsilon > 0$ small enough.

Now, by (8) and using the inequality (38), we find that

$$\sigma^{-\mu+1+\frac{3n}{4}} \left\|L_{\varepsilon,R,a}(v_1 - v_2) - L_g^{u_{\varepsilon,R,a}}(v_1 - v_2)\right\|_{(0,\alpha),[\sigma,2\sigma]}$$
$$\leq C\sigma^{d+1-\mu}\|v_1 - v_2\|_{(2,\alpha),[\sigma,2\sigma]} \leq \sigma^{d+1+\gamma-\mu}\|v_1 - v_2\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})},$$

and since $1 + d + \gamma - \overline{\gamma} > 0$, this implies that

$$\begin{aligned} &\left\|L_{\varepsilon,R,a}(v_1 - v_2) - L_g^{u_{\varepsilon,R,a}}(v_1 - v_2)\right\|_{C^{0,\alpha}_{(\gamma-1-\frac{3n}{4},\overline{\gamma}-1-\frac{3n}{4})}(B_{r_\varepsilon}(0)\backslash\{0\})} \\ &\qquad \leq C{r_\varepsilon}^{1+d+\gamma-\overline{\gamma}}\|v_1 - v_2\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})}. \end{aligned} \tag{46}$$

Using (41) and the estimates $\|v^\top\|_{(2,\alpha),[\sigma,2\sigma]} \leq \tau\sigma^\gamma {r_\varepsilon}^{2+l-\gamma}$ and $\|v^\perp\|_{(2,\alpha),[\sigma,2\sigma]} \leq \tau\sigma^\gamma {r_\varepsilon}^{2+l-\gamma}$, for any $v \in C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})$ in the ball with radius $\tau r_\varepsilon^{2+l\overline{\gamma}}$, we get that

$$\left\|\int_0^1 \frac{d}{dt} Q_{\varepsilon,R,a}({r_\varepsilon}^{-\overline{\gamma}}|x|^{\overline{\gamma}} v_\phi + h + v_2 + t(v_1 - v_2))dt\right\|_{(0,\alpha),[\sigma,2\sigma]}$$
$$\leq C\left(\sigma^{\overline{\gamma}+\gamma-\frac{n}{2}} {r_\varepsilon}^{l-\overline{\gamma}-\delta_1} + \sigma^{\gamma+\overline{\gamma}-1-\frac{n}{2}} {r_\varepsilon}^{1-\overline{\gamma}} + \sigma^{\gamma+\overline{\gamma}+1-\frac{n}{2}} {r_\varepsilon}^{-1-\overline{\gamma}}\right.$$
$$\left.+\sigma^{-2-\frac{n}{2}+2\gamma} {r_\varepsilon}^{2-\gamma+l}\right) \|v_1 - v_2\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})}.$$

Therefore, we obtain that

$$\sigma^{-\mu+1+\frac{3n}{4}}\left\|\int_0^1\frac{d}{dt}Q_{\varepsilon,R,a}(r_\varepsilon{}^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi+h+v_2+t(v_1-v_2))dt\right\|_{(0,\alpha),[\sigma,2\sigma]} \tag{47}$$
$$\leq Cr_\varepsilon{}^{1+\gamma-\mu+\frac{n}{4}}\|v_1-v_2\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})}.$$

Note that $1+\gamma-\overline{\gamma}+n/4>0$.

Thus, by (46) and (47) we obtain that

$$\|H_\varepsilon(R,a,\phi,v_1)-H_\varepsilon(R,a,\phi,v_2)\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})}\leq\frac{1}{2}\|v_1-v_2\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})}, \tag{48}$$

for $\varepsilon>0$ small enough. Therefore, using (45) and (48) we obtain the result. □

We summarize the main result of this section in the next theorem.

Theorem 3.2. *Let $\gamma=\delta_{n,2}+1-n/4-\varepsilon_1$ and $\overline{\gamma}=n/4+1+\varepsilon_1$, where $\varepsilon_1>0$ is a small constant. There exists a constant $\varepsilon_0\in(0,1)$ such that for each $\varepsilon\in(0,\varepsilon_0)$, $\kappa>0$, $\tau>0$, $|b|<1/2$, $a\in\mathbb{R}^n$, $\delta_1,\delta_2,l\in\mathbb{R}_+$ small and $\phi\in C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})^\perp$ with $3\delta_2>\max\{\delta_1,l\}$, $|a|r_\varepsilon^{1-\delta_2}\leq 1$ and $\|\phi\|_{(2,\alpha),r_\varepsilon}\leq\kappa r_\varepsilon^{2+l-\delta_1}$, there exists a solution $U_{\varepsilon,R,a,\phi}\in C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})$ for the equation*

$$\begin{cases} H_g(u_{\varepsilon,R,a}+r_\varepsilon^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi+h+U_{\varepsilon,R,a,\phi})=0 & \text{in} \quad B_{r_\varepsilon}(0)\backslash\{0\}\\ \pi_{r_\varepsilon}((r_\varepsilon^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi+U_{\varepsilon,R,a,\phi})|_{\partial B_{r_\varepsilon}(0)})=\phi & \text{on} \quad \partial B_{r_\varepsilon}(0)\end{cases},$$

such that

$$\|U_{\varepsilon,R,a,\phi}\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})}\leq\tau r_\varepsilon^{2+l-\overline{\gamma}} \tag{49}$$

and

$$\|U_{\varepsilon,R,a,\phi_1}-U_{\varepsilon,R,a,\phi_0}\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})}\leq Cr_\varepsilon^{\delta_4-\overline{\gamma}}\|\phi_1-\phi_0\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})}, \tag{50}$$

for some small constant $\delta_4>0$.

Proof. The solution $U_{\varepsilon,R,a,\phi}$ is the fixed point of the map $N_\varepsilon(R,a,\phi,\cdot)$ given by Proposition 3.2 with the estimate (49). Using the fact that $U_{\varepsilon,R,a,\phi}$ is a fixed point of the map $N_\varepsilon(R,a,\phi,\cdot)$ we can show that

$$\begin{aligned}&\|U_{\varepsilon,R,a,\phi_1}-U_{\varepsilon,R,a,\phi_0}\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})}\\ &\leq 2\|N_\varepsilon(R,a,\phi_1,U_{\varepsilon,R,a,\phi_0})-N_\varepsilon(R,a,\phi_0,U_{\varepsilon,R,a,\phi_0})\|_{C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_{r_\varepsilon}(0)\backslash\{0\})}\\ &\leq C\left\|L_g^{u_{\varepsilon,R,a}}(r_\varepsilon^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_{\phi_1-\phi_0})\right\|_{C^{0,\alpha}_{(\gamma-1-\frac{3n}{4},\overline{\gamma}-1-\frac{3n}{4})}(B_{r_\varepsilon}(0)\backslash\{0\})}\\ &+\left\|\int_0^1\frac{d}{dt}Q_{\varepsilon,R,a}(U_{\varepsilon,R,a,\phi_2}+r_\varepsilon^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_{\phi_t}+h)dt\right\|_{C^{0,\alpha}_{(\gamma-1-\frac{3n}{4},\overline{\gamma}-1-\frac{3n}{4})}(B_{r_\varepsilon}(0)\backslash\{0\})},\end{aligned}$$

where $\phi_t=\phi_0+t(\phi_1-\phi_0)$.

From this and the estimates given by the proof of the Proposition 3.2 it follows (50). □

We will write the full conformal factor of the resulting constant scalar curvature metric with respect to the metric g as

$$\mathcal{A}_\varepsilon(R,a,\phi) := u_{\varepsilon,R,a} + r_\varepsilon^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi + h + U_{\varepsilon,R,a,\phi}, \tag{51}$$

in conformal normal coordinates. The previous analysis says that the metric

$$\hat{g} = \mathcal{A}_\varepsilon(R,a,\phi)^{\frac{8}{n-4}}g$$

is defined in $\overline{B_{r_\varepsilon}(p)}\backslash\{p\} \subset M$, it is complete and has $\sigma_2(A_{\hat{g}}) = n(n-1)/8$. The completeness follows from the estimate $\mathcal{A}_\varepsilon(R,a,\phi) \geq c|x|^{\frac{4-n}{4}}$, for some positive constant c.

4. Exterior Analysis

In contrast with the previous section in which we worked with conformal normal coordinates, in this section it is better to work with the constant σ_2-curvature metric g_0, since in this case the constant functions 1 satisfies $H_{g_0}(1) = 0$. Hence, in this section (M^n, g_0) is an n-dimensional nondegenerate closed Riemannian manifold with $\sigma_2(A_{g_0}) = n(n-1)/8$. Therefore, by (2) we find that $R_{g_0} \neq 0$ in M.

4.1. **Analysis in $M\backslash B_r(p)$.** Let $r_1 \in (0,1)$ be a fixed constant. Let $\Psi : B_{r_1}(0) \to M$ be a coordinate system with respect to $g = \mathcal{F}^{\frac{8}{n-4}}g_0$ on M centered at a point $p \in M$, where $\mathcal{F}$ is defined in the beginning ot the Section 3.1. This function satisfies the expansion $\mathcal{F} = 1 + O(|x|^2)$ in g-normal coordinates at p. Note that, in these coordinates, we have $(g_0)_{ij} = \delta_{ij} + O(|x|^2)$, since $g_{ij} = 1 + O(|x|^2)$.

We start this section remember a result from [17] (see also [19] and [32]).

Proposition 4.1. *Assume that $\varphi \in C^{2,\alpha}(\mathbb{S}^{n-1}_r)$ and let $\mathcal{Q}_r(\varphi)$ be the only solution of*

$$\begin{cases} \Delta\mathcal{Q}_r(\varphi) = 0 & in \quad \mathbb{R}^n\backslash B_r(0) \\ \mathcal{Q}_r(\varphi) = \varphi & on \quad \partial B_r(0) \end{cases}$$

which tends to 0 at ∞. Then there exists a contant $C > 0$, that does not depend on r, such that

$$\|\mathcal{Q}_r(\varphi)\|_{C^{2,\alpha}_{1-n}(\mathbb{R}^n\backslash B_r(0))} \leq Cr^{n-1}\|\varphi\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_r)},$$

if φ is L^2-orthogonal to the constant functions. Moreover, if $\varphi = \sum_{j=1}^{\infty}\varphi_i$, where the function φ_i belongs to the eigenspace associated to the eigenvalue $i(i+n-2)$, then

$$\mathcal{Q}_r(\varphi)(x) = \sum_{j=1}^{\infty} r^{n+j-2}|x|^{2-n-j}\varphi_i.$$

Here, the space $C^{2,\alpha}_{1-n}(\mathbb{R}^n\backslash B_r(0))$ is defined as the colection of function $u \in C^{2,\alpha}_{loc}(\mathbb{R}^n\backslash B_r(0))$ for which the weighted Hölder norm

$$\|u\|_{C^{2,\alpha}_{1-n}(\mathbb{R}^n\backslash B_r(0))} = \sup_{r\leq\sigma}\sigma^{n-1}\|u\|_{(2,\alpha),[\sigma,2\sigma]}$$

is finite.

Consider a number $r > 0$ smaller then r_1. Let $\varphi \in C^{2,\alpha}(\mathbb{S}^{n-1}_r)$ be a function L^2-orthogonal to the constant functions. Remember that for each $s \in (0, r_1)$ we defined $M_s = M\backslash\Psi(B_s(0))$ in the Section 2.4. Let $u_\varphi \in C^{2,\alpha}_\nu(M_r)$ be a function such that $u_\varphi \equiv 0$ in M_{r_1} and

$$u_\varphi \circ \Psi = r^{-\overline{\gamma}}|x|^{\overline{\gamma}}\mathcal{Q}_r(\pi_r(\varphi))\eta + \eta\mathcal{Q}_r(\varphi - \pi_r(\varphi)),$$

where η is a smooth radial function equal to 1 in $B_{3r}(0)$, equal to zero in $\mathbb{R}^n\backslash B_{4r}(0)$ and with the estimates $|\partial_r\eta(x)| \leq c|x|^{-1}$ and $|\partial_r^2\eta(x)| \leq c|x|^{-2}$ for all $x \in B_{r_1}(0)$. This implies that $\|\eta\|_{(2,\alpha),[\sigma,2\sigma]} \leq c$, for every $r \leq \sigma \leq r_1$. Hence, we get that $u_\varphi = \varphi$ on $\partial B_r(p)$ and by Proposition 4.1 we have

$$\|u_\varphi\|_{C^{2,\alpha}_\nu(M_r)} \leq cr^{-\nu}\|\varphi\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_r)}, \tag{52}$$

for all $\nu \geq 1-n$. The function u_φ is defined in this way because of the second term in the right hand side of (51).

Finally, define a function $G_p \in C^\infty(M\backslash\{p\})$ by $G_p \circ \Psi = \eta|x|^{2-\frac{n}{2}}$ in $B_{r_1}(p)$ and equal to zero in M_{r_1}.

Our goal in this section is to solve the equation

$$H_{g_0}(1 + \Lambda G_p + u_\varphi + v) = 0 \quad \text{on} \quad M_r, \tag{53}$$

for some $r > 0$, $\Lambda \in \mathbb{R}$ and $\varphi \in C^{2,\alpha}(\mathbb{S}^{n-1}_r)$, with $1 + \Lambda G_p + u_\varphi + v > 0$, where H_{g_0} is defined in (7).

To solve this equation we linearize H_{g_0} about 1 to get

$$H_{g_0}(1 + \Lambda G_p + u_\varphi + v) = L^1_{g_0}(\Lambda G_p + u_\varphi + v) + Q^1_{g_0}(\Lambda G_p + u_\varphi + v), \tag{54}$$

since $H_{g_0}(1) = 0$, where $L^1_{g_0}$ is defined in (11) and

$$Q^1_{g_0}(u) = \int_0^1\int_0^1 \frac{d}{ds}L^{1+tsu}_{g_0}(u)dsdt. \tag{55}$$

Therefore, if $L^1_{g_0}$ has a right inverse G_{r,g_0}, then by (54), a solution of the equation (53) is a fixed point of the map $\mathcal{M}_r(\Lambda,\varphi,\cdot) : C^{2,\alpha}_\nu(M_r) \to C^{2,\alpha}_\nu(M_r)$ given by

$$\mathcal{M}_r(\Lambda,\varphi,v) = -G_{r,g_0}\left(Q^1_{g_0}(\Lambda G_p + u_\varphi + v) + L^1_{g_0}(\Lambda G_p + u_\varphi)\right), \tag{56}$$

where $\Lambda \in \mathbb{R}$ and $\varphi \in C^{2,\alpha}(\mathbb{S}^{n-1}_r)$.

4.2. **Inverse for the operator $L^1_{g_0}$.** To find a right inverse for $L^1_{g_0}$ we will follow the method of Jleli in [17].

First let us recall the following result from [17] (see also [18]).

Lemma 4.1. *Assume that $\nu \in (1-n, 2-n)$ is fixed and that $0 < 2r < s \leq r_1$. Then there exists an operator*

$$\tilde{G}_{r,s} : C^{0,\alpha}_{\nu-2}(\Omega_{r,s}) \to C^{2,\alpha}_\nu(\Omega_{r,s})$$

such that, for all $f \in C^{0,\alpha}_\nu(\Omega_{r,s})$, the function $w = \tilde{G}_{r,s}(f)$ is a solution of

$$\left\{\begin{array}{rcccc} \Delta w & = & f & \text{in} & B_s(0)\backslash B_r(0) \\ w & = & 0 & \text{on} & \partial B_s(0) \\ w & \in & \mathbb{R} & \text{on} & \partial B_r(0) \end{array}\right. .$$

In addition,

$$\|\tilde{G}_{r,s}(f)\|_{C^{2,\alpha}_\nu(\Omega_{r,s})} \leq C\|f\|_{C^{0,\alpha}_{\nu-2}(\Omega_{r,s})},$$

for some constant $C > 0$ that does not depend on s and r.

Note that, since $R_{g_0} \neq 0$, in the previous lemma we can consider $-\frac{n-4}{8(n-2)}R_{g_0}\Delta$ instead of Δ. Therefore, by a perturbation argument we obtain the next lemma.

Lemma 4.2. *Assume that ν and $\eta > 0$ are fixed numbers with ν and $\nu-\eta$ in $(1-n, 2-n)$. Let $0 < 2r < s \le r_1$ be constants. Then, for $r_1 > 0$ small enough, there exists an operator*

$$G_{r,s} : C^{0,\alpha}_{\nu-2}(\Omega_{r,s}) \to C^{2,\alpha}_{\nu}(\Omega_{r,s})$$

such that, for all $f \in C^{0,\alpha}_{\nu}(\Omega_{r,s})$, the function $w = G_{r,s}(f)$ is a solution of

$$\begin{cases} L^1_{g_0} w &= f \quad \text{in} \quad B_s(0)\backslash B_r(0) \\ w &= 0 \quad \text{on} \quad \partial B_s(0) \\ w &\in \mathbb{R} \quad \text{on} \quad \partial B_r(0) \end{cases}.$$

In addition,

$$\|G_{r,s}(f)\|_{C^{2,\alpha}_{\nu}(\Omega_{r,s})} \le C r^{-\eta} \|f\|_{C^{0,\alpha}_{\nu-2}(\Omega_{r,s})}, \tag{57}$$

for some constant $C > 0$ that does not depend on s and r.

Proof. Note that by (11) we get that

$$\begin{aligned} L^1_{g_0}(\tilde{G}_{r,s}(v)) - v &= -\frac{n-4}{8(n-2)} R_g(\Delta_{g_0} - \Delta)\tilde{G}_{r,s}(v) - \frac{n(n-1)(n-4)}{8}\tilde{G}_{r,s}(v) \\ &+ \frac{n-4}{4(n-2)}\langle Ric_{g_0}, \nabla^2_{g_0}\tilde{G}_{r,s}(v)\rangle_{g_0}. \end{aligned}$$

Thus, by

$$\sigma^{2-\nu+\eta}\|L^1_{g_0}(\tilde{G}_{r,s}(v)) - v\|_{(0,\alpha),[\sigma,2\sigma]} \le C\sigma^{-\nu+\eta}\|\tilde{G}_{r,s}(v)\|_{(2,\alpha),[\sigma,2\sigma]}$$

we obtain that

$$\|L^1_{g_0}(\tilde{G}_{r,s}(v)) - v\|_{C^{0,\alpha}_{\nu-2-\eta}(\Omega_{r,s})} \le Cs^{\eta}\|\tilde{G}_{r,s}(v)\|_{C^{2,\alpha}_{\nu}(\Omega_{r,s})} \le Cs^{\eta}\|v\|_{C^{0,\alpha}_{\nu-2}(\Omega_{r,s})}.$$

For $s > 0$ small enough there is an inverse $(L^1_{g_0} \circ \tilde{G}_{r,s})^{-1} : C^{0,\alpha}_{\nu-2}(\Omega_{r,s}) \to C^{2,\alpha}_{\nu-2-\eta}(\Omega_{r,s})$ with bounded norm. Besides, the operator $\tilde{G}_{r,s} : C^{0,\alpha}_{\nu-2-\eta}(\Omega_{r,s}) \to C^{2,\alpha}_{\nu}(\Omega_{r,s})$ satisfies the condition

$$\|\tilde{G}_{r,s}(f)\|_{C^{2,\alpha}_{\nu}(\Omega_{r,s})} \le Cr^{-\eta}\|f\|_{C^{0,\alpha}_{\nu-2-\eta}(\Omega_{r,s})}.$$

Therefore, we have the right inverse $G_{r,s} := \tilde{G}_{r,s} \circ (L^1_{g_0} \circ \tilde{G}_{r,s})^{-1} : C^{0,\alpha}_{\nu-2}(\Omega_{r,s}) \to C^{2,\alpha}_{\nu}(\Omega_{r,s})$, with the norm estimate (57). □

Theorem 4.3. *Assume that ν and $\eta > 0$ are fixed numbers with ν and $\nu-\eta$ in $(1-n, 2-n)$. There exists $r_2 < \frac{1}{4}r_1$, such that, for all $r \in (0, r_2)$ we can define an operator $G_{r,g_0} : C^{0,\alpha}_{\nu-2}(M_r) \to C^{2,\alpha}_{\nu}(M_r)$ with the property that for all $f \in C^{0,\alpha}_{\nu-2}(M_r)$ the function $w = G_{r,g_0}(f)$ solves*

$$L^1_{g_0}(w) = f,$$

in M_r with $w \in \mathbb{R}$ constant on $\partial B_r(p)$. In addition

$$\|G_{r,g_0}(f)\|_{C^{2,\alpha}_{\nu}(M_r)} \le Cr^{-\eta}\|f\|_{C^{0,\alpha}_{\nu-2}(M_r)},$$

where $C > 0$ does not depend on r.

Proof. The proof is analogous to the proof of Proposition 13.28 in [17].

In the Lemma 4.2 we can take $s = r_1$ with $r_1 > 0$ small enough. Let $f \in C^{0,\alpha}_{\nu-2}(M_r)$ and define a function $w_0 \in C^{2,\alpha}_{\nu}(M_r)$ by $w_0 := \chi_1 G_{r,r_1}(f|_{\Omega_{r,r_1}})$ where χ_1 is a smooth, radial function equal to 1 in $B_{\frac{1}{2}r_1}(p)$, vanishing in M_{r_1} and satisfying

$|\partial_r \chi_1(x)| \le c|x|^{-1}$ and $|\partial_r^2 \chi_1(x)| \le c|x|^{-2}$ for all $x \in B_{r_1}(0)$. From this it follows that $\|\chi_1\|_{(2,\alpha),[\sigma,2\sigma]}$ is uniformly bounded in σ, for every $r \le \sigma \le \frac{1}{2}r_1$. Thus,

$$\sigma^{-\nu}\|w_0\|_{(2,\alpha),[\sigma,2\sigma]} \le C\|G_{r,r_1}(f|_{\Omega_{r,r_1}})\|_{C^{2,\alpha}_\nu(\Omega_{r,r_1})} \le Cr^{-\eta}\|f\|_{C^{0,\alpha}_{\nu-2}(M_r)}.$$

Since w_0 vanishes in M_{r_1}, we get that

$$\|w_0\|_{C^{2,\alpha}_\nu(M_r)} \le Cr^{-\eta}\|f\|_{C^{0,\alpha}_{\nu-2}(M_r)}, \tag{58}$$

where the constant $C > 0$ is independent of r and r_1.

Since $w_0 = G_{r,r_1}(f|_{\Omega_{r,r_1}})$ in $\Omega_{r,\frac{1}{2}r_1}$, the function

$$h := f - L^1_{g_0}(w_0) \tag{59}$$

is supported in $M_{\frac{1}{2}r_1}$. We can consider that h is defined on the whole M with $h \equiv 0$ in $B_{\frac{1}{2}r_1}(p)$, and using that $L^1_{g_0}$ is bounded, we get that

$$\begin{aligned} \|h\|_{C^{0,\alpha}(M)} &= \|h\|_{C^{0,\alpha}(M_{\frac{1}{2}r_1})} \le C_{r_1}\|h\|_{C^{0,\alpha}_{\nu-2}(M_r)} \\ &\le C_{r_1}(\|f\|_{C^{0,\alpha}_{\nu-2}(M_r)} + \|w_0\|_{C^{2,\alpha}_\nu(M_r)}). \end{aligned}$$

By (58) we obtain that

$$\|h\|_{C^{0,\alpha}(M)} \le C_{r_1}r^{-\eta}\|f\|_{C^{0,\alpha}_{\nu-2}(M_r)}, \tag{60}$$

with the constant $C_{r_1} > 0$ independent of r.

Since g_0 is nondegenerate, then $L^1_{g_0} : C^{2,\alpha}(M) \to C^{0,\alpha}(M)$ has a bounded inverse. This implies that we can define the function $w_1 := \chi_2(L^1_{g_0})^{-1}(h)$, where χ_2 is a smooth, radial function equal to 1 in M_{2r_2}, vanishing in $B_{r_2}(p)$ and satisfying $|\partial_r \chi_2(x)| \le c|x|^{-1}$ and $|\partial_r^2 \chi_2(x)| \le c|x|^{-2}$ for all $x \in B_{2r_2}(0)$ and some $r_2 \in (r, \frac{1}{4}r_1)$ to be chosen later. This implies that $\|\chi_2\|_{(2,\alpha),[\sigma,2\sigma]}$ is uniformly bounded in σ, for every $r \le \sigma \le \frac{1}{2}r_1$.

Hence, by (60) and the fact that $(L^1_{g_0})^{-1}$ is bounded, we get that

$$\|w_1\|_{C^{2,\alpha}_\nu(M_r)} \le C_{r_1}r^{-\eta}\|f\|_{C^{0,\alpha}_{\nu-2}(M_r)}, \tag{61}$$

since $\nu < 0$, where the constant $C_{r_1} > 0$ is independent of r and r_2.

Now, define an application $F_{r,g_0} : C^{0,\alpha}_{\nu-2}(M_r) \to C^{2,\alpha}_\nu(M_r)$ as $F_{r,g_0}(f) := w_0 + w_1$. By (58) and (61) we obtain that

$$\|F_{r,g_0}(f)\|_{C^{2,\alpha}_\nu(M_r)} \le C_{r_1}r^{-\eta}\|f\|_{C^{0,\alpha}_{\nu-2}(M_r)}, \tag{62}$$

where the constant $C_{r_1} > 0$ does not depend on r and r_2.

Therefore we get the following

i) In Ω_{r,r_2} we have that $w_0 = G_{r,r_1}(f|_{\Omega_{r,r_1}})$ and $w_1 = 0$. Therefore
$$L^1_{g_0}(F_{r,g_0}(f)) = f.$$

ii) In $\Omega_{r_2,2r_2}$ we have that $w_0 = G_{r,r_1}(f|_{\Omega_{r,r_1}})$ and $w_1 = \chi_2 L^{-1}_{g_0}(h)$. Hence
$$L^1_{g_0}(F_{r,g_0}(f)) = f + L^1_{g_0}(\chi_2(L^1_{g_0})^{-1}(h)).$$

iii) In M_{2r_2} we have that $w_1 = (L^1_{g_0})^{-1}(h)$ and by (59) we obtain
$$L^1_{g_0}(F_{r,g_0}(f)) = L^1_{g_0}(w_0) + h = f.$$

Thus, using the boundedness of $L^1_{g_0}$ and (60) we get

$$\begin{aligned}\|L^1_{g_0}(F_{r,g_0}(f))-f\|_{(0,\alpha),[\sigma,2\sigma]} &\leq \|L^1_{g_0}(\chi_2(L^1_{g_0})^{-1}(h))\|_{(0,\alpha),[\sigma,2\sigma]}\\ &\leq C_{r_1}r_2^{-3}r^{-\eta}\|f\|_{C^{0,\alpha}_{\nu-2}(M_r)},\end{aligned}$$

where the constant $C_{r_1}>0$ does not depend on r.

Therefore

$$\|L^1_{g_0}(F_{r,g_0}(f))-f\|_{C^{0,\alpha}_{\nu-2}(M_r)} \leq C_{r_1}r^{-\eta}r_2^{-1-\nu}\|f\|_{C^{0,\alpha}_{\nu-2}(M_r)},$$

since $1-n<\nu<2-n$ implies that $2-\nu>0$ and $-1-\nu>0$, for some constant $C_{r_1}>0$ independent of r and r_2. If we consider $r_2=2r$, then

$$\|L^1_{g_0}(F_{r,g_0}(f))-f\|_{C^{0,\alpha}_{\nu-2}(M_r)} \leq C_{r_1}r^{-1-\nu-\eta}\|f\|_{C^{0,\alpha}_{\nu-2}(M_r)}. \tag{63}$$

The assertion follows from a perturbation argument by (62) and (63). ☐

4.3. **Constant σ_2-curvature metrics on $M\backslash B_r(p)$.** Now we will show that the map $\mathcal{M}_r(\Lambda,\varphi,\cdot)$, given by (56), is a contraction. As a consequence the fixed point depends continuously on the parameters r, Λ and φ.

Proposition 4.2. *Let $\nu\in(1-n,2-n)$ and $\eta>0$ small enough. Let β, γ, δ_4, δ_5 and l be fixed positive constants such that $l>\max\{\delta_5,2\delta_4\}$. There exists $r_2\in(0,r_1/4)$ such that if $r\in(0,r_2)$, $\Lambda\in\mathbb{R}$ with $|\Lambda|^2\leq r^{n+l+\delta_5}$ and $\varphi\in C^{2,\alpha}(\mathbb{S}^{n-1}_r)$ is a function L^2-orthogonal to the constant functions with $\|\varphi\|_{(2,\alpha),r}\leq\beta r^{2+l-\delta_4}$, then there exists a fixed point of the map $\mathcal{M}_r(\Lambda,\varphi,\cdot)$ in the ball of radius $\gamma r^{2+l-\nu}$ in $C^{2,\alpha}_\nu(M_r)$.*

Proof. By (56) and Theorem 4.3 it follows that

$$\begin{aligned}\|\mathcal{M}_r(\Lambda,\varphi,0)\|_{C^{2,\alpha}_\nu(M_r)} &\leq Cr^{-\eta}\Big(\|Q^1_{g_0}(\Lambda G_p+u_\varphi)\|_{C^{0,\alpha}_{\nu-2}(\Omega_{r,r_1})}\\ &+ \|L_g(\Lambda G_p+u_\varphi)\|_{C^{0,\alpha}_{\nu-2}(\Omega_{r,r_1})}\Big),\end{aligned}$$

for some constant $C>0$ independent of r, since the functions G_p, u_φ and h are equal to zero in $M\backslash B_{4r}(p)$.

By (55) we get that

$$\|Q^1_g(\Lambda G_p+u_\varphi)\|_{(0,\alpha),[\sigma,2\sigma]} \leq C\sigma^{-4}\|\Lambda G_p+u_\varphi\|^2_{(2,\alpha),[\sigma,2\sigma]}$$

and then by (52) we obtain that

$$\begin{aligned}\sigma^{2-\nu}\|Q^1_g(\Lambda G_p+u_\varphi)\|_{(0,\alpha),[\sigma,2\sigma]} &\leq C\sigma^{-2-\nu}\|\Lambda G_p+u_\varphi\|^2_{(2,\alpha),[\sigma,2\sigma]}\\ &\leq C(r^{l-\delta_5}+r^{l-2\delta_4})r^{2+l-\nu},\end{aligned}$$

since $G_p=O(|x|^{2-\frac{n}{2}})$ and $u_\varphi=O(|x|^{1-n})$. Thus we get that

$$\|Q^1_{g_0}(\Lambda G_p+u_\varphi)\|_{C^{0,\alpha}_{\nu-2}(\Omega_{r,s})}\leq r^{\delta_6}r^{2+l-\nu},$$

with $\delta_6>0$. Now, by (11) we obtain that

$$\|L^1_g(\Lambda G_p+u_\varphi)\|_{(0,\alpha),[\sigma,2\sigma]}\leq C(\sigma^{d-3}+1)\|\Lambda G_p+u_\varphi\|_{(2,\alpha),[\sigma,2\sigma]},$$

and this implies that

$$\begin{aligned}\sigma^{2-\nu}\|L_g(\Lambda G_p+u_\varphi)\|_{(0,\alpha),[\sigma,2\sigma]} &\leq C(\sigma^{d-1-\nu}+\sigma^{2-\nu})\|\Lambda G_p+u_\varphi\|_{(2,\alpha),[\sigma,2\sigma]}\\ &\leq C\left(r^{d-1-\frac{l}{2}+\frac{\delta_5}{2}}+r^{2-\frac{l}{2}+\frac{\delta_5}{2}}+r^{d-1-\delta_4}+r^{2-\delta_4}\right)r^{2+l-\nu}.\end{aligned}$$

Therefore, choosing $\eta > 0$ small enough, we obtain that

$$\|\mathcal{M}_r(\Lambda, \varphi, 0)\|_{C^{2,\alpha}_\nu(M_r)} \leq \frac{1}{2}\gamma r^{2+l-\nu}. \tag{64}$$

Now, note that

$$\begin{aligned}
&\|\mathcal{M}(v_1) - \mathcal{M}(v_2)\|_{C^{2,\alpha}_\nu(M_r)} \\
&\leq Cr^{-\eta}\|Q^1_{g_0}(\Lambda G_p + u_\varphi + v_1) - Q^1_{g_0}(\Lambda G_p + u_\varphi + v_2)\|_{C^{0,\alpha}_{\nu-2}(M_r)} \\
&= Cr^{-\eta}\Big(\|Q^1_{g_0}(v_1) - Q^1_{g_0}(v_2)\|_{C^{0,\alpha}(M_{r_1})} \\
&+ \|Q^1_{g_0}(\Lambda G_p + u_\varphi + v_1) - Q^1_{g_0}(\Lambda G_p + u_\varphi + v_2)\|_{C^{0,\alpha}_{\nu-2}(\Omega_{r,r_1})}\Big).
\end{aligned}$$

Since $r > 0$ is small and $\|v_1\|_{C^{2,\alpha}_\nu(M_r)} < \gamma r^{2+l-\nu}$, then

$$\begin{aligned}
&\|Q^1_{g_0}(v_1) - Q^1_{g_0}(v_2)\|_{C^{0,\alpha}(M_{r_1})} \\
&\leq C(\|v_1\|_{C^{2,\alpha}_\nu(M_r)} + \|v_2\|_{C^{2,\alpha}_\nu(M_r)})\|v_1 - v_2\|_{C^{2,\alpha}_\nu(M_r)} \\
&\leq Cr^{2+l-\nu}\|v_1 - v_2\|_{C^{2,\alpha}_\nu(M_r)}.
\end{aligned} \tag{65}$$

We have that

$$\begin{aligned}
&\|Q^1_{g_0}(\Lambda G_p + u_\varphi + v_1) - Q^1_{g_0}(\Lambda G_p + u_\varphi + v_2)\|_{(0,\alpha),[\sigma,2\sigma]} \\
&\leq C\sigma^{-4}\left(|\Lambda|\sigma^{2-\frac{n}{2}} + \|u_\varphi\|_{(2,\alpha),[\sigma,2\sigma]} + \sigma^{2+l}\right)\|v_1 - v_2\|_{(2,\alpha),[\sigma,2\sigma]} \\
&\leq C\left(\sigma^{-2-\frac{n}{2}}r^{\frac{n}{2}+\frac{l}{2}+\frac{\delta_5}{2}} + \sigma^{-3-n}r^{1+n+l-\delta_4} + \sigma^{-2+l}\right)\|v_1 - v_2\|_{(2,\alpha),[\sigma,2\sigma]}.
\end{aligned}$$

Then

$$\begin{aligned}
&\sigma^{2-\nu}\|Q^1_{g_0}(\Lambda G_p + u_\varphi + v_1) - Q^1_{g_0}(\Lambda G_p + u_\varphi + v_2)\|_{(0,\alpha),[\sigma,2\sigma]} \\
&\leq C\left(r^{(l+\delta_5)/2} + r^{l-\delta_4}\right)\|v_1 - v_2\|_{C^{2,\alpha}_\nu(M_r)},
\end{aligned}$$

and this together with (65) implies that

$$\|\mathcal{M}_r(\Lambda, \varphi, v_1) - \mathcal{M}_r(\lambda, \varphi, v_2)\|_{C^{2,\alpha}_\nu(M_r)} \leq \frac{1}{2}\|v_1 - v_2\|_{C^{2,\alpha}_\nu(M_r)}, \tag{66}$$

for $r > 0$ small enough. Therefore, by (64) and (66) we obtain the result. □

By Proposition 4.2 we get the main result of this section.

Theorem 4.4. *Let $\nu \in (1-n, 2-n)$ and $\eta > 0$ small enough. Let β, γ, δ_4, δ_5 and l be fixed positive constants such that $l > \{\delta_5, 2\delta_4\}$. There exists $r_2 > 0$ such that if $r \in (0, r_2)$, $\Lambda \in \mathbb{R}$ with $|\Lambda|^2 \leq r^{n+l+\delta_5}$ and $\varphi \in C^{2,\alpha}(\mathbb{S}^{n-1}_r)$ is a function L^2-orthogonal to the constant functions with $\|\varphi\|_{(2,\alpha),r} \leq \beta r^{2+l-\delta_4}$, then there exists a solution $V_{\Lambda,\varphi} \in C^{2,\alpha}_\nu(M_r)$ to the problem*

$$\begin{cases} H_{g_0}(1 + \Lambda G_p + u_\varphi + V_{\Lambda,\varphi}) = 0 & \text{in} \quad M_r \\ (u_\varphi + V_{\Lambda,\varphi}) \circ \Psi|_{\partial B_r(0)} - \varphi \in \mathbb{R} & \text{on} \quad \partial M_r \end{cases}.$$

Moreover,

$$\|V_{\Lambda,\varphi}\|_{C^{2,\alpha}_\nu(M_r)} \le \gamma r^{2+l-\nu} \tag{67}$$

and

$$\|V_{\Lambda,\varphi_1} - V_{\Lambda,\varphi_2}\|_{C^{2,\alpha}_\nu(M_r)} \le C r^{\delta_6-\nu}\|\varphi_1-\varphi_2\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_r)}, \tag{68}$$

for some constant $\delta_6 > 0$ small enough independent of r.

Proof. The solution $V_{\Lambda,\varphi}$ is the fixed point of $\mathcal{M}_r(\Lambda,\varphi,\cdot)$ given by Proposition 4.2 with the estimate (67). The inequality (68) follows similarly to (50), that is, it follows by the estimates obtained by the proof of the Proposition 4.2. □

If g is the metric given in the previous section, then there is a function f such that $g_0 = f^{\frac{8}{n-4}}g$ and in the normal coordinate system centered at p with respect to g we have $f = 1+O(|x|^2)$, in fact, $f = 1/\mathcal{F}$. We will denote the full conformal factor of the resulting constant σ_2-curvature metric in M_r with respect to the metric g as $\mathcal{B}_r(\Lambda,\varphi)$, that is, the metric

$$\tilde{g} = \mathcal{B}_r(\Lambda,\varphi)^{\frac{8}{n-4}}g$$

has $\sigma_2(A_g) = n(n-1)/8$, where

$$\mathcal{B}_r(\Lambda,\varphi) := f + \Lambda f G_p + f u_\varphi + f V_{\Lambda,\varphi}.$$

5. Gluing the initial data

In the previous sections we have constructed two families of constant σ_2-curvature, one of them is a family of complete metrics defined on a punctured geodesic ball $B_{r_\varepsilon}(p)\backslash\{p\}$ and the other family is defined on the complement $M_{r_\varepsilon} := M\backslash B_{r_\varepsilon}(p)$. Both families have prescribed Dirichlet boundary data and they depend on $n+2$ parameters.

In this section we examine suitable choices of the parameter sets on each piece so that the Cauchy data can be made to match up to be C^1 at the boundary of $B_{r_\varepsilon}(p)$. In this way we obtain a weak solution to $H_{g_0}(u) = 0$ on $M\backslash\{p\}$. Since g_0 is conformal to some 2-admissible metric, then we can use elliptic regularity to show that the glued solution is a smooth metric.

To do this we will split the equation that the Cauchy data must satisfy in an equation corresponding to the high frequencies space, another one corresponding to the space spanned by the constant functions, and n equations corresponding to the space spanned by the coordinate functions.

By the Theorem 3.2 there exists a family of metrics in $\overline{B_{r_\varepsilon}(p)}\backslash\{p\}$, for small enough $r_\varepsilon = \varepsilon^s > 0$ with $0 < s < 1$, satisfying the following

$$\hat{g} = \mathcal{A}_\varepsilon(R,a,\phi)^{\frac{8}{n-4}}g,$$

with $\sigma_2(A_{\hat{g}}) = n(n-1)/8$. Here we have that

$$\mathcal{A}_\varepsilon(R,a,\phi) = u_{\varepsilon,R,a} + r_\varepsilon^{-\overline{\gamma}}|x|^{\overline{\gamma}}v_\phi + h + U_{\varepsilon,R,a,\phi},$$

in g-conformal normal coordinates centered at p, where

- I1) $R^{\frac{4-n}{4}} = 2(1+b)\varepsilon^{\frac{4-n}{4}}$ with $|b| \le 1/2$;
- I2) $\phi \in C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})^\perp$ with $\|\phi\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \le \kappa r_\varepsilon^{2+l-\delta_1}$, $l > 0$ and $\delta_1 > 0$ is small and $\kappa > 0$ is some constant to be chosen later;
- I3) $|a| r_\varepsilon^{1-\delta_2} \le 1$ where $\delta_2 > 0$ is a small number with $3\delta_2 > \max\{\delta_1, l\}$;

I4) $h = \dfrac{1}{2}\left((1-\overline{\gamma})r_\varepsilon^{-\overline{\gamma}-1}|x|^{\overline{\gamma}+1} + (\overline{\gamma}+1)r_\varepsilon^{-\overline{\gamma}+1}|x|^{\overline{\gamma}-1}\right)\overline{f}$, with $\overline{f} = O(|x|^2)$;

I5) $U_{\varepsilon,R,a,\phi} \in C^{2,\alpha}_{(\gamma,\overline{\gamma})}(B_r(0)\backslash\{0\})$ satisfies the inequalities (49) and (50) and the condition $\pi_{r_\varepsilon}(U_{\varepsilon,R,a,\phi}|_{\partial B_{r_\varepsilon}(0)}) = 0$;

I6) $\gamma = \delta_{n,2} + 1 - n/4 - \varepsilon_1$ and $\overline{\gamma} = n/4 + 1 + \varepsilon_1$ with $\varepsilon_1 > 0$ a small constant, where $\delta_{n,2}$ is defined in (26).

Also, by Theorem 4.4 there exists a family of metrics in M_{r_ε}, for small enough $r_\varepsilon > 0$, given by

$$\tilde{g} = \mathcal{B}_{r_\varepsilon}(\lambda,\varphi)^{\frac{8}{n-4}}g$$

with $\sigma_2(A_{\tilde{g}}) = n(n-1)/8$. Here we have that

$$\mathcal{B}_{r_\varepsilon}(\Lambda,\varphi) = f + \Lambda f G_p + f u_\varphi + f V_{\lambda,\varphi},$$

in g-conformal normal coordinates centered at p, where

E1) $f = 1 + \overline{f}$ with $\overline{f} = O(|x|^2)$;

E2) $\Lambda \in \mathbb{R}$ with $|\Lambda|^2 \leq r_\varepsilon^{n+l+\delta_5}$, with l and δ_5 constants such that $l > \delta_5$;

E3) $\varphi \in C^{2,\alpha}(\mathbb{S}^{n-1}_r)$ is a function L^2-orthogonal to the constant functions and with $\|\varphi\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_r)} \leq \beta r_\varepsilon^{2+l-\delta_4}$ where β is a positive constant to be chosen later and δ_4 is a constant such that $2\delta_4 < l$;

E4) $V_{\Lambda,\varphi} \in C^{2,\alpha}_\nu(M_{r_\varepsilon})$ is a function such that on $\partial M_{r_\varepsilon}$ it is constant and it satisfies the inequalities (67) and (68).

Our purpose is to show that there are parameters $R \in \mathbb{R}_+$, $a \in \mathbb{R}^n$, $\Lambda \in \mathbb{R}$ and functions φ and ϕ in $C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})$ such that

$$\left\{\begin{array}{lcl} \mathcal{A}_\varepsilon(R,a,\phi) & = & \mathcal{B}_{r_\varepsilon}(\Lambda,\varphi) \\ \partial_r\mathcal{A}_\varepsilon(R,a,\phi) & = & \partial_r\mathcal{B}_{r_\varepsilon}(\Lambda,\varphi) \end{array}\right. \tag{69}$$

on $\partial B_{r_\varepsilon}(p)$.

First, we take the function $\overline{f}$ in I4) equal to the respective function in E1) and δ_1 in I2) equal to δ_4 in E3). Now, if we take ω and ϑ in the ball of radius $r_\varepsilon^{2+l-\delta_1}$ in $C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})$, with ω belonging to the space spanned by the coordinate functions, ϑ belonging to the high frequencies space, and we define $\varphi := \omega + \vartheta$, then we can apply Theorem 4.4 to define the function $\mathcal{B}_{r_\varepsilon}(\Lambda,\varphi)$, since $\|\varphi\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \leq 2r_\varepsilon^{2+l-\delta_1}$.

By the definition of the function h in I4), we get that

$$\pi_{r_\varepsilon}(\mathcal{A}_\varepsilon(R,a,\phi)) = \phi + \pi_{r_\varepsilon}(u_{\varepsilon,R,a} + \overline{f})$$

and by the definition of u_φ and G_p in Section 4.1, we obtain that

$$\pi_{r_\varepsilon}(\mathcal{B}_r(\Lambda,\varphi)) = \vartheta + \pi_{r_\varepsilon}(\overline{f} + \Lambda\overline{f}|x|^{2-\frac{n}{2}} + \overline{f}u_\varphi + \overline{f}V_{\Lambda,\varphi}).$$

Here we used that $\pi_{r_\varepsilon}(u_\varphi|_{\mathbb{S}^{n-1}_{r_\varepsilon}}) = \vartheta$, $\pi_{r_\varepsilon}(V_{\Lambda,\varphi}|_{\mathbb{S}^{n-1}_{r_\varepsilon}}) = 0$ and $f = 1 + \overline{f}$.

Define

$$\begin{array}{rcl} \phi_\vartheta & := & \pi_{r_\varepsilon}((\mathcal{B}_{r_\varepsilon}(\Lambda,\omega+\vartheta) - u_{\varepsilon,R,a} - \overline{f})|_{\mathbb{S}^{n-1}_{r_\varepsilon}}) \\ & = & \pi_{r_\varepsilon}((\Lambda\overline{f}|x|^{2-\frac{n}{2}} + \overline{f}u_{\omega+\vartheta} + \overline{f}V_{\Lambda,\omega+\vartheta} - u_{\varepsilon,R,a})|_{\mathbb{S}^{n-1}_{r_\varepsilon}}) + \vartheta. \end{array} \tag{70}$$

We have to derive an estimate for $\|\phi_\vartheta\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})}$. Note that, by (23), we get that

$$\pi_{r_\varepsilon}(u_{\varepsilon,R,a}|_{\mathbb{S}^{n-1}_{r_\varepsilon}}) = O(|a|^2 r_\varepsilon^2), \tag{71}$$

since $r_\varepsilon = \varepsilon^s$ and $R^{\frac{4-n}{4}} = 2(1+b)\varepsilon^{\frac{4-n}{4}}$, with $s \in (0,1)$ small enough and $|b| \leq 1/2$, implies that $R < r_\varepsilon$. If a is a point in $\mathbb{R}^n$ with $|a|^2 \leq r_\varepsilon^l$ then we have that $|a| r_\varepsilon^{1-\delta_2} \leq r_\varepsilon^{1+\frac{l}{2}-\delta_2}$ with $1 + l/2 - \delta_2 > 0$. This implies that $|a| r_\varepsilon^{1-\delta_2} \leq 1$ for $\varepsilon > 0$ small enough. Furthermore, since $|a|^2 r_\varepsilon^2 \leq r_\varepsilon^{2+l}$, we can show that

$$\|\pi_{r_\varepsilon}(u_{\varepsilon,R,a}|_{\mathbb{S}^{n-1}_{r_\varepsilon}})\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \leq C r_\varepsilon^{2+l}, \tag{72}$$

for some constant $C > 0$ independent of ε, R and a.

Observe that, E2) implies

$$\|\pi_{r_\varepsilon}((\Lambda \overline{f} |x|^{2-\frac{n}{2}})|_{\mathbb{S}^{n-1}_{r_\varepsilon}})\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \leq C r_\varepsilon^{2+l}. \tag{73}$$

Now, using (52), (67), (70) and the fact that $\overline{f} = O(|x|^2)$, we deduce that

$$\|\phi_\vartheta - \vartheta\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \leq c r_\varepsilon^{2+l}. \tag{74}$$

Therefore

$$\|\phi_\vartheta\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \leq c r_\varepsilon^{2+l-\delta_1}, \tag{75}$$

for every $\vartheta \in \pi(C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon}))$ in the ball of radius $r_\varepsilon^{2+l-\delta_1}$, where $c > 0$ is some constant that does not depend on ε. Remember that $l > \delta_4 = \delta_1$.

We observe here that the function h in (51) it is important to find the estimate (75). Otherwise, the term $\pi_{r_\varepsilon}(\overline{f})$ would appear in the second equality in (70) and instead of the exponent $2 + l - \delta_1$ we would get only $2 - \delta_1$. But, for our purpose we need an exponent bigger than $2 - \delta_1$.

Now, by the estimate (75) we can apply the Theorem 3.2 with κ in I2) equal to the constant c in (75). Thus $\mathcal{A}_\varepsilon(R, a, \phi_\vartheta)$ is well defined. By the definition (70) immediately yields

$$\pi_{r_\varepsilon}(\mathcal{A}_\varepsilon(R, a, \phi_\vartheta)|_{\mathbb{S}^{n-1}_{r_\varepsilon}}) = \pi_{r_\varepsilon}(\mathcal{B}_{r_\varepsilon}(\Lambda, \omega + \vartheta)|_{\mathbb{S}^{n-1}_{r_\varepsilon}}).$$

We project the second equation of the system (69) on the high frequencies space, the space of functions which are L^2-orthogonal to $e_0, \ldots, e_n$. This yields a nonlinear equation which can be written as

$$r_\varepsilon \partial_r (v_\vartheta - u_\vartheta) + \mathcal{S}_\varepsilon(a, b, \Lambda, \omega, \vartheta) = 0, \tag{76}$$

on $\partial_r B_{r_\varepsilon}(0)$, where

$$\begin{aligned} \mathcal{S}_\varepsilon(a, b, \Lambda, \omega, \vartheta) &= r_\varepsilon \partial_r v_{\phi_\vartheta - \vartheta} + r_\varepsilon \partial_r \pi_{r_\varepsilon}(u_{\varepsilon,R,a}|_{\mathbb{S}^{n-1}_{r_\varepsilon}}) \\ &+ r_\varepsilon \partial_r \pi_{r_\varepsilon}((U_{\varepsilon,R,a,\phi_\vartheta} - \Lambda \overline{f} G_p - \overline{f} u_{\omega+\vartheta})|_{\mathbb{S}^{n-1}_{r_\varepsilon}}) \\ &- r_\varepsilon \partial_r \pi_{r_\varepsilon}((\overline{f} V_{\Lambda,\omega+\vartheta})|_{\mathbb{S}^{n-1}_{r_\varepsilon}}). \end{aligned}$$

The map $\mathcal{Z} : C^{2,\alpha}(\mathbb{S}^{n-1})^\perp \to C^{0,\alpha}(\mathbb{S}^{n-1})^\perp$ defined by

$$\mathcal{Z}(\vartheta) := \partial_r (v_\vartheta - \mathcal{Q}_1(\vartheta)),$$

is an isomorphism (see [17], proof of Proposition 8 in [25] and proof of Proposition 2.6 in [29]). Here $\mathcal{Q}_r$ is given by Proposition 4.1. On the other hand, for any $\vartheta \in C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})^\perp$ we can show that

$$\mathcal{Z}(\overline{\vartheta}) = r_\varepsilon \partial_r (v_\vartheta - \mathcal{Q}_{r_\varepsilon}(\vartheta))(r_\varepsilon \cdot),$$

where $\overline{\vartheta} = \vartheta(r_\varepsilon \cdot)$, see [32] for more details. Therefore a solution to the equation (76) is a fixed point of the map $\mathcal{H}_\varepsilon(a,b,\Lambda,\omega,\cdot) : \mathcal{D}_\varepsilon \to C^{2,\alpha}(\mathbb{S}^{n-1})^\perp$ given by

$$\mathcal{H}_\varepsilon(a,b,\Lambda,\omega,\vartheta) = -\mathcal{Z}^{-1}(\mathcal{S}_\varepsilon(a,b,\Lambda,\omega,\vartheta_{r_\varepsilon})(r_\varepsilon \cdot)),$$

where $\mathcal{D}_\varepsilon := \{\vartheta \in \pi_1(C^{2,\alpha}(\mathbb{S}^{n-1})); \|\vartheta\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} \leq r_\varepsilon^{2+l-\delta_1}\}$ and $\vartheta_{r_\varepsilon}(x) := \vartheta(r_\varepsilon^{-1}x)$.

Lemma 5.1. *There exists a constant $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0,\varepsilon_0)$, $a \in \mathbb{R}^n$, $b,\Lambda \in \mathbb{R}$ with $|a|^2 \leq r_\varepsilon^l$, $|b| \leq 1/2$ and $|\Lambda|^2 \leq r_\varepsilon^{n+l+\delta_5}$, and each $\omega \in C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})$ in the space spanned by the coordinate functions and with norm bounded by $r_\varepsilon^{2+l-\delta_1}$, then the map $\mathcal{H}_\varepsilon(a,b,\Lambda,\omega,\cdot)$ has a fixed point in $\mathcal{D}_\varepsilon$.*

Proof. First note that by (74), ϕ_0 satisfies

$$\|\phi_0\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \leq c r_\varepsilon^{2+l},$$

where the constant $c > 0$ is independent of ε.

From (31), (49), (52), (67), (72) and (73) and the fact that $\overline{f} = O(|x|^2)$ we obtain that

$$\|\mathcal{S}_\varepsilon(a,b,\lambda,\omega,0)\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \leq c r_\varepsilon^{2+l},$$

for some constant $c > 0$ independent of ε. This implies that

$$\|H_\varepsilon(a,b,\Lambda,\omega,0)\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} \leq \frac{1}{2} r_\varepsilon^{2+l-\delta_1}, \tag{77}$$

for $\varepsilon > 0$ small enough.

Now, if $\vartheta_1, \vartheta_2 \in \mathcal{D}_\varepsilon$, then

$$\begin{aligned}
&\|\mathcal{H}_\varepsilon(a,b,\Lambda,\omega,\vartheta_1) - \mathcal{H}_\varepsilon(a,b,\Lambda,\omega,\vartheta_2)\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} \\
&\quad \leq C \left(\left\| r_\varepsilon \partial_r v_{\phi_{\vartheta_{r_\varepsilon,1}} - \vartheta_{r_\varepsilon,1} - (\phi_{\vartheta_{r_\varepsilon,2}} - \vartheta_{r_\varepsilon,2})} \right\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \right. \\
&\qquad + \left\| r_\varepsilon \partial_r \pi_{r_\varepsilon}((U_{\varepsilon,R,a,\phi_{\vartheta_{r_\varepsilon,1}}} - U_{\varepsilon,R,a,\phi_{\vartheta_{r_\varepsilon,2}}})|_{\mathbb{S}^{n-1}_{r_\varepsilon}}) \right\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \\
&\qquad + \left\| r_\varepsilon \partial_r \pi_{r_\varepsilon}((\overline{f}(V_{\Lambda,\omega+\vartheta_{r_\varepsilon,1}} - V_{\Lambda,\omega+\vartheta_{r_\varepsilon,2}}))|_{\mathbb{S}^{n-1}_{r_\varepsilon}}) \right\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \\
&\qquad \left. + \left\| \| r_\varepsilon \partial_r \pi_{r_\varepsilon}((\overline{f} u_{\vartheta_{r_\varepsilon,1} - \vartheta_{r_\varepsilon,2}})|_{\mathbb{S}^{n-1}_{r_\varepsilon}}) \right\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \right).
\end{aligned} \tag{78}$$

By (70) we get

$$\phi_{\vartheta_{r_\varepsilon,1}} - \vartheta_{r_\varepsilon,1} - (\phi_{\vartheta_{r_\varepsilon,2}} - \vartheta_{r_\varepsilon,2}) = \pi_{r_\varepsilon}((\overline{f} u_{\vartheta_{r_\varepsilon,1} - \vartheta_{r_\varepsilon,2}} + \overline{f}(V_{\lambda,\omega+\vartheta_{r_\varepsilon,1}} - V_{\lambda,\omega+\vartheta_{r_\varepsilon,2}}))|_{\mathbb{S}^{n-1}_{r_\varepsilon}}).$$

Using the inequalities (52) and (68) and the fact that $\overline{f} = O(|x|^2)$, we obtain that

$$\|\phi_{\vartheta_{r_\varepsilon,1}} - \vartheta_{r_\varepsilon,1} - (\phi_{\vartheta_{r_\varepsilon,2}} - \vartheta_{r_\varepsilon,2})\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \leq c r_\varepsilon^{\delta_7} \|\vartheta_{r_\varepsilon,1} - \vartheta_{r_\varepsilon,2}\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})},$$

for some constants $\delta_7 > 0$ and $c > 0$ that does not depend on ε. Using (31) we have that

$$\|r_\varepsilon \partial_r v_{\phi_{\vartheta_{r_\varepsilon,1}} - \vartheta_{r_\varepsilon,1} - (\phi_{\vartheta_{r_\varepsilon,2}} - \vartheta_{r_\varepsilon,2})}\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})} \leq c r_\varepsilon^{\delta_7} \|\vartheta_1 - \vartheta_2\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}. \tag{79}$$

By (50) and (68) we conclude that

$$\|U_{\varepsilon,R,a,\phi_{\vartheta_{r_\varepsilon,1}}} - U_{\varepsilon,R,a,\phi_{\vartheta_{r_\varepsilon,2}}}\|_{(2,\alpha),[\frac{1}{2}r_\varepsilon, r_\varepsilon]} \leq C r_\varepsilon^{\delta_4} \|\vartheta_{r_\varepsilon,1} - \vartheta_{r_\varepsilon,2}\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})}$$

and

$$\|V_{\Lambda,\omega+\vartheta_{r_\varepsilon,1}} - V_{\Lambda,\omega+\vartheta_{r_\varepsilon,2}}\|_{(2,\alpha),[r_\varepsilon,2r_\varepsilon]} \leq Cr_\varepsilon^{\delta_6}\|\vartheta_{r_\varepsilon,1} - \vartheta_{r_\varepsilon,2}\|_{C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})},$$

for some $\delta_4 > 0$ and $\delta_6 > 0$ independent of ε. From this, (52) and the fact that $f = 1 + \overline{f}$, we derive an estimate as (79) for the other terms in (78). Therefore we get

$$\|\mathcal{H}_\varepsilon(a,b,\Lambda,\omega,\vartheta_1) - \mathcal{H}_\varepsilon(a,b,\Lambda,\omega,\vartheta_2)\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} \leq \frac{1}{2}\|\vartheta_1 - \vartheta_2\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}, \tag{80}$$

for all $\vartheta_1, \vartheta_2 \in \mathcal{D}_\varepsilon$, since $\varepsilon > 0$ is small enough. By (77) and (80) we get the result. □

Therefore there exists a unique solution of (76) in the ball of radius $r_\varepsilon^{2+l-\delta_1}$ in $C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})$. We denote by $\vartheta_{\varepsilon,a,b,\Lambda,\omega}$ the solution given by Lemma 5.1. Since this solution is obtained through the application of fixed point theorem for contraction mappings, it is continuous with respect to the parameters ε, a, b, Λ and ω.

Now, recall that $R^{\frac{4-n}{4}} = 2(1+b)\varepsilon^{\frac{4-n}{4}}$ with $|b| \leq 1/2$. Hence, using (71) and Corollary 2.1 and 2.2 we can show that

$$\begin{aligned} u_{\varepsilon,R,a}(r_\varepsilon\theta) &= 1 + b + \frac{\varepsilon^{\frac{n-4}{2}}}{4(1+b)}{r_\varepsilon}^{2-\frac{n}{2}} + \left(\frac{n-4}{2}u_{\varepsilon,R}(r_\varepsilon\theta) + r\partial_r u_{\varepsilon,R}(r_\varepsilon\theta)\right)a\cdot x \\ &+ O(|a|^2r_\varepsilon^2) + O(\varepsilon^{\frac{n+4}{2}}{r_\varepsilon}^{-\frac{n}{2}}), \end{aligned}$$

where the last term does not depend on $\theta \in \mathbb{S}^{n-1}$. Hence, we have

$$\begin{aligned} \mathcal{A}_\varepsilon(R,a,\phi_{\vartheta_{\varepsilon,a,b,\Lambda,\omega}})(r_\varepsilon\theta) = 1 + b &+ \frac{\varepsilon^{\frac{n-4}{2}}}{4(1+b)}{r_\varepsilon}^{2-\frac{n}{2}} + v_{\phi_{\vartheta_{\varepsilon,a,b,\lambda,\omega}}}(r_\varepsilon\theta) \\ &+ \left(\frac{n-4}{2}u_{\varepsilon,R}(r_\varepsilon\theta) + r_\varepsilon\partial_r u_{\varepsilon,R}(r_\varepsilon\theta)\right)r_\varepsilon a\cdot\theta + \overline{f}(r_\varepsilon\theta) \\ &+ U_{\varepsilon,R,a,\phi_{\vartheta_{\varepsilon,a,b,\lambda,\omega}}}(r_\varepsilon\theta) + O(|a|^2r_\varepsilon^2) + O(\varepsilon^{\frac{n+4}{2}}r_\varepsilon^{-\frac{n}{2}}). \end{aligned} \tag{81}$$

In g-conformal normal coordinate system in the neighborhood of $\partial M_{r_\varepsilon}$, namely $\Omega_{r_\varepsilon,\frac{1}{2}r_1}$, we have

$$\begin{aligned} \mathcal{B}_{r_\varepsilon}(\Lambda,\omega+\vartheta_{\varepsilon,a,b,\Lambda,\omega})(r_\varepsilon\theta) = 1 &+ \Lambda r_\varepsilon^{2-\frac{n}{2}} + u_{\omega+\vartheta_{\varepsilon,a,b,\lambda,\omega}}(r_\varepsilon\theta) \\ &+ (\overline{f}u_{\omega+\vartheta_{\varepsilon,a,b,\Lambda,\omega}})(r_\varepsilon\theta) + (fV_{\Lambda,\omega+\vartheta_{\varepsilon,a,b,\Lambda,\omega}})(r_\varepsilon\theta) \\ &+ O(|\Lambda|r_\varepsilon^{4-\frac{n}{2}}) + \overline{f}(r_\varepsilon\theta). \end{aligned} \tag{82}$$

We now project the system (69) on the set of functions spanned by the constant function. This yields the equations

$$\left\{\begin{aligned} b + \left(\frac{\varepsilon^{\frac{n-4}{2}}}{4(1+b)} - \Lambda\right)r_\varepsilon^{2-\frac{n}{2}} &= \mathcal{H}_{0,\varepsilon}(a,b,\Lambda,\omega) \\ \left(2-\tfrac{n}{2}\right)\left(\frac{\varepsilon^{\frac{n-4}{2}}}{4(1+b)} - \Lambda\right)r_\varepsilon^{2-\frac{n}{2}} &= r_\varepsilon\partial_r\mathcal{H}_{0,\varepsilon}(a,b,\Lambda,\omega) \end{aligned}\right., \tag{83}$$

where $\mathcal{H}_{0,\varepsilon}$ and $\partial_r\mathcal{H}_{0,\varepsilon}$ are continuous maps. Using (81), (82), the estimates (49), (52), (67) and E2) we can show that

$$\mathcal{H}_{0,\varepsilon}(a,b,\Lambda,\omega)=O(r_\varepsilon^{2+l}) \quad \text{and} \quad r_\varepsilon\partial_r\mathcal{H}_{0,\varepsilon}(a,b,\Lambda,\omega)=O(r_\varepsilon^{2+l}). \tag{84}$$

Using these last estimates we obtain the following result.

Lemma 5.2. *There exists a constant $\varepsilon_2>0$ such that if $\varepsilon\in(0,\varepsilon_2)$, $a\in\mathbb{R}^n$ with $|a|^2\le r_\varepsilon^l$ and $\omega\in C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})$ in the space spanned by the coordinate functions and with norm bounded by $r_\varepsilon^{2+l-\delta_1}$, then the system (83) has a solution $(b,\Lambda)\in\mathbb{R}^2$, with $|b|\le 1/2$ and $|\Lambda|^2\le r_\varepsilon^{n+l+\delta_5}$.*

Proof. Define a continuous map $\mathcal{G}_{\varepsilon,a,\omega}:\mathcal{D}_{0,\varepsilon}\to\mathbb{R}^2$ by

$$\begin{aligned}\mathcal{G}_{\varepsilon,a,\omega}(b,\Lambda) \quad := \quad & \left(\frac{2r_\varepsilon}{n-4}\partial_r\mathcal{H}_{0,\varepsilon}(a,b,\Lambda,\omega)+\mathcal{H}_{0,\varepsilon}(a,b,\Lambda,\omega),\right.\\ & \left.\frac{\varepsilon^{\frac{n-4}{2}}}{4(1+b)}+\frac{2r_\varepsilon^{\frac{n}{2}-1}}{n-4}\partial_r\mathcal{H}_{0,\varepsilon}(a,b,\Lambda,\omega)\right),\end{aligned}$$

where $\mathcal{D}_{0,\varepsilon}:=\left\{(b,\Lambda)\in\mathbb{R}^2;|b|\le 1/2 \text{ and } |\Lambda|^2\le r_\varepsilon^{n+l+\delta_5}\right\}$.

Then, using (84) and the fact that $r_\varepsilon=\varepsilon^s$, with $s\in(0,1)$ small enough, we can show that $\mathcal{G}_{\varepsilon,a,\omega}(\mathcal{D}_{0,\varepsilon})\subset\mathcal{D}_{0,\varepsilon}$, for small enough $\varepsilon>0$. With the previous estimates we can show that this last map is a contraction. Hence it has a fixed point which depends continuously on the parameters ε, a and ω. It is easy to show that this fixed point is a solution of the system (83). □

From now on we will work with the fixed point given by Lemma 5.2 and we will write simply as (b,Λ), without subscript.

Finally, we project the system (69) over the space of functions spanned by the coordinate functions. It will be convenient to decompose ω in

$$\omega=\sum_{i=1}^n\omega_ie_i, \quad \text{where} \quad \omega_i=\int_{\mathbb{S}^{n-1}}\omega(r_\varepsilon\cdot)e_i. \tag{85}$$

Note that $|\omega_i|\le c_n\sup_{\mathbb{S}^{n-1}_{r_\varepsilon}}|\omega|$. Thus, by (81), (82) and Proposition 4.1, for each $i=1,\dots,n$ we get the system

$$\left\{\begin{array}{rcl} F(r_\varepsilon)r_\varepsilon a_i-\omega_i & = & \mathcal{H}_{i,\varepsilon}(a,\omega)\\ G(r_\varepsilon)r_\varepsilon a_i-(1-n)\omega_i & = & r_\varepsilon\partial_r\mathcal{H}_{i,\varepsilon}(a,\omega),\end{array}\right. \tag{86}$$

where

$$F(r_\varepsilon):=\frac{n-4}{2}u_{\varepsilon,R}(r_\varepsilon\theta)+r_\varepsilon\partial_ru_{\varepsilon,R}(r_\varepsilon\theta)$$

and

$$G(r_\varepsilon):=\frac{n-4}{2}u_{\varepsilon,R}(r_\varepsilon\theta)+\frac{n}{2}r_\varepsilon\partial_ru_{\varepsilon,R}(r_\varepsilon\theta)+r_\varepsilon^2\partial_r^2u_{\varepsilon,R}(r_\varepsilon\theta).$$

The maps $\mathcal{H}_{i,\varepsilon}$ and $\partial_r\mathcal{H}_{i,\varepsilon}$ are continuous. By the previous estimates we get that

$$\mathcal{H}_{i,\varepsilon}(a,\omega)=O(r_\varepsilon^{2+l}) \quad \text{and} \quad r_\varepsilon\partial_r\mathcal{H}_{i,\varepsilon}(a,\omega)=O(r_\varepsilon^{2+l}). \tag{87}$$

Lemma 5.3. *There exists a constant $\varepsilon_2 > 0$ such that if $\varepsilon \in (0, \varepsilon_2)$ then the system (86) has a solution $(a, \omega) \in \mathbb{R}^n \times C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})$ with $|a|^2 \leq r_\varepsilon^l$ and ω given by (85) with norm bounded by $r_\varepsilon^{2+l-\delta_1}$.*

Proof. We observe that by the Corollary 2.1 and the fact that $R^{\frac{4-n}{4}} = 2(1+b)\varepsilon^{\frac{4-n}{4}}$ we obtain that

$$G(r_\varepsilon) + (n-1)F(r_\varepsilon) = \frac{n(n-4)}{2}(1+b) + O(\varepsilon^{\frac{n-4}{2}(1-s)}),$$

where $s > 0$ is a fixed small number and $|b| \leq 1/2$. Thus, for $\varepsilon > 0$ small enough, we can define a continuous map $\mathcal{K}_{i,\varepsilon} : \mathcal{D}_{i,\varepsilon} \to \mathbb{R}^2$ by

$$\mathcal{K}_{i,\varepsilon}(a_i, \omega_i) := \big(r_\varepsilon^{-1}(G(r_\varepsilon) + (n-1)F(r_\varepsilon))^{-1}(r_\varepsilon \partial_r H_{i,\varepsilon}(a,\omega) + (n-1)H_{i,\varepsilon}(a,\omega)),$$

$$(G(r_\varepsilon) + (n-1)F(r_\varepsilon))^{-1} F(r_\varepsilon)(r_\varepsilon \partial_r H_{i,\varepsilon}(a,\omega) + (n-1)H_{i,\varepsilon}(a,\omega)) - H_{i,\varepsilon}(a,\omega)\Big),$$

where $\mathcal{D}_{i,\varepsilon} := \{(a_i, \omega_i) \in \mathbb{R}^2; |a_i|^2 \leq n^{-1} r_\varepsilon^l$ and $|\omega_i| \leq n^{-1} k_i^{-1} r_\varepsilon^{2+l-\delta_1}\}$ and $k_i = \|e_i\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}$.

By (87) we obtain that $\mathcal{K}_{i,\varepsilon}(\mathcal{D}_{i,\varepsilon}) \subset \mathcal{D}_{i,\varepsilon}$, for small enough $\varepsilon > 0$. By the Brouwer's fixed point theorem there exists a fixed point for each map $\mathcal{K}_{i,\varepsilon}$.

Therefore, we obtain a solution $(a, \omega) \in \mathbb{R}^n \times C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})$ of the system (86). $\square$

Now we are ready to prove the main theorem of this paper.

Theorem 5.4. *Let (M^n, g_0) be a compact Riemannian manifold nondegenerate with dimension $n \geq 5$, g_0 conformal to some 2-admissible metric and the σ_2-curvature equal to $n(n-1)/8$. Let $\{p_1, \ldots, p_m\}$ a set of points in M such that $\nabla^j_{g_0} W_{g_0}(p_i) = 0$ for $j = 0, \ldots, \left[\frac{n-4}{2}\right]$ and $i = 1, \ldots, m$, where W_{g_0} is the Weyl tensor of the metric g_0. Then, there exist a constant $\varepsilon_0 > 0$ and a one-parameter family of complete metrics g_ε on $M \backslash \{p_1, \ldots, p_m\}$ defined for $\varepsilon \in (0, \varepsilon_0)$ such that*

1. *each g_ε is conformal to g_0 and has constant σ_2-curvature equal to $n(n-1)/8$;*
2. *g_ε is asymptotically Delaunay near each point p_i, for all $i = 1, \ldots, m$;*
3. *$g_\varepsilon \to g_0$ uniformly on compact sets in $M \backslash \{p_1, \ldots, p_m\}$ as $\varepsilon \to 0$.*

Proof. First we prove the theorem for $m = 1$ and then we will explain the minor changes that need to be made in order to deal with more than one singular point.

We keep the previous notations. Using the Theorem 3.2 we find a family of constant scalar curvature metrics in the punctured geodesic ball $\overline{B_{r_\varepsilon}(p)} \backslash \{p\}$, for small enough $\varepsilon > 0$, given by

$$\hat{g} = \mathcal{A}_\varepsilon(R, a, \phi)^{\frac{8}{n-4}} g,$$

with the parameters $R \in \mathbb{R}^+$, $a \in \mathbb{R}^n$ and $\phi \in C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})^\perp$ satisfying the conditions I1 to I5 in the page 31.

Using the Theorem 4.4 we obtain a family of constant scalar curvature metrics in $M \backslash B_{r_\varepsilon}(p)$, for small enough $\varepsilon > 0$, given by

$$\tilde{g} = \mathcal{B}_{r_\varepsilon}(\Lambda, \varphi)^{\frac{8}{n-4}} g,$$

with the parameters $\Lambda \in \mathbb{R}$ and $\varphi \in C^{2,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})$ satisfying the conditions E1 to E4 in the page 32.

Using the Lemmas 5.1, 5.2 and 5.3 we conclude that there exists a constant $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are parameters R_ε, a_ε, ϕ_ε, Λ_ε and φ_ε for which the functions $\mathcal{A}_\varepsilon(R_\varepsilon, a_\varepsilon, \phi_\varepsilon)$ and $\mathcal{B}_{r_\varepsilon}(\Lambda_\varepsilon, \varphi_\varepsilon)$ coincide up to order one in $\partial B_{r_\varepsilon}(p)$.

Since g_0 is conformal to some 2-admissible metric, we can use elliptic regularity to show that the function $\mathcal{U}_\varepsilon$ defined by $\mathcal{U}_\varepsilon := \mathcal{A}_\varepsilon(R_\varepsilon, a_\varepsilon, \phi_\varepsilon)$ in $B_{r_\varepsilon}(p)\backslash\{p\}$ and $\mathcal{U}_\varepsilon := \mathcal{B}_{r_\varepsilon}(\Lambda_\varepsilon, \varphi_\varepsilon)$ in $M\backslash B_{r_\varepsilon}(p)$ is a positive smooth function in $M\backslash\{p\}$. Moreover, since $\mathcal{A}_\varepsilon(R_\varepsilon, a_\varepsilon, \phi_\varepsilon) \geq c|x|^{\frac{4-n}{4}}$, for some constant $c > 0$, then the function $\mathcal{U}_\varepsilon$ tends to infinity on approach to p with sufficiently fast rate to ensure that the metric $g_\varepsilon := \mathcal{U}_\varepsilon^{\frac{8}{n-4}} g$ is complete in $M\backslash\{p\}$.

Since the metric g is conformal to the metric g_0, then g_ε is a one-parameter family of complete smooth metric defined in $M\backslash\{p\}$ conformal to g_0 with Delaunay-type ends and σ_2-curvature equal to $n(n-1)/8$. In fact, by Theorem 3.2 and 4.4 the metric g_ε satisfies i), ii) and iii).

To prove the general case, we will just explain the minor changes that need to be made. We direct the reader to [32] for more details.

The interior analysis is done around at each point p_i as before in the Section 3, where we can find a family of complete metrics defined in $B_{r_{\varepsilon_i}}(p)\backslash\{p\}$, with $\varepsilon_i = t_i\varepsilon$, $\varepsilon > 0$, $t_i \in (\delta, \delta^{-1})$ and $\delta > 0$ fixed, $i = 1, \ldots, m$.

In order to get a family of metrics as in the Section 4 we need to make some changes. First we consider conformal normal coordinates around at point p_i. Then we define the spaces $C^{l,\alpha}_\nu(M\backslash\{p_1, \ldots, p_m\})$ and $C^{2,\alpha}_\nu(M_r)$ as in the Section 6 in [32]. For each $\varphi = (\varphi_1, \ldots, \varphi_m)$, where $\varphi_i \in C^{2,\alpha}(\mathbb{S}^{n-1}_r)$ is a function L^2-orthogonal to the constant functions, we define $u_\varphi \in C^{2,\alpha}_\nu(M_r)$ such that near each point p_i, the function u_φ is as in the Section 4.1. Then we prove a theorem analogous to the Theorem 4.3 in this context. Finally, we use this result to obtain a family of metrics with constant σ_2-curvature in the complement in M of the union of balls centered at each p_i. Therefore, using again that g_0 is conformal to some 2-admissible metric, we can use elliptic regularity to get the general result. $\square$

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