Exercises for Algebra 2
List 9 To hand in at 19.10.2020

Exercise 1.
Let $\mathbb{F}_p[x,y]$ be the polynomial ring in two variables x and y and $\mathbb{F}_p(x,y)$ its fraction field. Let $\sqrt[p]{x}$ be a root of $T^p - x$ and $\sqrt[p]{y}$ be a root of $T^p - y$.

1. Show that $\mathbb{F}_p(\sqrt[p]{x}, \sqrt[p]{y})$ is a field extension of $\mathbb{F}_p(x,y)$ of degree p^2.
2. Show that $a^p \in \mathbb{F}_p(x,y)$ for every $a \in \mathbb{F}_p(\sqrt[p]{x}, \sqrt[p]{y})$.
3. Conclude that the field extension $\mathbb{F}_p(\sqrt[p]{x}, \sqrt[p]{y}) \,/\, \mathbb{F}_p(x,y)$ has no primitive element and that it has infinitely many intermediate extensions.

Exercise 2.
Consider the purely transcendental extension $K = \mathbb{F}_3(x)/\mathbb{F}_3$ of transcendence degree 1, and let $\overline{K}$ be an algebraic closure of K. Let $a \in \overline{K}$ be a root of $f = T^3 - x$ and $b \in \overline{K}$ a root of $g = T^2 - 2$. Find the separable closure E of K in $K(a,b)$. What are the degrees $[K(a,b) : E]$ and $[E : K]$? What are the corresponding separable degrees and inseparable degrees?

Exercise 3.
Let $K = K_0(c_0, \dots, c_{n-1})$ be a rational function field in $c_0, \dots, c_{n-1}$ over K_0 and $f = T^n + c_{n-1}T^{n-1} + \dots + c_0 \in K[T]$. Let L be the splitting field of f and $a_1, \dots, a_n$ its roots. Show that $K_0(a_1, \dots, a_n)$ is a rational function field in $a_1, \dots, a_n$ over K_0.

Remark: This shows that we can remove the corresponding hypothesis from Thm. 4.6.9 of the lecture (Abel's theorem).

Exercise 4.
Let S and T be two transcendental bases of a field extension L/K. Let $\mathcal{S}$ be the collection of all bijections $\alpha : S' \to T'$ between subsets $S' \subset S$ and $T' \subset T$, partially ordered by the rule that $\alpha_1 \leq \alpha_2$ for $\alpha_i : S'_i \to T'_i$ in $\mathcal{S}$ $(i = 1,2)$ whenever $S'_1 \subset S'_2$, $T'_1 \subset T'_2$ and $\alpha_1(s) = \alpha_2(s)$ for all $s \in S'_1$. Show that every chain in $\mathcal{S}$ (which is a linearly ordered subset) has an upper bound in $\mathcal{S}$.

***Exercise 5.**
Let K be a field of characteristic different from 2 and $K(x)$ a rational function field in x over K. Let $f = T^2 - x^3 - x \in K(x)$ and L the splitting field of f over $K(x)$. Show that L/K is *not* purely transcendental.