Exercises for Algebra 2
List 6 To hand in at 28.9.2020

Exercise 1. Let L be the splitting field of $T^3 - 2$ over $\mathbb{Q}$. Show that $\sqrt[3]{2}$, $\sqrt{-3}$ and ζ_3 are elements of L. Calculate $\mathrm{N}_{L/\mathbb{Q}}(a)$ and $\mathrm{Tr}_{L/\mathbb{Q}}(a)$ for $a = \sqrt[3]{2}$, $a = \sqrt{-3}$ and $a = \zeta_3$. Calculate $\mathrm{N}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(\zeta_3)$ and $\mathrm{Tr}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(\zeta_3)$.

Exercise 2.
Let L/K be a finite Galois extension and let

$$M_a : \begin{array}{ccc} L & \longrightarrow & L \\ b & \longmapsto & a \cdot b \end{array}$$

be the K-linear map associated with an element $a \in L$. Show that the trace of M_a equals $\mathrm{Tr}_{L/K}(a)$ and that the determinant of M_a equals $\mathrm{N}_{L/K}(a)$.

Hint: Deduce the claim from the special cases that $a \in K$ and that L is the splitting field of the minimal polynomial of a over K, using Exercise 1 from List 2.

Exercise 3.
Let L be the splitting field of $f = T^4 - 3$ over $\mathbb{Q}$. What is the Galois group of $L/\mathbb{Q}$? Make a diagram of all subgroups of $\mathrm{Gal}(L/\mathbb{Q})$ that illustrates which subgroups are contained in others. Describe which intermediate extensions E/F of $L/\mathbb{Q}$ (i.e. $\mathbb{Q} \subset E \subset F \subset L$) are cyclotomic and which are Kummer extensions.

Hint: Find the four roots $a_1, \dots, a_4 \in \mathbb{C}$ of f. Which permutations of $a_1, \dots, a_4$ extend to field automorphisms of L?

Exercise 4.
Let p, q be distinct prime numbers.

1. Describe an irreducible polynomial $f \in \mathbb{F}_p[T]$ of degree p.
2. For $i = 1, \dots, 5$, consider the extensions $K(a_i)/K$ of $K = \mathbb{F}_p(x) = \mathrm{Frac}\, \mathbb{F}_p[x]$ where x is an indeterminate over $\mathbb{F}_p$ and $a_i \in \overline{K}$ is a root of f_i for

$$f_1 = \sum_{i=0}^{q-1} T^i, \quad f_2 = \sum_{i=0}^{p-1} T^i, \quad f_3 = T^q - x, \quad f_4 = T^p - x, \quad f_5 = T^p - T - x.$$

   Which of the extensions $K(a_i)/K$ are separable, normal, cyclotomic, Kummer and Artin-Schreier?

***Exercise 5.** By Exercise 2, we can extend the definition of the trace $\mathrm{Tr}_{L/K} : L \to K$ to any finite field extension L/K: the trace $\mathrm{Tr}_{L/K}(a)$ of an element $a \in L$ is defined as the trace of the K-linear map $M_a : L \to L$ that is defined by $M_a(b) = ab$.

1. Let K be of characteristic p and $L = K(a)$ where a is a root of $f = T^p - b \in K[T]$, which we assume to be irreducible. Show that for every $i = 1, \dots, p-1$, the minimal polynomial of a^i over K is $f_i = T^p - b^i$, and conclude that all elements $a, \dots, a^{p-1} \in L$ are inseparable over K. Show that $\mathrm{Tr}_{L/K}(a^i) = 0$ for all $i = 0, \dots, p-1$. Conclude that $\mathrm{Tr}_{L/K} : L \to K$ is constant zero.

   Remark: You can use without proof that $\mathrm{Tr}_{L/K}(b+c) = \mathrm{Tr}_{L/K}(b) + \mathrm{Tr}_{L/K}(c)$.

2. Assume that L/K is separable. Show that $\mathrm{Tr}_{L/K} : L \to K$ is not constant zero.

   Remark: You can use without proof that $\mathrm{Tr}_{L/K} = \mathrm{Tr}_{E/K} \circ \mathrm{Tr}_{L/E}$ for any intermediate field E of L/K, and that the normal closure L^{norm} of L/K is a separable extension of K if L/K is separable.

3. Assum that L/K is not separable. Show that $\mathrm{Tr}_{L/K} : L \to K$ is constant zero.

   Hint: Use the separable closure of K in L to reduce the claim to the case that $[L : K]_s = 1$, and deduce this special case from Lemma 3.2.1 of the lecture.