Exercises for Algebra 2
List 4 To hand in at 14.9.2020

Exercise 1.
Let $\zeta_3 = e^{2\pi i/3} \in \mathbb{C}$ be a primitive third root of unity, i.e. $\zeta_3^3 = 1$, but $\zeta_3 \neq 1$. Which of the field extensions $\mathbb{Q}(\zeta_3)$, $\mathbb{Q}(\sqrt[3]{5})$ and $\mathbb{Q}(\zeta_3, \sqrt[3]{5})$ of $\mathbb{Q}$ are Galois? What are the respective automorphism groups over $\mathbb{Q}$? Find all subgroups of $\mathrm{Aut}(\mathbb{Q}(\zeta_3, \sqrt[3]{5}))$ and all intermediate extensions of $\mathbb{Q}(\zeta_3, \sqrt[3]{5})/\mathbb{Q}$, and draw the corresponding diagrams.

Exercise 2.
Calculate the Galois groups of the splitting fields of the following polynomials over $\mathbb{Q}$.

1. $f_1 = T^3 - 1$;
2. $f_2 = T^3 - 2$;
3. $f_3 = T^3 + T^2 - 2T - 1$.

Hint: $\zeta_7^i + \zeta_7^{7-i}$ is a root of f_3 for $i = 1, 2, 3$.

Exercise 3.
Let K be a field and G a finite subgroup of the multiplicative group $K^\times$. Show that G is cyclic, which can be done along the following lines.

1. Let $\varphi(d)$ be the number of generators of a cyclic group of order d. Show for $n \geq 1$ that
$$\sum_{d|n} \varphi(d) \quad = \quad n.$$
Remark: The function $\varphi(d)$ is called *Euler's φ-function.*
2. Let $G_d \subset G$ be the subset of elements of order d. Show that G_d is empty if d is not a divisor of n and that G_d has exactly $\varphi(d)$ elements if it is not empty.
Hint: Use that $T^d - 1$ has at most d roots in a field.
3. Let n be the cardinality of G. Conclude that G must have an element of order n and that G is cyclic.

Exercise 4 (Cyclotomic polynomials)**.**
Let $\mu_\infty = \{\zeta \in \overline{\mathbb{Q}} | \zeta^n = 1 \text{ for some } n \geq 1\}$. Define

$$\Phi_d \;=\; \prod_{\substack{\zeta\in\mu_\infty \\ \text{of order } d}} (T - \zeta).$$

1. Show that $\prod_{d|n} \Phi_d = T^n - 1$ for $n \geq 1$.
2. Show that Φ_d has integral coefficients, i.e. $\Phi_d \in \mathbb{Z}[T]$.
3. Let $\zeta \in \mu_\infty$ be of order d. Show that Φ_d is the minimal polynomial of ζ over $\mathbb{Q}$.
4. Conclude that $\deg \Phi_d = \varphi(d)$ and that Φ_d is irreducible in $\mathbb{Z}[T]$.
5. Show that $\Phi_d = T^{d-1} + \cdots + T + 1$ if d is prime.
6. Calculate Φ_d for $d = 1, \ldots, 12$.

The polynomial Φ_d is called the *d-th cyclotomic polynomial.*

***Exercise 5.**
Show that $\mathbb{R}$ has no non-trivial field automorphisms.