Exercises for Algebra 2
List 2 To hand in at 31.8.2020

Exercise 1.
Let L/K be a finite field extension and $a \in L$ algebraic over K. Let $f(T) \in K[T]$ be the minimal polynomial of a over K. Show that the minimal polynomial of the K-linear map

$$\begin{array}{rccc} M_a: & L & \longrightarrow & L \\ & b & \longmapsto & a \cdot b \end{array}$$

is equal to f.

Exercise 2.
Let L/K be a finite field extension. Then there are elements $a_1, \ldots, a_n \in L$ such that $L = K(a_1, \ldots, a_n)$.

Exercise 3.
Let L/K be a field extension and $a_1, \ldots, a_n \in L$. Show that $K(a_1, \ldots, a_n)/K$ is algebraic if and only if $a_1, \ldots, a_n$ are algebraic over K.

Exercise 4.

1. Consider $\sqrt[3]{2} \in \mathbb{R}$. Show that $\sqrt[3]{2}$ is algebraic over $\mathbb{Q}$ and find its minimal polynomial. What is the degree $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$?
2. Let $\zeta_3 = e^{2\pi i/3} \in \mathbb{C}$. Show that ζ_3 is algebraic over $\mathbb{Q}$ and find its minimal polynomial. What is the degree $[\mathbb{Q}(\zeta_3) : \mathbb{Q}]$?
3. What is the degree of $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ over $\mathbb{Q}$?

Remark: The element ζ_3 is called a *primitive third root of unity.*

*__Exercise 5.__ [1]
Let K be an infinite field and $\overline{K}$ an algebraic closure of K. Show that $\overline{K}$ has the same cardinality as K. What is the cardinality of $\overline{K}$ if K is finite?

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[1] The starred exercises are bonus exercises. It is encouraged to work on these exercises and to hand in solution. The resulting points count as a bonus.