Exercises for Algebra 2
List 14 To hand in at 23.11.2020

Exercise 1.
Let G be a group acting on a set X and $\mathbb{C}^X$ the corresponding permutation representation. Show that the contragredient representation of $\mathbb{C}^X$ is isomorphic to $\mathbb{C}^X$.

Exercise 2.
Let $V_1, \ldots, V_5$ be pairwise non-isomorphic irreducible complex representations of S_4. Calculate the decomposition of $V_i \otimes V_j$ into irreducible components for all $i, j \in \{1, \ldots, 5\}$.

Bonus: Find an explicit description $K_0(S_4) = \mathbb{Z}[T_1, \ldots, T_r]/(f_1, \ldots, f_s)$ of the Grothendieck group of S_4 (as a ring!) in terms of generators $T_1, \ldots, T_r$ and relations $f_1, \ldots, f_s$; cf. the bonus exercises from lists 12 and 13.

Exercise 3.
Let G be a finite group.

1. Let χ and χ' be simple characters and $\chi'(e) = 1$. Show that $\chi \cdot \chi'$ is simple.
2. Define $\chi^*(g) = \overline{\chi(g)}$. Show that χ^* is a character with $(\chi^*)^* = \chi$. Show that χ^* is simple if and only if χ is simple.
3. Let $\sigma : G \to G$ be a group automorphism and χ a character. Define $\chi^\sigma(g) = \chi(\sigma(g))$. Show that χ^σ is a character, and that χ^σ is simple if and only if χ is simple.
4. Conclude from the previous parts of this exercise that if for a given dimension d, there is a unique simple character χ with $\chi(e) = d$, then
   a) $\chi(g) = 0$ if there is a simple character χ' with $\chi'(e) = 1$ and $\chi'(g) \neq 1$;
   b) $\chi(g) \in \mathbb{R}$ for all $g \in G$;
   c) $\chi(\sigma(g)) = \chi(g)$ for all automorphisms σ of G.

Exercise 4.
Let $G = \{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{F}_3) \,|\, a, b, d \in \mathbb{F}_3 \}$ be the subgroup of upper triangular matrices.

1. Determine all conjugacy classes of G.
2. Show that $N = \{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \,|\, b \in \mathbb{F}_3 \}$ is a normal subgroup of G and that $G^{\mathrm{ab}} = G/N$.
3. Determine all one dimensional characters of G.
4. Let X be the conjugacy class of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Show that G acts by conjugation on X, which defines a permutation representation $\mathbb{C}^X$. Show that decomposes into the direct sum of the one dimensional trivial representation and an irreducible 2-dimensional representation.
5. Complete the character table of G.

Exercise 5 (Bonus)**.**
Show that the elements e, $(12)(34)$, (123), (12345) and (12354) form a complete set of representatives for the conjugacy classes of A_5 and show that the character table of A_5 is

	e	$(12)(34)$	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$
χ_3	3	-1	0	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

This can be done along the following steps:

1. Calculate the size of each conjugacy class.
2. The trivial character χ_1 comes for free.
3. Calculate the character χ of the permutation representation of A_5 on 5 elements. Show that $\langle \chi, \chi_1 \rangle = 1$ and that $\chi_4 := \chi - \chi_1$ is a simple character.
4. Let $d_i = \chi_i(e)$. Determine the only possibility for the values of d_2, d_3 and d_5 such that $60 = \sum_{i=1}^{5} d_i^2$.
5. Since $(12)(34)$ has order 2, the only possible eigenvalues in each representation are ± 1. Conclude that $\chi_i(e)$ are odd integers for $i = 2, 3, 5$ and that $|\chi_i(e)| \leq 3$ for $i = 2, 3$ and $|\chi_5(e)| \leq 5$. Use the orthogonality relations for the first two columns of the character table to determine the only possible values of $\chi_i((12)(34))$ for $i = 2, 3, 5$.
6. Show that the conjugation with an element of S_5 defines an automorphism of A_5. Conclude that $\sigma(12345) = (12354)$ for some automorphism σ of A_5. Use Exercise 2 to show that $\chi_5(12345) = \chi_5(12354)$. Use the row orthogonality relations to determine the only possible values of χ_5.
7. Use the column orthogonality relations to find the only possible values of $\chi_i((123))$ for $i = 2, 3$.
8. Use the row orthogonality relations to compute the missing values of χ_2 and χ_3.

Hint: A solution can be found at https://groupprops.subwiki.org/wiki/Determination_of_character_table_of_alternating_group:A5.