Exercises for Algebra 2
List 11 To hand in at 2.11.2020

Exercise 1.
Let G be a finite group of order n and K a field whose characteristic does not divide n. Show that

$$\pi(v) \;=\; \frac{1}{n}\cdot\sum_{g\in G} g.v$$

defines a G-invariant projection $\pi : V \to V^G$, i.e. π is surjective and $\pi\circ\pi = \pi$. Conclude that π is the zero map if and only if there is no G-equivariant homomorphism from the 1-dimensional trivial representation to V.

Exercise 2.
Show that the action of S_4 on the vertices of the regular tetrahedron defines an irreducible 3-dimensional real representation V_3 of S_4. Show that the permutation representation of S_4 on $\mathbb{R}^4$, by permuting $\{1,2,3,4\}$, is isomorphic to the direct sum of V_3 with the trivial 1-dimensional representation.

Exercise 3.
Establish the following basic facts on the group algebra $K[G]$.

1. Complete the proof of Proposition 1.1.3: show that $\mathrm{Mod}^{\mathrm{fg}}(K[G])$ and $\mathrm{Rep}_K(G)$ are equivalent categories.
2. Show that $K[G_1\times G_2]$ and $K[G_1]\otimes_K K[G_2]$ are isomorphic rings for finite groups G_1 and G_2.
3. Formulate and prove a universal property for $K[G]$.

Exercise 4.
Let K be a field that contains a primitive n-th root of unity ζ_n.

1. Let G be a cyclic group of order n with generator g. Show that for every $k = 1,\dots,n$, the map

   $$\begin{array}{rccc}\pi_k: & K[G] & \longrightarrow & K\\ & \sum c_i g^i & \longmapsto & \sum c_i\zeta_n^{ki}\end{array}$$

   is a ring homomorphism.
2. Show that $\pi = (\pi_1,\dots,\pi_n) : K[G]\to K^n$ is a ring isomorphism.

   Hint: Note that the restrictions $\chi_k : G\to K$ of π_k to G are characters in the sense of the first part of the course, which are linearly independent by Theorem 4.4.3. Why does this imply that the kernel of π is trivial?
3. Conclude that $K[G]\simeq K^n$ for every finite abelian group G of order n if $\zeta_n\in K$.

***Exercise 5** (not to hand in)**.** Recall the following notions from category theory: categories and functors; monomorphisms, epimorphisms and isomorphism; product and coproduct; initial and terminal object; kernel and cokernel; image and coimage; adjoint functors and equivalences of categories.