Exercises for Algebra 2
List 10 To hand in at 26.10.2020

Exercise 1.
Let G be a finite group, K a field and V a finite dimensional K-vector space. Let $\theta : G \times V \to V$ be a *K-linear group action*, i.e. a map that satisfies

$$e.v = v, \quad (gh).v = g.(h.v), \quad g.(v+w) = g.v + g.w \quad \text{and} \quad g.(\lambda v) = \lambda(g.v)$$

for all $g, h \in G$, $\lambda \in K$ and $v, w \in V$. Show that there is a unique K-linear representation $\rho : G \to \mathrm{GL}(V)$ such that $\theta(g, v) = \rho(g)v$ for all $g \in G$ and $v \in V$. Show that every K-linear representation $\rho : G \to \mathrm{GL}(V)$ is of this form, i.e. $\rho(g)v = \theta(g, v)$ for some K-linear group action $\theta : G \times V \to V$.

Exercise 2.
Determine all 1-dimensional real representations of the symmetric group S_n for $n \geq 2$.

Exercise 3.
Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group with $i^2 = j^2 = k^2 = -1$ and $ij = k$. Show that there exists a unique 3-dimensional real representation $\rho : Q_8 \to \mathrm{GL}_3(\mathbb{R})$ with

$$\rho(i) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \rho(j) \;=\; \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Show that ρ is not irreducible and determine $\ker \rho$.

Exercise 4.
Consider the natural action of S_3 on $X = \{1, 2, 3\}$ and the associated permutation representation $\rho : S_3 \to \mathrm{GL}_3(\mathbb{C})$. Determine all subrepresentations of ρ. Which subrepresentations are irreducible?

*__Exercise 5.__
Let G be a finite group and $[G, G] = \langle\, aba^{-1}b^{-1} \mid a, b \in G \,\rangle$ its *commutator subgroup.* Show that $[G, G]$ is a normal subgroup of G and that $G^{\mathrm{ab}} = G/[G, G]$ is an abelian group. Show that every group homomorphism $\varphi : G \to H$ into an abelian group H factors into the quotient map $\pi : G \to G^{\mathrm{ab}}$ followed by a uniquely determined group homomorphism $\varphi^{\mathrm{ab}} : G^{\mathrm{ab}} \to H$. Conclude that $[G, G] \subset \ker \rho$ for every 1-dimensional representation ρ of G.